2. Graph expansion and eigenvalues

From now on, unless otherwise stated, we always assume that a graph is d-regular graphs with possibly parallel edges and loops.

Definition 2.1. *For $S \subseteq V(G)$, the edge boundary $\partial S = E(S, \overline{S})$ is the set of edges from S to $\overline{S} = V \setminus S$. The edge expansion ratio of G, denoted by $h(G)$, is defined as*

$$h(G) = \min_{S, |S| \leq n/2} \frac{|\partial S|}{|S|}.$$

This h is sometimes called the Cheeger constant of the graph.

We can further define the followings as well.

Definition 2.2.

$$\Psi_E(G,k) = \min_{S \subseteq V, |S| \leq k} \frac{|E(S,\overline{S})|}{|S|}, \;\; \Psi'_V(G,k) \min_{S \subseteq V, |S| \leq k} \frac{|N(S)|}{|S|}, \;\; \textit{and} \;\; \Psi_V(G,k) \min_{S \subseteq V, |S| \leq k} \frac{|N(S) \setminus S|}{|S|}.$$

Let's consider edge expansion as vertex expansion is more difficult to deal with.

Note that we want $h(G)$ be at least something. On the other hand, every connected graph has $h(G) > 0$, more specifically $h(G) \geq 2/n$. However, this is very weak. So we want to find a lower bound independent of the size of the graph. In other words, we want to find a family of graphs with uniform lower bound on $h(G)$.

Definition 2.3. *A sequence $\{G_i\}$ of d-regular graphs $\{G_i : i \in \mathbb{N}\}$ of size increasing with i is a family of expander graphs if there exists $\varepsilon > 0$ such that $h(G_i) \geq \varepsilon$ for all i.*

When we construct an expander graph, we want to construct one with (roughly) the size we wish for. Furthermore, we want to construct them efficiently (in time polynomial on the input) and perhaps wishes to be able to determine k-th neighbor of a given vertex efficiently. This can be more precise as follows.

Definition 2.4. *Let $\{G_i\}_i$ be a family of expander graphs where G_i is an n_i-vertex d-regular graph, and n_i increasing not to fast. (E.g. $n_{i+1} \leq n_i^2$.)*

1. *The family is called* **mildly explicit** *if there is an algorithm that generates the j-th graph in the family G_j in time polynomial in j.*
2. *The family is called* **very explicit** *if there is an algorithm that on input of an integer i, a vertex $v \in V(G_i)$ and $k \in [d]$ computes the k-th neighbor of the vertex v in the graph G_i. This algorithm 's run time should be polynomial in its input length, the number of bits needed to express the triple (i, v, k).*

Consider the following examples.

Margulis in 1973 came up with the following example of expander grahs. Consider a graph on vertex set $\mathbb{Z}_m \times \mathbb{Z}_m$, where the neighbors of (x, y) are

$$(x+y,y), (x-y,y), (x,y+x), (x,y-x), (x+y-1,y), (x-y+1,y), (x,y+x+1), (x,x-y-1).$$

This family is very explicit as one can efficiently computes the neighbors of a given vertex.

Another example is to consider the vertex set $\mathbb{Z}_p$ for a prime p, and make x adjacent to $x-1, x+1$ and its multiplicative inverse x^{-1} in $\mod p$. This family is a family of expander graphs (the proof depends on deep result on Number theory). However, this is only mildly explicit, since we are not able to generate a large primes deterministically.

Now we turn into an algebraic definition of expansion. The adjacency matrix of a graph G, denoted by $A = A(G)$ is an $n \times n$ matrix whose (u, v)-entry is the number of edges in G between vertex u and vertex v. As it is real symmetric matrix, it has n real eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. We can also find orthonormal eigenvectors $v_1, \ldots, v_n$ associated with the eigenvalues, i.e. $Av_i = \lambda_i v_i$. We call these eigenvalues of $A(G)$ as Spectrum of the graph G.

