Webpage: homepages.warwick.ac.uk/staff/H.Liu.9/topic-comb-sdu2021fall.html

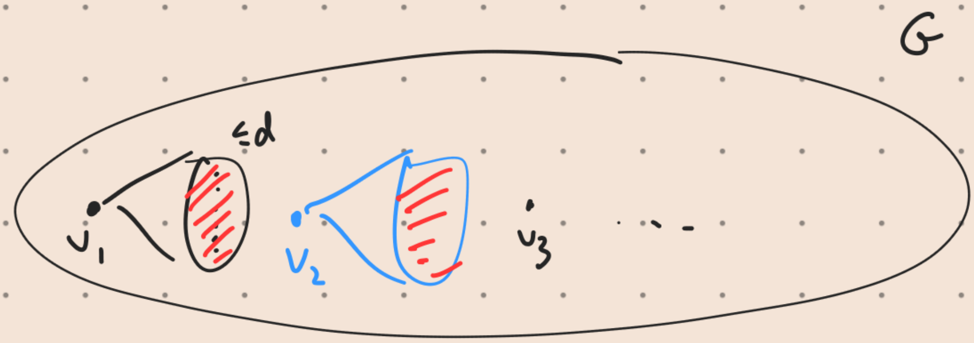
Lecture 1   Gibbs point process

Topic: indep number of Δ-free graphs.

$\alpha(G)$ = indep # of a graph G.

• **Obs**: $\forall G$, $\Delta(G) \leq d \Rightarrow \alpha(G) \geq \frac{n}{d+1}$
n-vertex

Pf:



generate an indep $I = \{v_1, v_2, \dots\}$

$|I| \geq \frac{n}{d+1}$. $\blacksquare$

Exercise: Prove that $\forall$ n-vx G w/ $d(G) \leq d$

$\Rightarrow \alpha(G) \geq \frac{n}{d+1}$.

Rmk: $\frac{n}{d+1}$ bound is optimal. $\Rightarrow d(G) = d$

$\alpha(G) = \frac{n}{d+1}$

Consider G = disj union of K_{d+1}

$\frac{n}{d+1}$ many $\blacksquare$

Extremal problem

- Discrete family: n-vx ave. deg $= d$ graphs $\mathcal{G}$ (Δ-free)
- parameter: indep # of a graph
- min. $\alpha(G)$: $G \in \mathcal{G}$
- min $\alpha(G) = \dfrac{n}{d+1}$ ← improve?

Meta problem (after extremal one is solved)

What if we forbid graphs that look like extremal structures, can we improve the bound?

Since disj union of cliques have lots of Δs. What if we add a Δ-free condition?

- Ajtai - Komlós - Szemerédi 80s
- Shearer: $\forall$ n-vx Δ-free graph G w/ $d(G) = d \Rightarrow \alpha(G) \geq (1+o(1)) \cdot \dfrac{\log d}{d} \cdot n$.

Gain a $\log d$ - factor improvement from Δ-freeness.

Exercise: Use Shearer's bdd to derive

$$R(3, k) \leq (1+o(1)) \frac{k^2}{\log k}.$$

↑ Δ-clique Ramsey #

Today: (Shearer / Alon)

Def: $\bar{\alpha}(G)$ = average size of all indep sets in G.

Thm $\forall$ n-vx Δ-free G w/ $\Delta(G) \le d$,

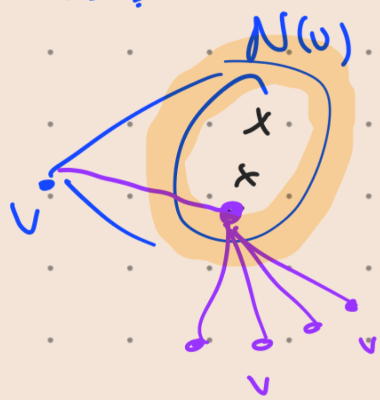
$$\bar{\alpha}(G) \geqslant (1+o(1)) \frac{\log_2 d}{4d} \cdot n.$$

Idea: (double-counting) Sample $\underset{\uparrow}{\overset{\text{indep set}}{I}} \sim \overset{\text{all indep sets in } G}{\mathcal{I}(G)}$ uniformly $\Rightarrow$ $\bar{\alpha}(G) = \mathbb{E}|I|$.

bound $\mathbb{E}|I|$ from two points of views:

1) $v \in I$ or not

2) How $N(v) \cap I$ looks like.

