5. The Margulis construction

In this section, we explicitly construct an expander. Let

$$T_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let G_n be the graph with the vertex set $(\mathbb{Z}_n)^2$ where each vertex $v \in (\mathbb{Z}_n)^2$ is adjacent to $T_1 v, T_2 v, T_1 v + e_1, T_2 v + e_2$. This yields an 8-regular graph G_n.

Theorem 5.1 (Gabber-Galil 1981)**.** *The graph G_n satisfies $\lambda(G) \leq 5\sqrt{2} < 8$.*

Margulis in 1973 proved expansion for a closely related family of graphs but could not give an explicit lower bound on the spectral gap. Gabber-Galil proved a lower bound on the spectral gap.

We wil prove the following weaker bound

Theorem 5.2. *The graph G_n satisfies $\lambda(G) \leq 7.3 < 8$.*

For an adjacency matrix A of $G = G_n$ and vector f on $V(G)$, we have

$$f^T A f = 2 \sum_{xy \in E(G_n)} f(x) f(y).$$

As $\lambda(G) = \max_{f \perp \mathbf{1}} \frac{f^T A f}{\|f\|^2}$, we only to prove the following:

$$\sum_{z \in \mathbb{Z}_n^2} f(z) \left[f(T_1 z) + f(T_1 z + e_1) + f(T_2 z) + f(T_2 z + e_2) \right] \leq 3.65 \sum f^2(x). \tag{5.1}$$

This is a real-valued function defined on an abelian group $\mathbb{Z}_n^2$. So, we want to analyze this function to obtain the above inequality.

5.1. Fourier analysis.

As a detour, we learn about Fourier analysis on abelian group. What we want to to analyze a function $g : \mathbb{Z}_n^2 \to \mathbb{C}$. Consider the collection $\mathcal{F}$ of all complex-valued functions defined on an abelian group H, which is $\mathbb{Z}_n^2$ in our case. Although our function is real-valued, we want to consider it as a complex-valued function, because collection of such functions form a linear space with a good orthonormal basis which is called characters. Then $\mathcal{F}$ forms a linear space, and this can be identified with a vector space $\mathbb{C}^{n^2}$ by considering g same as the vector $(g(x))_{x \in H}$. In other words, we consider a function $\delta_x : H \to \mathbb{C}$ such that $\delta_x(y) = 1$ if $x = y$ and 0 otherwise, and consider $\{\delta_x : x \in H\}$ as a basis to express functions on $\mathcal{F}$.

On the other hand, another choice of basis yields many useful properties. Especially, the following characters forms a basis.

Definition 5.3. *A* character *of an abelian group H is a homomorphism $\chi : H \to \mathbb{C}^*$, that is $\chi(g + h) = \chi(g)\chi(h)$ for all $h, g \in H$.*

The characters of a $\mathbb{Z}_n^2$ are exactly $\{\chi_b : b \in \mathbb{Z}_n^2\}$ where $\chi_b(a_1, a_2) = exp(\frac{2\pi(a_1 b_1 + a_2 b_2)i}{n})$. We write $w = exp(2\pi i/n)$, then $\chi_b(a) = w^{\langle a, b \rangle}$. We define the inner product of two vectors (=functions) as follows to ensures that our characters are unit vectors.

$$\langle f, g \rangle = \frac{1}{|H|} \sum_{x \in H} \overline{f(x)} g(x).$$

One can easily check that $\langle \chi_a, \chi_b \rangle = 0$ if $a \neq b$. So characters form an orthonormal basis on $\mathcal{F}$, hence any complex function can be expressed as a linear combinations of characters.

Proposition 5.4. *Every finite abelian group H has $|H|$ distinct characters which can be naturally indexed as $\{\chi_x : x \in H\}$. They form an orthonormal basis of $\mathcal{F}$. Thus every $f : H \to \mathbb{C}$ can be uniquely expressed as $f = \sum_{x\in H} \hat{f}(x)\chi_x$, where $\hat{f} : H \to \mathbb{C}$ is the discrete Fourier transform of f.*

$$\hat{f}(x) = \langle f, \chi_x\rangle = \frac{1}{|H|}\sum_{y\in H} \overline{f(y)}\chi_x(y).$$

In our case of $H = \mathbb{Z}_n^2$, we have $\hat{f}(x) = \frac{1}{n^2}\sum_b \overline{f(b)}w^{b_1x_1+b_2x_2}$. The following basic properties are easy to derive.