This spectrum contains much information about G. For example, we can easily check that $\lambda_1 = d$ and $v_1 = \frac{1}{\sqrt{n}}\mathbf{1} = (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})^T$. The graph is is connected iff $\lambda_1 > \lambda_2$, and it is bipartite iff $\lambda_1 = -\lambda_n$.

Moreover, spectrum is related to edge-expansion of graphs. To see why, recall that the edge-expansion is essentially counting the edges between S and $\overline{S}$. Consider the expression $1_S^T A 1_T$, where 1_S is the 0/1-vector whose i-th coordinate is 1 if and only if $i \in S$.. This expression counts the number of edges between S and T. Also, this value has relation with spectrum, as eigenvectors form an orthonormal basis.

Let $\lambda(G) = \max\{|\lambda_2|, |\lambda_n|\}$. The following expander mixing lemma shows that how edge distribution and spectrum are related.

Lemma 2.5 (Expander mixing lemma)**.** *Let G be a d-regular graph with n vertices and let $\lambda = \lambda(G)$. Then for all $S, T \subseteq V$, we have*

$$||E(S,T)| - \frac{d|S||T|}{n}| \le \lambda\sqrt{|S||T|}.$$

Proof. Proof is from the following simple observation: $|E(S,T)| = 1_S^T A 1_T$ where 1_S^T and 1_T are the characteristic vectors of S and T. (Its v-th coordinate is 1 if v is in the set and zero otherwise)

As $v_1, \dots, v_n$ are orthonormal basis, we can express

$$1_S = \sum_{i\in[n]} \alpha_i v_i, 1_T = \sum_{i\in[n]} \beta_i v_i.$$

Then as $v_1, \dots, v_n$ are orthonormal basis, we have

$$|E(S,T)| = 1_S^T A 1_T = \sum_{i,j} \alpha_i\beta_j v_i A v_j = \sum_i \lambda_i\alpha_i\beta_i.$$

Since $\alpha_1 = \langle 1_S, \frac{1}{\sqrt{n}}\mathbf{1}\rangle = \frac{|S|}{\sqrt{n}}$ and $\beta_1 = \frac{|T|}{\sqrt{n}}$ and $\lambda_1 = d$, we have

$$|E(S,T)| = d\frac{|S||T|}{n} + \sum_{i=2}^{n} \lambda_i\alpha_i\beta_i.$$

As all λ_i with $i \neq 1$ has absolute value at most λ, we have

$$||E(S,T)| - d\frac{|S||T|}{n}| \le \lambda\sum_{i=2}^{n} |\alpha_i\beta_i| \le \lambda\sqrt{\sum_{i=1}^{n}\alpha_i^2}\sqrt{\sum_{i=1}^{n}\beta_i^2} \le \lambda\sqrt{|S||T|}.$$

This proves the lemma. □

One meaning of this lemma is that if λ is small, then $|E(S,T)|$ is close to $\frac{d|S||T|}{n}$. If we take $T = \overline{S}$ and $|S| < n/2$, then $\frac{d|S||T|}{n} > \frac{d}{2}|S|$.

One natural question is that whether converse of such a statement is true. Namely, if $|E(S,T)|$ is close to $\frac{d|S||T|}{n}$ for all S, T, then do we have small λ? Indeed, the following lemma is known. We will not prove this here.

Lemma 2.6 (Bilu, Linial, 2006)**.** *For d-regular graph G, if*

$$||E(S,T)| - \frac{d|S||T|}{n}| \le \rho\sqrt{|S||T|}$$

holds for every two disjoint sets S and T and $\rho > 0$, then $\lambda \le O(\rho(1 + \log(d/\rho)))$.

The expander mixing lemma indicates that if λ is small, then every sets of size at most $n/2$ must have a large edge-boundary. Furthermore, we have the following theorem.

