Lecture 12

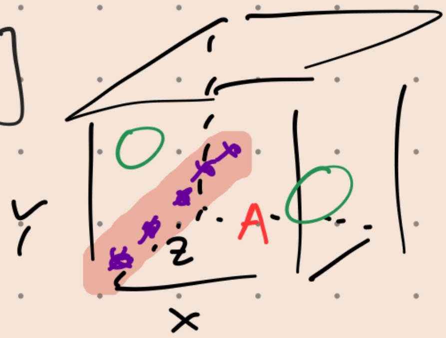
Recall:

- $f: X \times Y \to \mathbb{F}$ rank 1: $\forall (x,y) \in X \times Y$ $f(x,y) = u(x)v(y)$
  - $\exists u: X \to \mathbb{F}$
    $v: Y \to \mathbb{F}$
    $f = uv^T$

- $f: X \times Y \times Z \to \mathbb{F}$. slice rank 1 is s.t.
  - $u(x)\, v(y,z)$ OR
  - $u(y)\, v(x,z)$ OR
  - $u(z)\, v(x,y)$

  $sr(f) =$ min # rank 1 functs needed to express it as a linear combination.

Key lem [rank of diagonal hypermatrizes]



Let X be a finite set, $A \subseteq X$.
Let $\mathbb{F}$ be a field, and let
$f: X^3 \to \mathbb{F}$ s.t. $f(x,y,z) \neq 0 \iff x = y = z \in A$
$\implies sr(f) = |A|$.

- Often we want to (upper) bound the cardinality of a set, say A, with certain forbidden structure, say S-free.

Strategy Def polyn. $f: A^3 \to \mathbb{F}$ s.t.
using S-freeness

$f(x,y,z) \neq 0 \iff x=y=z \in A$

then $sr(f) = |A|$

<u>Thm</u> [Ellenberg-Gijswijt] $\forall$ capset $A \subseteq \mathbb{F}_3^n$, i.e.
no $\{x, x+r, x+2r\}$, $r \neq 0$, $\Rightarrow$ $|A| \leq O(2.756^n)$

<u>Obs</u>: $x, y, z \in \mathbb{F}_3^n$

$$x+y+z=0 \iff \begin{cases} \cdot\ x=y=z \\ \quad \text{OR} \\ \cdot\ x,y,z \text{ a line } x, x+r, x+2r \end{cases}$$

So for a capset A and $\forall x,y,z \in A$

$$x+y+z=0 \iff x=y=z \iff f(x,y,z) \neq 0$$

- if $x+y+z \neq 0 \Rightarrow \exists\, i \in [n]$ s.t. $x_i+y_i+z_i \neq 0$

want $f(x,y,z)=0$

$$1-(x_i+y_i+z_i)^2 = 0 \quad \text{in } \mathbb{F}_3$$

In $\mathbb{F}_p \ni x \neq 0$
$x^{p-1} = 1$

Def $f: A^3 \longrightarrow \mathbb{F}_3$ to be

$$f(x,y,z) = \prod_{i=1}^{n} \left(1-(x_i+y_i+z_i)^2\right)$$

then $f(x,y,z) \neq 0 \iff x=y=z \in A$.

So $sr(f) = |A|$

Claim: $sr(f)$ is $\leq 3R$, where

$$R = \sum_{\substack{a,b,c \geq 0 \\ a+b+c=n \\ b+2c \leq 2n/3}} \frac{n!}{a!b!c!}$$ is # of 0,1,2-vectors of length n s.t. that $v_1 + \cdots + v_n \leq 2n/3$.

Pf: Note that $f(x,y,z) = \prod_{i=1}^{n} \left(1 - (x_i + y_i + z_i)^2\right)$ is a $3n$-var. polyn. w/ total deg $\leq 2n$ and the deg of each var. (x_i, y_i, z_i) is ≤ 2.

Thus, f is a linear combination of monomials of the form

$$x^\alpha y^\beta z^\gamma = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} y_1^{\beta_1} \cdots y_n^{\beta_n} z_1^{\gamma_1} \cdots z_n^{\gamma_n}$$

where α, β, γ are 0,1,2-vectors of length n.

and $|\alpha| + |\beta| + |\gamma| \leq 2n$, $\left(|\alpha| = \alpha_1 + \cdots + \alpha_n\right)$.

- For any such $x^\alpha y^\beta z^\gamma$, by pigeonhole principle one of the exponent is $\leq 2n/3$ $\left(\begin{array}{l} \text{either } |\alpha| \leq 2n/3 \\ \text{or } |\beta| \leq \ldots \\ \text{or } |\gamma| \leq \ldots \end{array}\right)$

$f = f_1 + f_2 + f_3$ where

$$f_1(x,y,z) = \sum_{|\alpha| \leq 2n/3} c_\alpha x^\alpha \cdot g_\alpha(y,z) \quad \text{and}$$

$$f_2(x,y,z) = \sum_{|\beta| \leq 2n/3} c_\beta y^\beta g_\beta(x,z) \quad \text{and}$$

$$f_3(x,y,z) = \sum_{|\gamma| \le 2n/3} g_\gamma z^\gamma \, g_\gamma(x,y).$$

- Note that $\quad sr(f) \le \sum_{i=1}^{3} sr(f_i)$

where $sr(f_i) \le$ # 0,1,2-vect. of length n w./ coord.
sum up to $\le 2n/3$. $= R$

let a, b, c be # 0s, 1s, 2s in such vectors

- $a, b, c \geq 0$
- $a+b+c = n \Rightarrow R = \sum_{a,b,c} \frac{n!}{a!\,b!\,c!}$
- $b + 2c \le 2n/3$

