4. Extremal problems on spectrum

One natural question is how strong the expansion can be.

How big $\Psi_E(G,k)$ be? If we consider a vertex set U of size at most k which induces a connected subgraph of G. Then $G[U]$ contains at least $|U|-1$ edges, hence $\Psi_E(G,k) \leq (d-2)|U|+2$. Moreover, if we randomly partition $V(G)$ into two sets of equal size, we can show that there exists S with $|E(S,\overline{S})|/|S| \leq d/2+o(1)$. This shows that $h(G) \leq d/2+o(1)$. Alon in 1997 proved that the $h(G) \leq d/2 - c\sqrt{d}$ for all $d \geq 3$ for some absolute constant c. This is tight, as we know $h(G) \geq \frac{d-\lambda_2}{2}$ and there exist graphs with $\lambda_2(G) \leq O(\sqrt{d})$.

How about $\lambda_2(G)$? The following theorem finds a smallest possible value of $\lambda_2(G)$. As any (n,d)-graph has its diameter at least $\log_{d-1} n$, the following theorem implies that $\lambda_2(G) \geq 2\sqrt{d-1}(1-O(1/\log^2 n))$.

Theorem 4.1 (Alon-Boppana 91, Friedman 93)**.** *There exists a constant c such that every (n,d)-graph of diameter at least Δ satisfies*

$$\lambda_2(G) \geq 2\sqrt{d-1}(1-\frac{c}{\Delta^2}).$$

Proof. We will choose a vector f which is orthogonal to the largest eigenvector $v_1 = \frac{1}{n}(1,\dots,1)^T$, and compute Rayleigh quotient $\frac{f^TAf}{\|f\|^2}$. This value will be a lower bound of λ_2 as f is orthogonal to v_1. What we be a good choice for such a vector f? Intuitively, a graph with maximum possible expansion should be locally tree-like. If there are many short cycles, they are not helping a vertex set to expand. Hence, we will consider some vector which looks like an eigenvector of a d-regular tree. However, we want this vector to be orthogonal to v_1, so we consider a vector which looks like an eigenvector of a d-regular tree on two different regions, where it has positive value on one region and negative values on the other region. This will help us to obtain a vector orthogonal to v_1 as well as make them behaves like the eigenvector of d-regular tree.

Let $k = \lfloor \Delta/2 \rfloor - 1$ and choose two vertices s,t at distance Δ in G. For each $0 \leq i \leq k$,

$$S_i = \{v : dist(s,v) = i\},\ \ T_i = \{v : dist(t,v) = i\} \ \text{ and } Q = V(G) \setminus \bigcup_{0\leq i\leq k} (S_i \cup T_i). \quad (4.1)$$

Let $T(k)$ be the d-regular tree of height k with the root v_0 and let $A_{T(k)}$ be its adjacency matrix. Now we want to find an eigenvector for $A_{T(k)}$ corresponding to its largest eigenvalue.

We consider the vector $g : V(T(k)) \to \mathbb{R}$ where $g(v) = g_{\text{dist}(v_0,v)}$. In order for this to be an eigenvector with the eigenvalue μ, then it must satisfies that for $i \in [k]$

$$\mu g_0 = dg_1,\ \ \mu g_i = g_{i-1} + (d-1)g_{i+1} \text{ and } g_{k+1} = 0.$$

We consider a sequence satisfying these relation: $g_i = (d-1)^{-i/2}\sin((k+1-i)\theta)$ where θ will be determined later. Let $\mu = 2\sqrt{d-1}\cos\theta$. Then we have $g_{k+1} = 0$ and it satisfies

$$\begin{aligned} g_{i-1} + (d-1)g_{i+1} &= (d-1)^{-(i-1)/2}[\sin((k+2-i)\theta) + \sin((k+1-(i+1))\theta)] \\ &= \sqrt{d-1}(d-1)^{-i/2}[\sin((k+2-i)\theta) + \sin((k-i)\theta] \\ &= 2\sqrt{d-1}(d-1)^{-i/2}\sin((k+1-i)\theta)\cos(\theta) = \mu h_i \end{aligned}$$

Now, the relate $\mu g_0 = dg_1$ becomes

$$(2d-2)\cos(\theta)\sin((k+1)\theta) = d\sin(k\theta).$$

Take θ as the smallest positive root of this equation. As left side is bigger for positive θ very close to zero, and right side is bigger for $\theta = \pi/(k+1)$ such choice is possible with $\theta < \pi/(k+1) \leq 2\pi/(\Delta-1)$. Then for this choice of theta, the vector $(g_0,\dots,g_{k+1})$ satisfies the above relation. Moreover, we know that $\cos(\theta) > 1 - \frac{c}{\Delta^2}$ for some constant c.

