Lecture 7

Def: A **P-sunflower** is a family of P sets with identical pairwise intersection. That is,

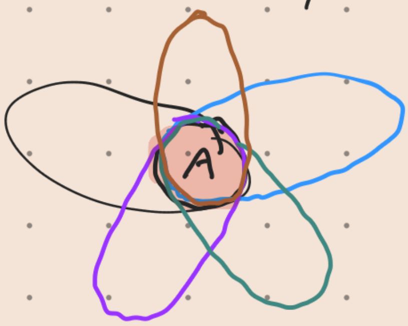
$\{S_1, \dots, S_p\}$ s.t. $\forall\, ij \in \binom{[P]}{2} \Rightarrow S_i \cap S_j = A$.

We call A the **core** of the sunflower

Note that A could be empty.

Here P = # of petals.

- core $\neq \phi$
- core $= \phi$



- k-graphs = k-uniform hypergraphs = k-sets system

Conj (Erdős-Rado sunflower conj)

Let $P \geq 3$. Then $\exists\, C = C(P)$ s.t.

$\forall$ k-graph $\mathcal{F}$ without a P-sunflower has

$\leq C^k$ edges.

- Alweiss-Lovett-Wu-Zhang: $\left(\log k \cdot (P \log\log k)^{O(1)}\right)^k$
- Rao, using Shannon noiseless coding thm,

$(O(1) \cdot P \log(Pk))^k$

• Tao, entropy pf.

Notations • X set, write $X_\delta = \mathrm{Binom}(X, \delta)$

is binomial random subsets of X w/ each element included w/ prob. δ, indep of others.

• A set, $W \sim A$

W uniform sampled from A.

• W, Z. r.v. write

$W \sim Z$ the same distribution.

§2.9.1 Reduction to spread hypergraphs via induction.

Take $R \geq 1$.

Def: (Spread) A k-graph $\mathcal{H}$ is R-spread if

$$\forall Z \neq \phi, \quad d_{\mathcal{H}}(Z) = \left| \{ S \in \mathcal{H} : S \supseteq Z \} \right| \leq \frac{|\mathcal{H}|}{R^{|Z|}}$$

• Reduction: We may consider spread hypergraphs.

Suppose we want to show:

sunflower-free $\Rightarrow$ $\leq R^k$ edges.

If $\mathcal{H}$ is not R-spread.

$\Rightarrow \exists Z$ s.t. $d(Z) \geq \dfrac{e(\mathcal{H})}{R^{|Z|}}$

Now if $e(\mathcal{H}) \geq R^k \Rightarrow d(Z) \geq R^{k-|Z|}$

consider link of Z: $(k-|Z|)$-uniform hypergraph with edge set $\{ S_i \setminus Z : S_i \in \mathcal{H}, S_i \supseteq Z \}$

Induct. on uniformity ✓. $\blacksquare$

- We will now find in a spread hypergraph p pairwise disjoint sets.

<u>Def</u>: A random set A is R-spread, $R>1$, if $\Pr(Z \subseteq A) \leq R^{-|Z|}$ $\forall Z$.

A hypergraph $\mathcal{H}$ is R-spread if the uniform random edge from $\mathcal{H}$ is R-spread.

§ 2.9.2. Incorporate Spreadness in entropy.

- Maximality $\Rightarrow$ X takes values in a set S_Y that depends on Y, then $H(X|Y) \leq \mathbb{E}_Y \log|S_Y|$.

$\Rightarrow$ (6) .... $H(B|A) \leq \mathbb{E}|A|$, $\forall$ random $B \subseteq$ random A

- Mutual info. of two r.v. X & Y.

$$I(X;Y) = H(X) + H(Y) - H(X,Y)$$
$$= H(X) - H(X|Y) = H(Y) - H(Y|X)$$
$$= H(X,Y) - H(X|Y) - H(Y|X).$$

<u>Monotonicity</u>
- if $Y = f(X) \Rightarrow I(X;Y) = H(Y)$

<u>indep.</u>
- if X, Y indep. $\Rightarrow I(X;Y) = 0$

- Let X be a r.v., an empirical seq $X_1, X_2, \cdots$ for X that take values from the same set as X and converging to X in distribution, that is,

$$n \sim [N], \quad \Pr(X_n = a) = \Pr(X = a) + o_N(1).$$

- Mutual info between a random set & its random subset.