To estimate R, say $a = (\alpha + o(1))n$
$b = (\beta + o(1))n$
$c = (\gamma + o(1))n$

Stirling's formula $\Rightarrow \frac{n!}{a!\,b!\,c!} = e^{h(\alpha,\beta,\gamma)n + o(n)}$

$$h(\alpha,\beta,\gamma) = \alpha \log\frac{1}{\alpha} + \beta \log\frac{1}{\beta} + \gamma \log\frac{1}{\gamma}.$$

$\Rightarrow$ set $N = e^{n(X + o(1))}$, where

$X = \max h(\alpha, \beta, \gamma)$
s.t. $\alpha, \beta, \gamma \geq 0$
$\alpha + \beta + \gamma = 1$
$\beta + 2\gamma \le 2/3$

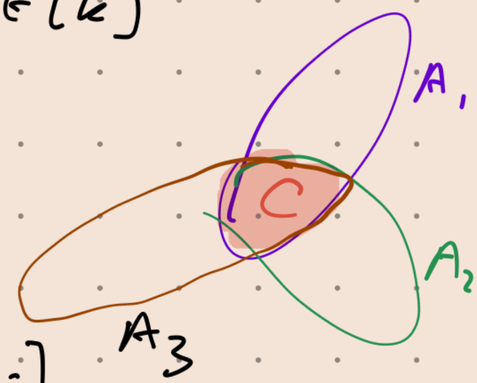
By Lagrange multiplier $\Rightarrow$ max attained.

$$\begin{cases} \alpha = \dfrac{32}{3(15+\sqrt{33})} \\ \beta = \dfrac{4(\sqrt{33}-1)}{3(15+\sqrt{33})} \\ \gamma = \dfrac{(\sqrt{33}-1)^2}{6(15+\sqrt{33})} \end{cases}$$

$h(\alpha, \beta, \gamma) \approx 1.01345 \Rightarrow O(2.756^n)$. $\blacksquare$

§5.3 Sunflower

<u>Def</u>: let $k \in \mathbb{N}$, sets $A_1, \ldots, A_k$ form a k-sunflower if they have common pairwise intersection, i.e. $\exists$ a set C s.t. $\forall$ distinct $i, j \in [k]$ $A_i \cap A_j = C$



<u>Conj</u> [Erdős - Szemerédi sunflower conj]

$\forall k$, $\exists$ $c = c(k) < 2$ s.t. $\forall A \subseteq 2^{[n]}$ with no k-sunflower has size $|A| \leq c^n$.

We shall see a sol^{n} for 3-sunflower..

<u>Thm</u> [Naslund - Sawin] $\forall A \subseteq 2^{[n]}$ 3-sunflower free collection of subsets of $[n]$

$$\Rightarrow |A| \leq 3n \sum_{k \leq n/3} \binom{n}{k} \leq \left(\frac{3}{2^{2/3}}\right)^{(1+o(1))n}.$$

- $\frac{3}{2^{2/3}} \approx 1.8898 < 2$.

Pf: • Identify $S \in A \iff \mathbb{1}_S \in \{0,1\}^n$ indicator funct.

So view A as a set of vectors in $\{0,1\}^n$

Need to constr. $f: A^3 \longrightarrow \mathbb{F}_2$ s.t.

$$f(x,y,z) \neq 0 \iff x = y = z \in A$$

then [key lem] $\Rightarrow$ $sr(f) \geq |A|$

• $\forall$ distinct 3 set $X, Y, Z \in A$

3-sunflower-free $\Rightarrow$ $Y \cap Z \setminus X \neq \emptyset$

$\Rightarrow \exists i$ s.t. $\{x_i, y_i, z_i\} = \{0,1,1\}$

$\Rightarrow 2 - (x_i + y_i + z_i) = 0$

• Problematic off-diag case:

X, X, Y and $X \subseteq Y$,

$\Rightarrow \forall i$, $\{x_i, y_i, z_i\}$ can only be $\{0,0,0\}$, $\{0,0,1\}$ or $\{1,1,1\}$

• To fix this, partition $A = \bigcup_{\ell=1}^{n} A_\ell$ where

A_ℓ consists of all sets of size ℓ.

Take ℓ max. $|A_\ell|$ so. $|A| \leq n \cdot |A_\ell|$.

Now define $f : A_\ell^3 \to \mathbb{F}_2$ as

$$f(x,y,z) = \prod_{i=1}^{n} \left(2 - (x_i + y_i + z_i)\right)$$

then $f(x,y,z) \neq 0 \iff x = y = z$

so $sr(f) = |A_\ell| \geq \frac{1}{n}|A|$

- f is spanned by monomials

$$x^\alpha y^\beta z^\gamma \quad \text{s.t.} \quad |\alpha| + |\beta| + |\gamma| \leq n$$
$$\alpha, \beta, \gamma \in \{0,1\}^n$$

$\Rightarrow$ one of them $\leq n/3$

group monom. by this smallest deg one

$f = f_1 + f_2 + f_3$, where

$$f_1 = \sum_{|\alpha| \leq n/3} c_\alpha x^\alpha g_\alpha(y,z)$$
$f_2 = \ldots$
$f_3 = \ldots$

$sr(f_i) \leq \#\{0,1\}^n$ vectors w/ $\leq n/3$ 1s.

$$= \sum_{k \leq n/3} \binom{n}{k}.$$

$$\Rightarrow \frac{1}{n}|A| \leq |A_\ell| = sr(f) \leq 3 \sum_{k \leq n/3} \binom{n}{k} \qquad \square$$