Note that with this choice of $0 < \theta < \pi/(k+1)$, this sequence is non-increasing in i and $g : V(T(k)) \to \mathbb{R}$ with $g(v) = h_{\text{dist}(v_0,v)}$ gives an eigenvector of $A_{T(k)}$ which is a nonnegative vector.

Now, let $f : V(G) \to \mathbb{R}$ as

$$f(v) = \begin{cases} c_1 g_i & \text{if } v \in S_i \\ -c_2 g_i & \text{if } v \in T_i \\ 0 & \text{otherwise.} \end{cases}$$

where c_1, c_2 are nonnegative constant chosen so that $\sum_{v \in V(G)} f(v) = 0$.

Claim 4. *We have $(Af)_v \geq \mu f_v$ for $v \in \bigcup_i S_i$ and $(Af)_v \leq \mu f_v$ for $v \in \bigcup_i T_i$.*

Proof. Let $v \in S_i$ for some $i > 0$. Assume it has p neighbors in S_{i-1} with $p \geq 1$ and q neighbors in S_i and $d - p - q$ neighbors in S_{i+1}. Then as g is an non-increasing sequence,

$$\begin{aligned}(Af)_v &= pc_1 g_{i-1} + qc_1 g_i + (d-p-q)c_1 g_{i+1} \\ &\geq c_1(g_{i-1} + (d-1)g_{i+1}) \geq c_1(A_{T_k} g)_i = c_1 \mu g_i = \mu f_v.\end{aligned}$$

Similar argument works for $v = s$ or $v \in \bigcup_i T_i$. □

By this claim, we have

$$f^T A f = \sum_{v \in V(G)} f_v (Af)_v = \sum_{v \in \bigcup S_i} f_v (Af)_v + \sum_{v \in \bigcup T_i} f_v (Af)_v + \sum_{v \in Q} f_v (Af)_v \geq \sum_{v \in \bigcup S_i \cup \bigcup T_i} f_v \mu f_v = \mu f^T f.$$

Note that $f_v = 0$ for $v \in Q$. Here f is orthogonal to $\mathbf{1}$ by the choice of c_1, c_2. Hence,

$$\lambda_2(A) \geq \frac{f^T A f}{\|f\|^2} \geq \mu = 2\sqrt{d-1}\cos\theta = 2\sqrt{d-1}(1 - c/\Delta^2).$$

□

Moreover, we know that a constant fraction of eigenvalues exceed $2\sqrt{d-1} - \varepsilon$ for any fixed $\varepsilon > 0$. Serre proved that there exists $c(\varepsilon, d) > 0$ such that at least $c(\varepsilon, d)n$ eigenvalues are at least $2\sqrt{d-1} - \varepsilon$. It's known that $c(\varepsilon, d) \geq (d-1)^{-\pi\sqrt{2/\varepsilon}}$. Cioabá in 2006 proved the following weaker bound.

Proof. Let A be the ajacency matrix of G. Let n_ε be the number of eigenvalues larger than $2\sqrt{d-1} - \varepsilon$. Let k be a constant which will be determined later.

We want to use the fact that $\text{trace}(A^k)$ is the sum of k-th power of eigenvalues. However, if all the eigenvalues are nonnegative, then we can upper bound this in terms of n_ε. To make sure all the eigenvalues of the matrix we consider is nonnegative, we consider $A + dI$ and compute the following.

$$\text{trace}(A + dI)^k = \sum_{i=1}^{n} (\lambda_i + d)^k \leq (2d)^k n_\varepsilon + (d + 2\sqrt{d-1} - \varepsilon)^k n. \tag{4.2}$$

On the other hand,

$$\text{trace}(A + dI)^k = \sum_{j=0}^{k} \binom{k}{j} \text{trace}(A^j) d^{k-j} \geq \sum_{\ell=0}^{\lfloor k/2 \rfloor} \binom{k}{2\ell} n t_{2\ell} d^{k-2\ell}.$$

Here $t_{2\ell}$ is the tree number of closed walks of length 2ℓ from a specific vertex in a d-regular graph with girth bigger than 2ℓ. (Or consider a d-regular tree with height ℓ, and consider the number of closed walks of length 2ℓ from its root.) As we can consider a homomorphism from the regular tree with height ℓ to G, then all closed walks map to closed walks. So this inequality holds.