Theorem 2.7 (Dodziuk 84, Alon-Milman 85, Alon 86)**.** *Let G be a d-regular graph with spectrum $\lambda_1 \geq \cdots \geq \lambda_n$. Then*

$$\frac{d-\lambda_2}{2} \leq h(G) \leq \sqrt{2d(d-\lambda_2)}.$$

Proof. We first show the left inequality. The proof is similar with the proof of expander mixing lemma. Let S be a set satisfying $h(G) = \frac{|E(S,\overline{S})|}{|S|}$ and $|S| \leq n/2$. Consider the vector $f = |\overline{S}|1_S - |S|1_{\overline{S}}$. Then we have

$$\|f\|^2 = |\overline{S}|^2|S| + |S|^2|\overline{S}| = |S||\overline{S}|(|S| + |\overline{S}|) = n|S||\overline{S}|$$
$$f^TAf = 2(|E(S)||\overline{S}|^2 + |E(\overline{S})||S|^2 - |S||\overline{S}|E(S,\overline{S})|).$$

Since G is d-regular, we have

$$2|E(S)| = d|S| - |E(S,\overline{S})| \text{ and } 2|E(\overline{S})| = d|\overline{S}| - |E(S,\overline{S})|.$$

Note that f is orthogonal to $\mathbf{1}$, which is an eigenvector for λ_1. Hence, we have

$$\lambda_2 \geq \frac{f^TAf}{\|f\|^2} = d - \frac{n|E(S,\overline{S})|}{|S||\overline{S}|} \geq d - 2h(G).$$

This conclude the left inequality. Here, $\frac{f^TAf}{\|f\|^2}$ is called Rayleigh quotient.

The other inequality is more difficult. Let g be the eigenvector associated with λ_2, we seek to find a cut $(S,\overline{S})$ from G. If g is looking like the vector f above (i.e. having two values, one positive one negative), then it would have been easy.

By choosing $-g$ if necessary, assume that g has at most half coordinates positive.

As g might not look like that, we let $f = g^+$ be the vector with $f_v = \max(g_v, 0)$ and let $V^+ = \{v : f_v > 0\}$, then $|V^+| \leq n/2$. Assume $V = [n]$ and $f_1 \geq \cdots \geq f_n$. Let

$$B_f = \sum_{xy\in E(G)} |f_x^2 - f_y^2|.$$

This expression has several merits. One is that this is related to the Cheeger constant as follows. This can be shown by the following

$$\begin{aligned}
B_f &= \sum_{xy\in E(G),x<y} (f_x^2 - f_y^2) = \sum_{xy\in E(G),x<y} \sum_{i=x}^{y-1} (f_i^2 - f_{i+1}^2) \\
&= \sum_{i=1}^{n-1} (f_i^2 - f_{i+1}^2)|E([i],\overline{[i]})| = \sum_{i\in V^+} (f_i^2 - f_{i+1}^2)|E([i],\overline{[i]})| \\
&\geq h \sum_{i\in V^+} (f_i^2 - f_{i+1}^2)i = h \sum_{i\in V^+} f_i^2 = h\|f\|^2. \qquad (2.1)
\end{aligned}$$

On the other hand, we can get upper bound on B_f using the relation $f_x^2 - f_y^2 \leq (f_x - f_y)(f_x + f_y)$.

We have

$$B_f = \sum_E |f_x^2 - f_y^2| = \sum_E |f_x + f_y||f_x - f_y| \leq \sqrt{\sum_E (f_x + f_y)^2 \sum_E (f_x - f_y)^2}.$$

We give an arbitrary orientation on G and let K be the incidence matrix of this oriented graph G with columns indexed by V and row indexed by E. If $e = \overrightarrow{uv} \in E(G)$, then $K_{ev} = 1, K_{eu} = -1$. Then we have $L = K^TK$ where $L = dI - A(G)$ is the Laplacian of G. Note that $(Kf)_{e=xy} = f(x) - f(y)$, hence $\|fK\|^2 = \sum_E (f(x) - f(y))^2$. As $K^TK = L$, we have $f^TLf = \|Kf\|^2$.