Proposition 5.5. *Let $f, g \in \mathcal{F}$. Then the following hold:*

(a) $\sum_{a\in H} f(a) = 0 \Leftrightarrow \hat{f}(0) = 0$.
(b) $\frac{1}{n^2}\sum_{x\in H} \overline{f(x)}g(x) = \langle f, g\rangle = \langle \sum_{x\in H} \hat{f}(x)\chi_x, \sum_{x\in H} \hat{g}(x)\chi_x\rangle = \sum_{x\in H} \overline{\hat{f}(x)}\hat{g}(x)$.
(c) Parseval's identity: $\frac{1}{n^2}\sum_{x\in H} |f(x)|^2 = \sum_{x\in H} |\hat{f}(x)|^2$.
(d) $f(a) = n^2\langle f, \delta_a\rangle = n^2\langle \sum_{x\in H} \hat{f}(x)\chi_x, \delta_a\rangle = \sum_{x\in H} \hat{f}(x)w^{-\langle a,x\rangle}$.
(e) If A is a nonsingular 2×2 matrix over $\mathbb{Z}_n$, $b \in \mathbb{Z}_n^2$ and $g(x) = f(Ax + b)$, then

$$\hat{g}(y) = w^{-\langle A^{-1}b,y\rangle}\hat{f}((A^{-1})^T y).$$

5.2. **A Proof of Theorem 5.2.** Now, assume f is a vector orthogonal to $\mathbf{1}$, meaning $\sum_z f(z) = 0 \Leftrightarrow \hat{f}(0, 0) = 0$. (Let $e_0 = (0, 0)$.) Hence, by (b) and (e), for $i \in [2], j \in \{0, i\}$, let $g(z) = f(T_iz + e_j)$, then we have (as f is a real vector, $f(z) = \overline{f(z)}$)

$$\begin{aligned}\sum_{z\in\mathbb{Z}_n^2} \overline{f(z)}f(T_iz + e_j) &= n^2\sum_{z\in\mathbb{Z}_n^2} \overline{\hat{f}(z)}\hat{g}(z) = n^2\sum_{z\in\mathbb{Z}_n^2} \overline{\hat{f}(z)}\omega^{-\langle T_i^{-1}e_j,z\rangle}\hat{f}((T_i^{-1})^T z)\\ &= \sum_{z\in\mathbb{Z}_n^2} \overline{f(z)}f(T_{3-i}^{-1}z)\omega^{-(z_1,z_2)\cdot e_j}.\end{aligned}$$

Here, $T_i^{-1}e_j = e_j$ for $i = j$. (Note that $A_i = A_{3-i}^T$.) Hence, (5.1) becomes

$$\sum_{z\in\mathbb{Z}_n^2} \overline{f(z)}\left[f(T_2^{-1}z)(1 + \omega^{-z_1}) + f(T_1^{-1}z)(1 + \omega^{-z_2})\right] \le 3.65\sum_{z\in\mathbb{Z}_n^2} |f(z)|^2.$$

Let g be a function with $g(z) = |f(z)|$. As we know $|1 + \omega^{-t}| = 2|\cos(\frac{\pi t}{n})|$, it suffices to prove

$$\sum_{z\in\mathbb{Z}_n^2} 2g(z)\left[g(T_2^{-1}z)|\cos(\frac{\pi z_1}{n})| + g(T_1^{-1}z)|\cos(\frac{\pi z_2}{n})|\right] \le 3.65\sum_{z\in\mathbb{Z}_n^2} g^2(z). \qquad (5.2)$$

As we have sum of squares on the right side, we want to bound the left side by sum of squares. One good way is to use inequality $2\alpha\beta \le \alpha^2 + \beta^2$. However, we also want to use the fact that $|\cos(\frac{\pi z_i}{n})|$ becomes small when z_i is closed to $n/2$. For this, we introduce a number γ, and use inequality $2\alpha\beta \le \gamma\alpha^2+\gamma^{-1}\beta^2$ with different γ for each z. Later, in some way, we want to group the terms on the left side, so that in each group, either $|\cos(\frac{\pi z_i}{n})|$ is small or the average value of γ multiplied is small. Assume we have $\gamma(z, z') = 1/\gamma(z', z)$, which we will specify later.

Using the inequality $2\alpha\beta \le \gamma\alpha^2 + \gamma^{-1}\beta^2$, the following inequality implies (5.2).

$$\sum_{z\in\mathbb{Z}_n^2}\sum_{i\in[2]} |\cos(\frac{\pi z_{3-i}}{n})|\left[\gamma(z, T_i^{-1}z)g^2(z) + \gamma(T_i^{-1}z, z)g^2(T_i^{-1}z)\right] \le 3.65\sum_{z\in\mathbb{Z}_n^2} g^2(z). \qquad (5.3)$$

Note that $(T_i z)_{3-i} = z_{3-i}$. Hence, we have

$$\sum_{z\in\mathbb{Z}_n^2} |\cos(\frac{\pi z_{3-i}}{n})|\gamma(T_i^{-1}z, z)g^2(T_i^{-1}z) = \sum_{T_i z\in\mathbb{Z}_N^2} |\cos(\frac{\pi (T_i z)_{3-i}}{n})|\gamma(z, T_i z)g^2(z)$$
$$= \sum_{z\in\mathbb{Z}_N^2} |\cos(\frac{\pi z_{3-i}}{n})|\gamma(z, T_i z)g^2(z).$$