Such a walk can be associated with $+1/-1$ sequence where $+1$ is an edge directed away from the starting vertex and -1 is directed towards the starting vertex. Such sign pattern satisfies that they sum up to 0 and the sum of each prefix is nonnegative. Hence, there are Catalan number $C_k = \frac{1}{k+1}\binom{2k}{k}$ many such sign pattern. And each sign pattern corresponds to at least $(d-1)^k$ such walks, as moving away from the starting vertex has at least $d-1$ choices (If we reach the starting point, then we have to choose one of d choices, not $d-1$ choicese.) Hence, we have $t_{2\ell} \geq \frac{1}{\ell+1}\binom{2\ell}{\ell}(d-1)^k = \Omega((2\sqrt{d-1})^{2\ell}\ell^{-3/2})$. Hence,

$$\begin{aligned}\operatorname{trace}(A+dI)^k &\geq \frac{c'}{k^{3/2}}\sum_{\ell=0}^{\lfloor k/2\rfloor}\binom{k}{2\ell}n(2\sqrt{d-1})^{2\ell}d^{k-2\ell} = (\frac{c'}{2k^{3/2}})n[(d+2\sqrt{d-1})^k + (d-2\sqrt{d-1})^k]\\ &\geq (\frac{c'}{2k^{3/2}})n(d+2\sqrt{d-1})^k\end{aligned}$$

for some constant c'. This with (4.2) imply that

$$\frac{n_\varepsilon}{n} \geq \frac{\frac{1}{2}c'k^{-3/2}(d+2\sqrt{d-1})^k - (d+2\sqrt{d-1}-\varepsilon)^k}{(2d)^k}.$$

Take $k \geq \frac{2d}{\varepsilon}\log\frac{d}{\varepsilon}$, then we have $n_\varepsilon \geq (d/\varepsilon)^{-3d/\varepsilon}n$.

□

The following question is open.

Question 4.2. *What is the largest function $c(\varepsilon, d)$ which makes the above true?*

For non-regular graphs, much less is known. Consider the lollipop graph L_n on $2n$ vertices and which is obtained from K_n and a path P_{n+1} by identifying some vertex of K_n with an end vertex of P_{n+1}. This has diameter $\Theta(n)$ but $\lambda(L_n) \leq 2$. Hence $\lambda(G) \geq 2\sqrt{d-1} - o(1)$ does not hold in general where d is the average degree of G. However, the following is known.

Theorem 4.3 (Hoory, 2005)**.** *Suppose that the average degree of G is at least d whenever a ball of radius r is deleted from the graph. Then $\lambda(G) \geq 2\sqrt{d-1}(1-\frac{c\log r}{r})$ for some absolute constant $c > 0$.*

Definition 4.4. *A d-regular graph G is* Ramanujan *if $\lambda(G) \leq 2\sqrt{d-1}$.*

Lubotzky-Phillips-Sarnak in 1988 and Margulis in 1988 independently proved that arbitrarily large d-regular Ramanujan graphs exist when $d-1$ is prime. Moreover they can be explicitly constructed. Morgenstern in 1994 extended this to the case when $d-1$ is a prime power. In 2015, Marcus, Spielman and Srivastava proved that there exists infinitely many bipartite Ramanujan graphs for all $d \geq 3$ and for any number of vertices. Cohen in 2016 showed that one can construct those graphs in polynomial time. Here, bipartite Ramanujan graphs are graphs whose all eigenvalues except $\pm d$ have absolute value at most $2\sqrt{d-1}$. One big conjecture is that for any $d \geq 3$, there exists arbitrarily large d-regular Ramanujan graphs.

The construction of Lubotzky-Phillips-Sarnak are roughly as follows. Let p, q be distinct primes that are congruent to 1 modulo 4. Then $X^{p,q}$ will be a $(p+1)$-regular graph.

Let G be a group and S be a subset of G that is closed under inversion. Cayley graph $C(G,S)$ is a graph with vertex set G and edge set $\{(x, xs) : x \in G, s \in S\}$. Let $G = PGL(2,q)$ be the group of 2 by 2 nonsingular matrices over $\mathbb{F}_q$ where two matrices A and sA are identified for $s \in \mathbb{F}_s - \{0\}$. Fix some integer i with $i^2 \equiv -1 (\mod q)$. Let

$$S = \left\{\begin{pmatrix} a_0 + ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - ia_1\end{pmatrix} : a_0^2 + a_1^2 + a_2^2 + a_3^2 = p \text{ with odd } a_0 > 0 \text{ and even } a_1, a_2, a_3\right\}.$$

By a theorem of Jacobi, there are $p+1$ solutions to the equation above. So $|S| = p+1$. Also S is closed under inversion as needed. Take a connected component of $C(G,S)$ containing the identity. Then it's a Ramanujan graph.