Lecture 10

Recall:

- $A \subseteq \mathbb{F}_p^n$, $\exists$ a non-zero polyn on $\mathbb{F}_p^n$ vanishing on A w./ $\deg \leq d$, where $\binom{n+d}{d} > |A|$

  b/c $\dim \left[\begin{matrix} \text{vect sp. of} \\ \deg \leq d \text{ polyn} \end{matrix} \right] = \binom{n+d}{d}$

Rephrase

- $A \subseteq \mathbb{R}^n$, $\exists$ a non-zero polyn on $\mathbb{R}^n$ vanishing on A w./ $\deg \lesssim_n |A|^{1/d}$

Notation $x \lesssim_d y \iff x = O_d(y)$

§4 Joint thm.

- Another important application of the polyn. method in discrete geometry.

Def: Given a set of lines in $\mathbb{R}^d$, a joint formed by these lines is a point that lies on d given lines, whose directions are linearly indep.

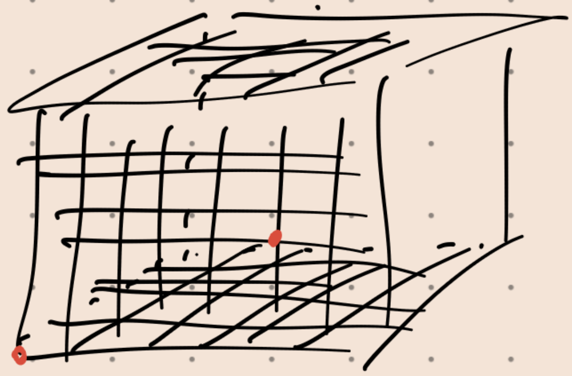
- We want to bound # joints formed by a set of lines.

<u>Thm</u> [Joint thm] The number of joints formed by N lines in $\mathbb{R}^d$ is $\lesssim_d N^{d/(d-1)}$.

<u>Rmk</u>: Sharp, $\mathbb{R}^3$

lines $N = 2t^2$

joints $= t^3 \geq \frac{1}{10} N^{3/2}$



In general, take S hyperplanes in $\mathbb{R}^d$ in general position $\Rightarrow$

$\forall$ $d-1$ of them intersect at a line

lines $= N = \binom{S}{d-1}$

$\forall$ d of them intersect at a <u>point</u> joint

joints $= \binom{S}{d} \gtrsim_d N^{d/(d-1)}$.

- Guth - Katz $\mathbb{R}^3$
- Quilodrán, Kaplan - Sharir - Shustin $\mathbb{R}^d$
- Carbery - Iliopoulou, all field $\mathbb{F}^d$.

Strategy: Suppose a non-zero low-deg polyn. f is heavily incident to given lines

$\Rightarrow$ f vanish on all these lines

$\Rightarrow$ info about ∇f $\lightning$.

Pf: Let J be the set of joints formed by a set L of N lines

Claim: $\exists$ a line $\ell \in L$ containing $\lesssim_d |J|^{1/d}$ joints

(Claim $\Rightarrow$ Thm) Take out one "light" line at a time, each time removing $\lesssim_d |J|^{1/d}$ joints.

$\Rightarrow$ $|J| \lesssim_d N \cdot |J|^{1/d}$ ☺

Pf of Claim: Supp. not, i.e. $\forall\ \ell \in L$, #joints on $\ell \gtrsim_d |J|^{1/d}$.

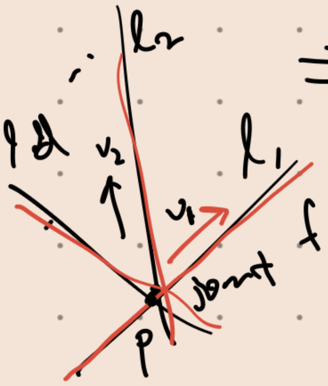
- Take a non-zero polyn. f in $\mathbb{R}^d$ that vanishes on J w/ min deg.

  #constr. $\leq |J|$ $\Rightarrow$ $\deg f \lesssim_d |J|^{1/d}$

- $\forall\ \ell \in L$, consider the restr. f_ℓ

  $\deg f_\ell \leq \deg f \lesssim_d |J|^{1/d} < |J \cap \ell|$

  $\Rightarrow$ f_ℓ vanish on the whole line ℓ.



$\Rightarrow$ f vanish on all lines in $\mathcal{L}$.

$\Rightarrow$ $\forall$ joint $p \in J$, let $\ell_1, \dots, \ell_d$ be the set of d linearly indep line going thru p, let v_i be direction of ℓ_i

Since f vanish on ℓ_i

$\Rightarrow$ (directional derivative) $\nabla f(p) \cdot v_i = 0$

Since $\{v_i\}_{i \in [d]}$ linearly indep. $\Rightarrow$ $\nabla f(p) = \vec{0}$

$\Rightarrow$ $\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_d} \end{pmatrix}$ vanises on J

$\Rightarrow$ all $\frac{\partial f}{\partial x_i}$ vanishes on J

Note that at least one $\frac{\partial f}{\partial x_i} \neq 0$-polyn.

o.w. if all 0-polyn. $\Rightarrow$ f = constant $\neq 0$

$\lightning$ f vanish on J

minimality of $\deg f$.

