Lecture 11

§5 Capset problem and slice rank.

<u>Def</u> A capset $A \subseteq \mathbb{F}_3^n$ is a set with no line $\{x, x+r, x+2r\}$, $x, r \in \mathbb{F}_3^n$, $r \neq 0$

The capset problem is to bound the maximum size of a capset in $\mathbb{F}_3^n$.

- Meshulam extended Roth's Fourier argu. $\leq O\left(\frac{3^n}{n}\right)$
- Croot-Lev-Pach : 3AP-free set in $\mathbb{Z}_4^n$ is exponentially small.
- Ellenberg-Gijwijt : a variation of CLP technique $\Rightarrow O(2.756^n)$
- lower bound: Edel $\geq (2.2174)^n$.
- Tao : a symm. variation of CLP which treats all 3 var. the same.

§5.1 Slice rank.

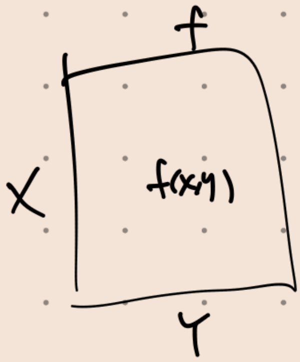
<u>Recall</u> : Let X, Y finite sets, $\mathbb{F}$ field.

Let $f : X \times Y \longrightarrow \mathbb{F}$ is of rank one

$\exists\, u: X \to \mathbb{F}$ & $v: Y \to \mathbb{F}$

if $\forall\, (x,y) \in X \times Y$, $f(x,y) = u(x)\, v(y)$

In general, the rank of a two-var. funct. is the min # of rank one funct. whose linear span contains it.



$f = uv^T$

i.e. $$f(x,y) = \sum_{i=1}^{k} u_i(x)\, v_i(y)$$

- $f: X \times Y \times Z \longrightarrow \mathbb{F}$

naturally: $$f(x,y,z) = \sum_{i=1}^{k} u_i(x)\, v_i(y)\, w_i(z)$$

<u>Instead</u> f has slice rank one if $\forall (x,y,z) \in X \times Y \times Z$

$$f(x,y,z) = u(x)\, v(y,z)$$

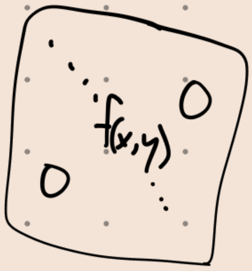
OR $$= u(y)\, v(x,z)$$

OR $$= u(z)\, v(x,y)$$

In general, the slice rank of f is the min # of slice rank one functs needed to write f as a linear combination.

$$f(x,y,z) = \sum_{i=1}^{r_1} u_i(x) v_i(y,z) + \sum_{i=r_1+1}^{r_2} u_i(y) v_i(x,z) + \sum_{i=r_2+1}^{r} u_i(z) v_i(x,y).$$

2-dim



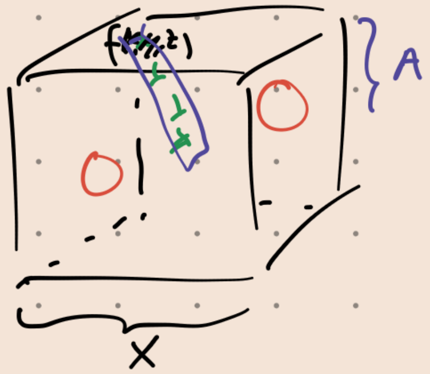
diag. matrix

rank = # non-zero diag. entries

$f(x,y) \neq 0 \iff x = y$

3-dim

$f(x,y,z) \neq 0 \iff x = y = z$



Def $sr(f)$ = slice rank

<u>Lem 1</u>: [rank of diag. hypermatrices]

Let X be finite set, $A \subseteq X$, $\mathbb{F}$ field

Let $f: X^3 \to \mathbb{F}$ s.t. $f(x,y,z) \neq 0 \iff x = y = z$ & $x \in A$

Then $sr(f) = |A|$.