Consider $A \sim 2^{[n]}$ & a random subset $B \subseteq A$.

$$\Rightarrow I(A;B) = H(A) - H(A|B)$$
$$= \mathbb{E}_b \left(H(A) - H(A \mid |B| = b) \right)$$
$$\overset{\text{max.}}{=} \mathbb{E}_b \left(n - (n-b) \right) = \mathbb{E}_b\, b = \mathbb{E}|B|.$$

- Spread set shares a lot more info. w/ its random subset.

Lem 2.20 Let A be a random R-spread set w/ $R>1$.

If A is uniformly chosen, then $\forall$ random $B \subseteq A$

$$\Rightarrow \quad I(A;B) \geq \log R \cdot \mathbb{E}|B|$$

Rmk: We will use this lem in contrapositive way.

WTS $\mathbb{E}|B|$ small $\Rightarrow$ enough to upp. bd. $I(A;B)$.

Lem 2.21 Let $W \sim \binom{X}{\delta|X|}$, $0<\delta<1$

$\frac{1}{\delta}$-spread

- $\forall$ random $B \subseteq W$, $H(W) - H(W|B) \geq \log\frac{1}{\delta} \cdot \mathbb{E}|B|$
- (Absorption) $\forall$ random $B \subseteq X$,

$$H(W \cup B) \leq H(W) + 1 + \left(1 + \log\frac{1}{\delta}\right) \cdot \mathbb{E}|B| \quad \cdots\cdots (7)$$

- Relative product. (X, X') in which X' is a cond. indep. copy of X subject to certain constraint $f(X) = f(X')$ for a given f.

$$\Rightarrow H(X) = H(X, f(X)) = H(X \mid f(X)) + H(f(X))$$

only relevant info to X from X' is $f(X)=f(X')$

$$= H(X \mid X') + H(f(X)) \quad \cdots\cdots (8)$$

Lem 2.22 Let $R > 1$, $0 < \delta < 1$, A be a random R-spread subset of X, and $V \sim X_\delta$ indep. of A.

Then $\exists$ another random $A' \sim A$ of X s.t.

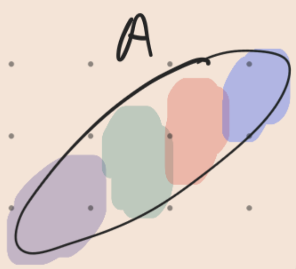
$$A' \setminus V \subseteq A \quad \text{and} \quad \mathbb{E}|A' \setminus V| \le \boxed{\frac{4 + \log 1/\delta}{\log R} \cdot \mathbb{E}|A|}.$$

Rmk: $\mathbb{E}|A \setminus V| = (1-\delta)\,\mathbb{E}|A|$, by passing to a $A' \sim A$ we can improve $(1-o(1))$ to $o(1)$ factor (if $R \gg 1/\delta$)

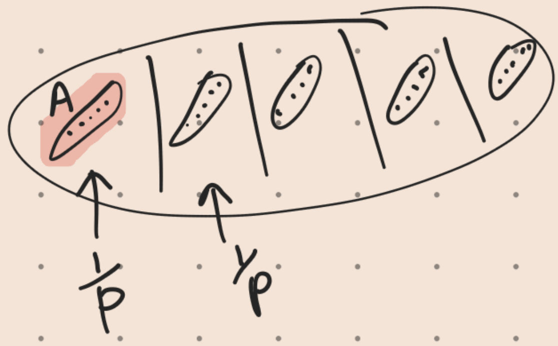
(Sketch)
Pf Lem 2.22 $\Rightarrow$ upp bd. $\le R^k$, where $R = (P\log(Pk))^C$
$\left((\log k)^{O(k)}\right)$

- $\mathcal{H}$, R-spread

  Sample A unif from $\mathcal{H}$ $\Rightarrow$ A R-spread.



