Lecture 15

Recall

Thm [Comb Null.] non-zero f on $\mathbb{F}_p^n$, $S_1, \dots, S_n \subseteq \mathbb{F}_p$

f has a term $x_1^{k_1} \cdots x_n^{k_n}$ w/ non-zero coeff. $\Rightarrow$ f cannot vanish on $\prod S_i$

Permanent lem • A $n \times n$ matrix over $\mathbb{F}$ w/ $\operatorname{per}(A) \neq 0$

$\forall\, b = (b_1, \dots, b_n) \in \mathbb{F}^n$, $\forall\, S_1, \dots, S_n \subseteq \mathbb{F}$ w/ $|S_i| = 2$

$\Rightarrow \exists\, x \in \prod_{i=1}^n S_i$ s.t. $Ax - b$ has no zero entry, i.e.

$\forall\, i \in [n]$, $(Ax)_i \neq b_i$.

Thm [Erdős - Ginzburg - Ziv] • p prime
• multiset A of elt in $\mathbb{Z}_p$ of size $|A| = 2p-1$

$\Rightarrow \exists$ sub-multiset of A w/ size p, whose elts sum up to 0.

Pf: • Order elt. of A in non-decreasing order as

$$0 \le a_1 \le a_2 \le \cdots \le a_p \le a_{p+1} \le \cdots \le a_{2p-1} \le p-1$$

We may assume $\forall\, i \in [p-1]$, $a_i \neq a_{i+p-1}$

for o.w. $a_i = a_{i+1} = \cdots = a_{i+p-1}$ which sum up to $p \cdot a_i = 0$ in $\mathbb{Z}_p$ ☺

• Let $S_i = \{a_i, a_{i+p-1}\}$ for all $i \in [p-1]$

$|S_i| = 2$

- Let $J = (1)_{(p-1)\times(p-1)}$ all-1 matrix

$$\Rightarrow \quad per(J) = (p-1)! \neq 0$$

- Let $b = (b_1, \dots, b_{p-1})$ be s.t.

$$B := \{b_1, \dots, b_{p-1}\} = \mathbb{Z}_p \setminus \{-a_{2p-1}\}$$

- Perm. lem $\Rightarrow \exists x \in \prod_{i=1}^{p-1} S_i$ s.t.

$$\forall i \in [p-1], \quad (Jx)_i \neq b_i$$

$$\Leftrightarrow \quad (Jx)_i = x_1 + \cdots + x_{p-1} \notin B$$

$$\Rightarrow \quad x_1 + \cdots + x_{p-1} = -a_{2p-1}$$

$$\Rightarrow \quad x_1 + \cdots + x_{p-1} + a_{2p-1} = 0$$

as $x_i \in S_i$, all S_i disjoint $\Rightarrow$ size-p multisubset w/ sum 0 $\blacksquare$

Part 3 Pseudorandomness

- In Szemerédi's regularity lem:

partition G into bounded # parts s.t.

b/w most pairs, we see pseudorandom graphs.

- pseudorandomness
- quasirandomness
- regularity
- uniformity

} refering to obj. that are random-like.

§ Quasirandom graphs.

- 80s introduced by Thomason, indep Chung-Graham-Wilson.
- Shall see some properties of graphs which at first look irrelevant, but turns out to be equiv. to one another.

<u>Notation</u>

- define edge density $p = \frac{e(G)}{\binom{n}{2}}$
- for adj. matrix A of G, write $\lambda_1, \ldots, \lambda_n$ w/
- $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ for eigenvalues of A.
- for $u, v \in V(G)$, write $d(u,v) = \text{codegree} = |N_G(u) \cap N_G(v)|$
- subgraph (homomorphism) density of H in G.

(# label homomorphisms)

$$t(H, G) = \frac{\#\text{ labeled copies of } H}{|V(G)|^{|V(H)|}} = \mathbb{P}(\text{a unif random map } f \text{ induce a copy of } H)$$

f is a homom.