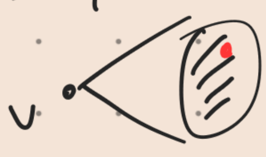


For 1) $$\mathbb{E}|I| = \sum_{v \in V(G)} \Pr(v \in I)$$

For 2) $$\mathbb{E}|I| \geqslant \frac{1}{d} \sum_{v \in V(G)} \boxed{\sum_{u \in N(v)} \Pr(u \in I)}$$

- Shall see these two bounds go in opposite directions, the desired bdd on $\bar{\alpha}(G)$ follows.

Rmk: Spatial Markov property drives the argument



whether or not $v \in I$ depends only on its "boundary cond"

Pf: • Let I be a uniform random indep. sets in G.

• Estimate $\mathbb{E}|I| = \sum_{v \in V(G)} \Pr(v \in I)$

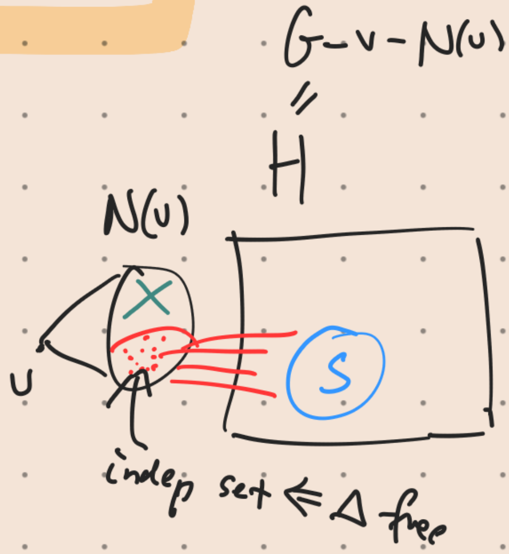
• Fix a "boundary cond." by conditioning that

$I \cap V(H) = S$

Define $X = N(v) \setminus N(S)$

$x = |X|$ random variable

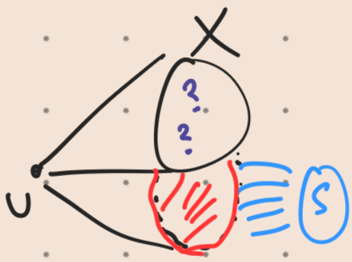
x : the number of vxs in $N(v)$ that are suitable to be added to I.



• Note $N(v)$ itself is an indep set due to Δ-freeness,
as I is uniformly at random chosen

$\Rightarrow$ v or any subset of X is equally likely to be included in I.

$$\Pr(v \in I \mid I \cap V(H) = S) = \frac{1}{1+2^x}$$



• Recall 2)

$$\mathbb{E}|I| \geq \frac{1}{d} \sum_{v \in V(G)} \underbrace{\sum_{u \in N(v)} \Pr(u \in I)}_{\hookrightarrow\ \mathbb{E}|N(v) \cap I|}$$

$$\mathbb{E}\left(|N(v) \cap I| \;\middle|\; I \cap V(H) = S \right)$$

$$= \frac{\frac{x}{2} \cdot 2^x}{1 + 2^x}$$

- So $\mathbb{E}|I| = \sum_{v \in V(G)} \Pr(v \in I) \geq \sum_{v \in V(G)} \frac{1}{d} \mathbb{E}|N(v) \cap I|$

$$\Rightarrow \quad \mathbb{E}|I| \geq \frac{1}{2} \sum_{v \in V(G)} \left(\Pr(v \in I) + \frac{1}{d} \mathbb{E}|N(v) \cap I| \right)$$

- To get rid of the conditioning, we min over x.

$$\Pr(v \in I) + \frac{1}{d} \mathbb{E}|N(v) \cap I| \geq \boxed{\min_{0 \leq x \leq d}} \max\left(\frac{1}{1+2^x}, \frac{1}{d} \cdot \frac{\frac{x}{2} \cdot 2^x}{1+2^x} \right)$$

Optimize $\Rightarrow$ min. occurs when two equal.

$$\Rightarrow \quad \mathbb{E}|I| \geq (1+o(1)) \frac{\log_2 d}{4d} \cdot n.$$

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