On the other hand, $\sqrt{\sum_E (f_x+f_y)^2} \le \sqrt{2\sum_E (f_x^2+f_y^2)} = \sqrt{2d\sum_V f_x^2} = \sqrt{2d}\|f\|$. Hence, we have

$$B_f \le \sqrt{2d}\sqrt{f^T L f}\|f\|. \tag{2.2}$$

Using this together with (2.1), we have

$$h(G)\|f\| \le B_f \le \sqrt{2d}\sqrt{f^T L f}\|f\|. \tag{2.3}$$

Again, we want to compute Rayleigh quotient of the laplccian on the right side.

Claim 2.

$$\frac{f^T L f}{\|f\|^2} \le d - \lambda_2.$$

Proof. For $x \in V^+$, we have

$$(Lf)_x = df_x - \sum_y A_{xy} f_y = dg_x - \sum_{y\in V^+} A_{xy} g_y \le dg_x - \sum_{y\in V} A_{xy} g_y = (Lg)_x = ((dI-A)g)_x = (d-\lambda_2)g_x.$$

As $f_x = 0$ for $x \notin V^+$, we have

$$f^T L f = \sum_{x\in V} f_x \cdot (Lf)_x \le (d-\lambda_2)\sum_{v\in V^+} g_x^2 = (d-\lambda_2)\sum_{v\in V} f_x^2 = (d-\lambda_2)\|f\|^2.$$

This proves the claim. □

Apply this claim to (2.3), then we obtain $h(G) \le \sqrt{2d\frac{f^T L f}{\|f\|^2}} \le \sqrt{2d(d-\lambda_2)}$. □

Note that the above theorem is best possible up to constant. It is not difficult to see that $h(G) \le d/2 + o(1)$ (see Section 4 for this), so the lower bound is tight for small λ_2. For large λ_2, one can consider a d-dimensional hypercube Q_d. It is not difficult to see that $h(Q_d) = 1$ while $\frac{d-\lambda_2}{2} = 1$.

For upper bound, for example, consider a cycle C_n. We have $h(C_n) = \Theta(1/n)$ while $\sqrt{2d(d-\lambda)} \simeq \Theta(1/n)$.

We can similarly consider vertex expansion.

Theorem 2.8 (Kahale, 1995)**.** *Let G be an n-vertex d-regular graph with $\lambda(G) \le \lambda$. Then there exists an absolute constant c such that every set S of size at most ρn satisfies*

$$\frac{|N(S)|}{|S|} \ge (d/2)\left(1 - \sqrt{1 - \frac{4(d-1)}{d^2\lambda^2}}\right)(1 - c\frac{\log d}{\log(1/\rho)}).$$

We prove the following weaker version.

Theorem 2.9 (Tanner, 1984)**.** *Let G be an n-vertex d-regular graph with $\lambda(G) \le \alpha d$. Then every set S of size at most ρn satisfies*

$$\frac{|N(S)|}{|S|} \ge \frac{1}{\rho(1-\alpha^2)+\alpha^2}.$$

Proof. Let S be a set of size ρn. We consider $\|\frac{1}{d}A1_S\|^2$, which has eigenvalues $\hat{\lambda}_i = \frac{\lambda_i}{d}$. By writing $1_S = \sum_i \alpha_i v_i$ where $v_1 = \frac{1}{\sqrt{n}}\mathbf{1}$ and $a_1 = \frac{|S|}{\sqrt{n}}$, we have

$$\|\frac{1}{d}A1_S\|^2 = \sum_{i=1}^n \hat{\lambda}_i^2\alpha_i^2 \le \frac{|S|^2}{n} + \sum_{i=2}^n \alpha^2\alpha_i^2 \le \frac{|S|^2}{n} + \alpha^2(\|1_S\|_2^2 - a_1^2) = \rho n(\rho + \alpha^2(1-\rho))$$

$$\|\frac{1}{d}A1_S\|^2 = \sum_{x\in N(S)} (\frac{|S\cap N(x)|}{d})^2 \ge \frac{|S|^2}{|N(S)|} = \rho n(\frac{|S|}{|N(S)|}).$$

Here, the last inequality follows from Cachy-Schwwartz since $\sum_{x \in N(S)} \frac{|S \cap N(X)|}{d} = |S|$. This proves what we want. $\square$