$f : V(H) \longrightarrow V(G)$ # maps $= |V(G)|^{|V(H)|}$

$$t_{ind}(H, G) = \frac{\#\text{ induced labeled copies of } H}{|V(G)|^{|V(H)|}}$$

$$= \mathbb{P}(\overset{\text{unif}}{f} \text{ induces an induced copy of } H)$$

Thm Let $p \in (0,1)$ and G be a d-regular graph w/ $d = pn$, then the following properties are equivalent.

- [induced subgraph count] $\forall H$,

  Exercise $\qquad t_{ind}(H,G) = p^{e(H)}(1-p)^{e(\bar{H})} + o(1)$ as $n \to \infty$

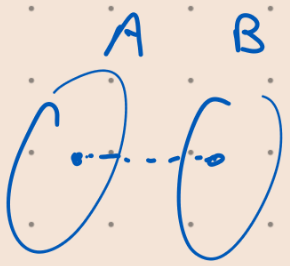
- [subgraph count] $\forall H$,

  Definition $\qquad t(H,G) = p^{e(H)} + o(1)$

- [4-cycle count] $t(C_4, G) \le p^4 + o(1)$

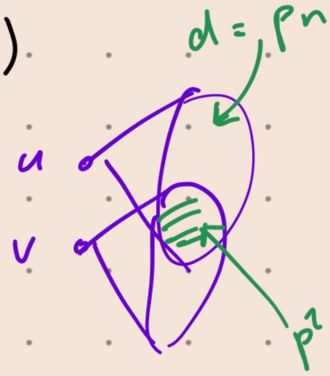
- [Spectral gap] $|\lambda_2| = o(n)$

  Expander mixing lemma



- [Discrepancy] $\forall A, B \subseteq V(G)$,

  $$e(A,B) = p|A|\cdot|B| + o(n^2)$$



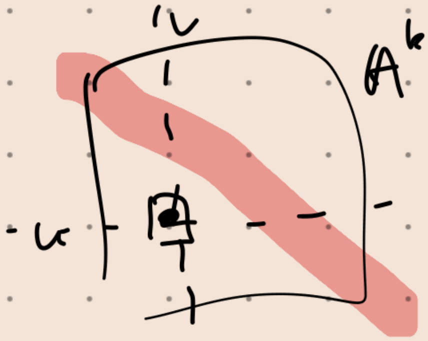
- [Codegree] $\sum_{u,v \in V(G)} \left| d(u,v) - p^2 n \right| = o(n^3)$

Plan

Pf [4-cycle count $\Rightarrow$ spect. gap]

idea {
This amounts to write C_4-count using trace of A^4
and the correct count of C_4 $\Rightarrow$ contribution from non-trivial eigenvalues to the trace is negligible.
}

Exer: $\forall\, u,v \in V(G)$, $\forall\, k \in \mathbb{N}$,



$$(A^k)_{u,v} = \#\text{ walks in } G \text{ from } u \text{ to } v.$$

- the trace of A^k

$$tr(A^k) = \sum_{i=[n]} \lambda_i^k$$
$$= \sum_{v \in V(G)} (A^k)_{v,v}$$

Counts # of closed walks of length k in G.

Among these walks, the degenerate ones (i.e. those not corresp. to C_k) is negligible, at most $O(n^{k-1}) = o(n^k)$

$\lambda_1 = d = pn$

- Thus, splitting the λ_1-term gives

$$\cancel{p^4 n^4} + o(n^4) \geq t(C_4, G)\, n^4 + \underbrace{o(n^4)}_{\text{degenerate 4-walks}} = tr(T^4) = \lambda_1^4 + \sum_{i \geq 2} \lambda_2^4$$
$$= \cancel{p^4 n^4} + \sum_{i \geq 2} \lambda_2^4$$

$$\Rightarrow |\lambda_i| = o(n) \quad \forall\, i \geq 2.$$