Iterate Lem 2.22 to cover $A' \sim A$
$\uparrow$ some edge in $\mathcal{H}$.



I.e. iterate Lem 2.22 so that $\mathbb{E}|A' \setminus V| < \frac{1}{p}$

Markov $\Rightarrow$ $\Pr(|A' \setminus V| \ge 1) \le \mathbb{E}|A' \setminus V| < 1/p$

then union bd. $\blacksquare$

Pf (Lem 2.22) • If A is empty, take $A' = A$. ✓.

May assume $\mathbb{E}|A| \geq 1$.

• View $X = [|X|]$, take $N_1, N_2 >> |X|$

• Pass binom. model $\Rightarrow$ unif model.

$$V \sim X_\delta \quad \Rightarrow \quad W \sim \binom{[N_2]}{\delta N_2}$$

Take empirical seq of A: $A_1, A_2, \ldots,$

$$n \sim [N_1] \quad \Rightarrow \quad \text{random set } A_n$$

$$\mathbb{E}|A_n| \geq \mathbb{E}|A| - o(1) \geq 1 - o(1).$$

Note that $W \cap [|X|] \xrightarrow{\text{dist.}} V$

Thus, as $N_1, N_2 \to \infty$, suffices to show

$$\exists n' \sim n \quad \mathbb{E}|A_{n'} \setminus W| \leq \frac{4 + \log \frac{1}{\delta}}{\log R} \mathbb{E}|A_n| + o(1).$$

Key idea Take a cond. indep copy (n', W') of (n, W) subject to

$$(9) \cdots\cdots \quad A_{n'} \cup W' = A_n \cup W \Rightarrow A_{n'} \setminus W \subseteq A_n \cap A_{n'}.$$

Thus, enough to show

$$\mathbb{E}|A_n \cap A_{n'}| \leq \frac{4 + \log \frac{1}{\delta}}{\log R} \mathbb{E}|A_n| + o(1).$$

By Lem 2.20 w/ $(A,B) = (A_n, A_n \cap A_{n'})$

suffices: $H(n) - H(n \mid A_n \cap A_{n'}) \leq (4 + \log\frac{1}{\delta})\mathbb{E}|A_n| + o(1)$ ....(10)

- bd $H(n)$ using $H(n,W)$

use (8) w/ $((n,W), (n',W'), A_n \cup W) = (X, X', f(X))$

$$H(n,W) = H(n,W \mid n',W') + H(A_n \cup W)$$

ind. $\cancel{H(n)+H(W)}$ //

$$\overset{(7)}{\leq} H(n,W \mid n',W') + \cancel{H(W)} + \underbrace{(2+\log\tfrac{1}{\delta})\mathbb{E}|A_n| + o(1)}$$ ......(11)

$$\Rightarrow H(n) \leq H(n,W \mid n',W') + \sim\sim\sim$$

- Relate $H(n,W \mid n',W')$ to $A_n \cap A_{n'}$.

$$H(n,W \mid n',W') \leq \underbrace{H(A_n \cap A_{n'} \mid n',W')}_{\leq \mathbb{E}|A_n|}$$ choose $A_n \cap A_{n'}$

$$+ H(n \mid A_n \cap A_{n'}, \cancel{n'}, \cancel{W'})$$ choose A_n

$$+ \underbrace{H(W \mid n, A_n \cap A_{n'}, n', W')}_{\leq \mathbb{E}|A_n|}$$ Choose W.

1st & 3rd term can be bdd by (6)

$$\Rightarrow H(n,W \mid n',W') \leq 2\mathbb{E}|A_n| + H(n \mid A_n \cap A_{n'})$$

plug this into (11) ▨