Hence (5.3) becomes

$$\sum_{z\in\mathbb{Z}_n^2}\sum_{i\in[2]} |\cos(\frac{\pi z_{3-i}}{n})| \left[\gamma(z, T_i^{-1}z) + \gamma(z, T_i z)\right] g^2(z) \le 3.65 \sum_{z\in\mathbb{Z}_n^2} g^2(z).$$

Now we want to show that for each $z \in \mathbb{Z}_N^2 - \{(0,0)\}$,

$$\sum_{i\in[2]} \cos(\frac{\pi z_{3-i}}{n})| \left[\gamma(z, T_i^{-1}z) + \gamma(z, T_i z)\right] \le 3.65. \tag{5.4}$$

Now we define γ to make this holds. To define γ for different choices for z, we define the following partial order. From now on, we always write an element $(z_1, z_2) \in \mathbb{Z}_n^2$ in such a way that $-n/2 \le z_i \le n/2$ for each $i \in [2]$. In other words, as we have $(z_1, z_2) = (z_1 + in, z_2 + jn)$ for all $i, j \in \mathbb{Z}$, there are many ways to write an element of $\mathbb{Z}_n^2$, but when we write (z_1, z_2), assume that we always have chosen so that z_i lies between $-n/2$ and $n/2$.

Note that if $|z_1| + |z_2| \ge n/2$, then $|\cos(\frac{\pi z_1}{n})| + |\cos(\frac{\pi z_2}{n})|$ is small. On the other hand, when $|z_1| + |z_2| \le n/2$, then many of $T_i^{\pm 1}z$ are more away from $(0,0)$ than z. So, we want to define $\gamma(z, z')$ in such a way that it is smaller than 1 when z' is more away from $(0,0)$ than z. This motivates the following definition or a partial order.

Definition 5.6. *We say $(z_1, z_2) > (z_1', z_2')$ if $|z_1| \ge |z_1'|$ and $|z_2| \ge |z_2'|$ and at least one inequality is strict.*

we let

$$\gamma((z_1, z_2), (z_1', z_2')) = \begin{cases} 5/4 & \text{if } (z_1, z_2) > (z_1', z_2') \\ 4/5 & \text{if } (z_1, z_2) < (z_1', z_2') \\ 1 & \text{otherwise.} \end{cases}$$

If $|z_1| + |z_2| > n/2$, then assume $z_1, z_2 \ge 0$. The other cases are similar. As cos is a decreasing function, we have

$$|\cos(\pi z_1/n)| + |\cos(\pi z_2/n)| \le |\cos(\pi z_1/n)| + |\cos(\pi(n/2 - z_1)/n)|$$
$$\le \cos(\pi z_1/n) + \sin(\pi z_1/n) \le \sqrt{2}.$$

As $\gamma(z_1, z_2) \le 5/4$, we have $\sum_{i\in[2]} \cos(\frac{\pi z_{3-i}}{n})| \left[\gamma(z, T_i^{-1}z) + \gamma(z, T_i z)\right] \le 2 \cdot 5/4 \cdot \sqrt{2} \le 2.5\sqrt{2} \le 3.5$. Now, consider (z_1, z_2) with $|z_1| + |z_2| \le n/2$.

Proposition 5.7. *For each $(z_1, z_2) \in \mathbb{Z}_n^2$ with $|z_1|+|z_2| \le n/2$, one of the following holds. Three of the four points $T_1 z, T_2 z, T_1^{-1}z, T_2^{-1}z$ are more than z in the partial order and one is less than z in the partial order. Two of the four points $T_1 z, T_2 z, T_1^{-1}, T_2^{-1}z$ are more than z and two are incomparable with z.*

Proof. If $|z_1| = |z_2|$, then we can easily check that the second case holds.

WLOG, assume $|z_1| > |z_2|$. As $(T_i^{\pm 1}z)_{3-i} = z_{3-i}$, we just have to check whether $(T_i^{\pm 1}z)_i$ is bigger than z_2. By symmetry assume $z_1 > z_2 \ge 0$. As $z_1 + z_2 \le n/2$, we have $|z_1 - 2z_2| < |z_1|$ so we have $T_1^{-1}z < z$. The other three points satisfy $T_1 z > z, T_2^{\pm 1}z$ since $|z_1 + 2z_2| > |z_1|$ and $|z_2 \pm 2z_1| > |z_2|$. Hence the proposition holds.

□

By this proposition, the left side of (5.4) is at most $\max\{3\cdot 4/5+5/4, 2\cdot 4/5+2\} \leq 3.65$. This proves what we desired.