<u>Notation</u> $\mathbb{1}_S$ = indicator funct for a set S

$$\mathbb{1}_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{o.w.} \end{cases}$$

$\mathbb{1}_a$ instead of $\mathbb{1}_{\{a\}}$

<u>pf</u>: Since $f(x,y,z) \neq 0 \iff x = y = z \in A$

$$\Rightarrow f(x,y,z) = \sum_{a \in A} \mathbb{1}_a(x) \mathbb{1}_a(y) \mathbb{1}_a(z) \cdot f(a,a,a).$$

$\Rightarrow \quad sr(f) \le |A|$

It is left show the oppo. direction $sr(f) \geq |A|$.
$\overset{\shortparallel}{r}$

- Supp $sr(f) = r$ and write

$$f(x,y,z) = \boxed{\sum_{i=1}^{r_1} u_i(x) v_i(y,z)} + \sum_{i=r_1+1}^{r_2} u_i(y) v_i(x,z) + \sum_{i=r_2+1}^{r} u_i(z) v_i(x,y)$$

w.l.o.g. $r_1 > 0$. NTS $r \geq |A|$

- We shall constr. a funct

$$g : Y \times Z \to \mathbb{F}$$

$$g(y,z) = \sum_{x \in X} h(x) f(x,y,z), \text{ where } h : X \to \mathbb{F}$$

and show the rank of g

$$\left.\begin{array}{l} (i) \geq |A| - r_1 \\ (ii) \leq r - r_1 \end{array}\right\} \Rightarrow r \geq |A| \text{ as desired}$$

- Claim $\exists\, h : X \to \mathbb{F}$
  - orthog. to all u_i, $i \le r_1$
    i.e. $\sum_{x \in X} h(x) \cdot u_i(x) = 0$
  - # zero entries in $h \le r_1$

$$(\overbrace{h(x), h(x'), \cdots}^{\le r_1})$$

Pf: - Consider the vect. sp. of funct. $h : X \to \mathbb{F}$ orthog. to all u_i : V

constr. $= r_1$

$\Rightarrow \quad \dim V \geq |X| - r_1$

- Let $M = \begin{pmatrix} — u_1 — \\ — u_2 — \\ \vdots \\ — u_{r_1} — \end{pmatrix}$, $\begin{pmatrix} h_1 \\ \vdots \\ h_{|X|} \end{pmatrix}$, $Mh = 0 \ \forall h \in V$

want to find h w./ $\leq r_1$ zeros.

$M_{r_1, |X|} \, h_{|X|,1} = 0_{r_1,1}$

var. $= |X|$

constr. $= r_1$

freedom $\geq |X| - r_1$

$\Rightarrow \exists h \in V$ w./ $\geq |X| - r_1$ non-zero entries

$g : Y \times Z \to \mathbb{F}$

- Recall $g(y,z) = \sum_x h(x) \cdot f(x,y,z)$

Claim g is diag. w./ diag. entry

$h(a) f(a,a,a)$, $\forall a \in A$, out of which

$\geq |A| - r_1$ non-zero. $\Rightarrow$ rank $g \overset{(i)}{\geq} |A| - r_1$

pf. Recall $f(x,y,z) \neq 0 \Leftrightarrow x = y = z = a \in A$.

$\Rightarrow \begin{cases} \forall y \neq z, & g(y,z) = 0 \\ \forall y = z \notin A, & g(y,z) = 0 \end{cases}$

As h has $\leq r_1$ zero entries $\Rightarrow g$ has

$\geq |A| - r_1$ non-zero diag. entries.

For (ii) rank $g \le r - r_1$.

$$g(y,z) = \sum_{x \in X} h(x)\, f(x,y,z)$$

$$= \sum_{x \in X} h(x) \left[\sum_{i=1}^{r_1} u_i(x)\, v_i(y,z) + \sum^{r_2} u_i(y)\, V_i(x,z) + \sum^{r} u_i(z)\, v_i(x,y) \right]$$

$h \perp u_i \Rightarrow 1^{st} = 0$

$$2^{nd} \text{ term} = \sum_{x \in X} h(x) \cdot \sum_{i=r_1+1}^{r_2} u_i(y) \cdot v_i(x,z)$$

$$= \sum_{i=r_1+1}^{r_2} \underline{u_i(y)} \underbrace{\sum_{x \in X} h(x) \cdot v_i(x,z)}$$ ← funct. of z

$\overset{(ii)}{\Rightarrow}$ rank $g \le r - r_1$ ▨

How to use slice rank?

- Supp. we have a set A w./ certain forbidden structure (line, 3-sunflower)

  we want to upp bdd $|A|$

Strategy: Def a (diag) poly. $f : A^3 \to \mathbb{F}$

(use the struct. info of A to)

s.t. $f(x,y,z) \neq 0 \iff x = y = z$

Lem 1 $\Rightarrow$ $sr(f) = |A|$

Then we try to upp bdd slice rank of f.