$\frac{\partial f}{\partial x_i} \neq 0$ $\lightning$ the choice of f $\blacksquare$.

- Different pf without taking derivative by Zhang.

- Instead of caring about polyn. vanish at a given pt., rather look at the Taylor series around that pt.

<u>Pf</u>: • L: N given lines, J the set of joints

$\Rightarrow$ non-zero polyn. f $\deg \lesssim_d |J|^{1/d}$ vanishing on J.

• Define ord./special joints on a line (wrt. f).

$\forall$ joint $p \in \ell$, T_0 affine linear map

$$T_0 : \begin{cases} \ell \longrightarrow x_d\text{-axis} \\ p \longmapsto 0 \\ f \longmapsto f \circ T_0^{-1} \end{cases}$$

<u>Ex</u> $\mathbb{R}^3$

$xy(z-a)$

l.h.p. $= -xya$

indep. of z

So $f \circ T_0^{-1}(0) = f(p) = 0$.

• p is ordinary on ℓ: if the lowest homogeneous part of $f \circ T_0^{-1}$ is indep of x_d.

& special o/w.

• well-defined, i.e. indep of choices of T_0

(b/c two such maps $T_0 = T_0' \circ T$ differ by a map T that is a scaling when restricted to x_d-axis)

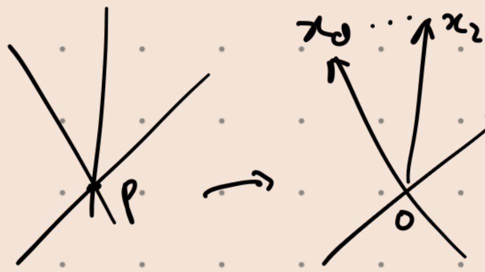
Shall see i) $\forall$ joint p is special on $\geq$ one line passing thru it.

ii) on $\forall$ line $l \in L$, $\leq \deg f$ many special joints.

Double-counting # special joints

$$|J| \overset{(i)}{\leq} \overset{(ii)}{\leq} N \cdot \deg f \lesssim_d N \cdot |J|^{1/d}$$

For (i), $\forall p \in J$, take T_0 $\begin{cases} p \mapsto 0 \\ \text{lines thru } p \mapsto x_i\text{-axis} \end{cases}$



$\Rightarrow$ the lowest homog. part of $f \circ T_0^{-1}$ depends on some $\underline{\underline{x_i}}$ o.w. it's a constant

$\Rightarrow f \circ T_0^{-1}(0) \neq 0$ ↯

p is special on the corresponding line.

For (ii).

<u>Claim</u> If p is special on l $\forall T_0$ $\begin{cases} l \mapsto x_d\text{-axis} \\ p \mapsto 0 \end{cases}$

then in $f \circ T_0^{-1}(x) = \sum_\alpha x^{(\alpha,0)} f_\alpha(x_d)$, where $\alpha = (\alpha_1, \dots, \alpha_{d-1})$, $\forall \alpha$ w/ $|\alpha| = \alpha_1 + \dots + \alpha_{d-1}$ s.t.

$f_\alpha(x_d) \neq 0$ - polyn.

then $f_\alpha(x_d)$ vanishes at $x_d = 0$.

[Claim $\Rightarrow$ (ii)] $\forall T$: $\begin{cases} \ell \mapsto x_d\text{-axis} \\ p \mapsto (0, 0, \dots, 0, p_T) \end{cases}$

$$\Rightarrow f \circ T^{-1}(x) = \sum_\alpha x^{(\alpha, 0)} f_\alpha(x_d - p_T)$$

$\forall \alpha$ w./ $\begin{cases} |\alpha| \text{ min} \\ f_\alpha \neq 0, \end{cases}$ p_T is a root of f_α.

$\Rightarrow$ # special p on ℓ $\leq \deg f_\alpha \leq \deg f \lesssim_d |J|^{1/d}$.

<u>pf (Claim)</u> Supp. not that $f_\alpha(0) \neq 0$.

- Note that $f_\alpha(0)$ is the constant term of f_α.
- by minimality of α, $f_\alpha(0) \cdot x^{(\alpha, 0)}$ is in lowest homog. part of $f \circ T_0^{-1}$

$\Rightarrow$ other terms in lowest hom. part are all indep of x_d, o.w. sum of power of $x_1, \dots, x_{d-1} < |\alpha|$ ↯ choice of α ▨