Lecture 6

Recall

$$\underset{(\text{large})}{G} \xrightarrow{\text{homomorphism}} \underset{(\text{fixed})}{H} \quad : \quad H\text{-colouring}$$

- $H = K_q$: proper q-colouring of G.
- $H =$  : indep. sets in G

Thm 2.13 $\forall$ n-vx d-reg bip. G

$$i(G) \le i(K_{d,d})^{n/2d}$$

\# indep sets in disj union of $K_{d,d}$

Pf: • Let $V(G) = \mathcal{O} \cup \mathcal{E}$, $|\mathcal{O}| = |\mathcal{E}| = n/2$ (as G is d-reg)

• $\mathcal{I}(\cdot)$ = all indep sets in G

$$X \sim \mathcal{I}(G), \quad Y \sim \mathcal{I}(K_{d,d})$$

max. $\Rightarrow$ $H(X) = \log i(G)$, $H(Y) = \log i(K_{d,d})$

Suffices: $\quad H(X) \le \frac{n}{2d} \cdot H(Y)$

• View X as the random vect. $(X_{\mathcal{O}}, X_{\mathcal{E}})$

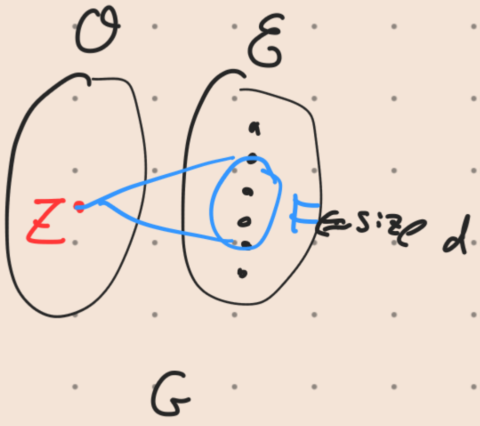
where $X_{\mathcal{O}} = (X_v : v \in \mathcal{O})$, $X_v = \mathbb{1}_{\{v \in X\}}$ . . . .

• chain rule

$$\Rightarrow \quad H(X) = H(X_{\mathcal{E}}) + H(X_{\mathcal{O}} \mid X_{\mathcal{E}})$$

• To bound $H(X_{\mathcal{E}})$, we use Shearer's lem.

Let $Z \sim \mathcal{O}$ & set



$$F = N(Z)$$

$\forall v \in \mathcal{E}$: $\Pr(v \in F) = \dfrac{d}{|\mathcal{E}|} = \dfrac{2d}{n} =: \mu$

• Shearer's lem

$$\implies H(X_{\mathcal{E}}) \le \frac{n}{2d} \mathbb{E}_F H(X_F)$$

$$= \frac{n}{2d} \mathbb{E}_Z H(X_{N(Z)})$$

$$= \frac{\cancel{n}}{\cancel{2}d} \cdot \frac{1}{\cancel{n/2}} \sum_{v \in \mathcal{O}} H(X_{N(v)})$$

$$\implies H(X_{\mathcal{E}}) \le \frac{1}{d} \sum_{v \in \mathcal{O}} H(X_{N(v)})$$

• $$H(\underbrace{X_{\mathcal{O}}}_{(X_v \,:\, v \in \mathcal{O})} | X_{\mathcal{E}}) \overset{\text{subadd.}}{\le} \sum_{v \in \mathcal{O}} H(X_v | X_{\mathcal{E}})$$

$$\overset{\text{dropping cond.}}{\le} \sum_{v \in \mathcal{O}} H(X_v | X_{N(v)})$$

in fact "=" holds

Spatial Markov property

• Thus $$H(X) \le \frac{1}{d} \sum_{v \in \mathcal{O}} \underbrace{\left(H(X_{N(v)}) + d \cdot H(X_v | X_{N(v)}) \right)}$$

Enough to show $\forall v \in \mathcal{O}$, $\le H(Y)$

$Y \sim \mathcal{I}(K_{d,d})$

We will define a random indep set W in $K_{d,d}$

s.t. $H(X_{N(v)}) + d \cdot H(X_v \mid X_{N(v)}) = H(W) \overset{\max}{\leq} H(Y)$

$H(W_B) + H(W_A \mid W_B) = H(W)$

Sample W from $\mathcal{I}(K_{A,B})$:

$K_{d,d}$

- its marginal dist. on B same as $X_{N(v)}$, i.e. $W_B \overset{\text{identical}}{\sim} X_{N(v)}$

- each vx $u \in A$ has the same cond. marginal dist. as $X_v \mid X_{N(v)}$, $W_u \mid W_B \overset{\text{identical}}{\sim} X_v \mid X_{N(v)}$. $\blacksquare$

§ 2.8 Counting matroids

A matroid is a pair $(E, \mathcal{B})$, where

- E finite set ground set
- $\mathcal{B} \subseteq 2^E$ nonempty base

satisfying the following axiom:

Base exchange: $\forall\, B, B' \in \mathcal{B}$, $\forall e \in B \setminus B'$

$\Rightarrow \exists f \in B' \setminus B$ s.t. $B \setminus \{e\} \cup \{f\} \in \mathcal{B}$

- Subsets of base are independent set.
- Subsets of E are dependent sets if they not indep.

Fact: Base exchange $\Rightarrow$ all base $B \in \mathcal{B}$ have the same size $=$ which we call rank of the matroid.

Def: • $m_{n,r}$ = # matroids of rank r on ground set $[n]$

• m_n = # all matroids on $[n]$ $= \sum_{r=0}^{n} m_{n,r}$.

Trivially $m_n \leq 2^{2^n} \iff \log\log m_n \leq n$
$(\mathcal{B} \subseteq 2^E)$ $\quad m_n \geq 2^{\frac{1}{2}\binom{n}{n/2}}$

• Knuth: $\log\log m_n \geq n - \frac{3}{2}\log n - O(1)$.

Thm 2.14 [Bansal-Pendavingh-van der Pol]

$$\log\log m_n \leq n - \frac{3}{2}\log n + \log\log n + O(1)$$

• Strategy is to use entropy (Shearer's lem) to bound matroids of higher ranks by rank-2 matroids

• Let's first estimate $m_{n,2}$.

• $m_{n,0} = 1$, $\forall n$ as only one is $\mathcal{B} = \{\phi\}$

• $m_{n,1} = 2^n - 1$, as $\forall$ non-empty $\mathcal{B} \subseteq \binom{[n]}{1}$ satisfies base exchange

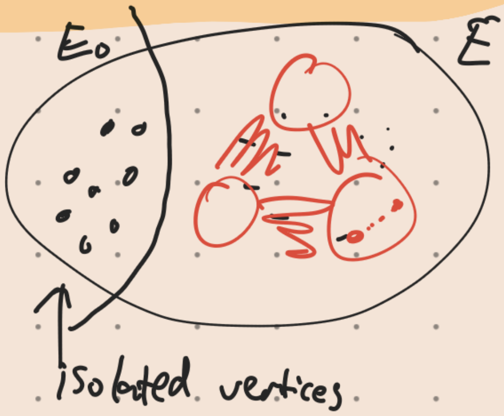
Only need $\quad m_{n,2} \leq n^n$

Lem 2.15 (in a rank-2 matroid, $e, f \in E$, e, f being depend. is an equiv. relation)

Lem 2.15 Let $(E, \mathcal{B})$ be rank-2, then

$\exists$ partition $E = E_0 \cup E_1 \cup \dots \cup E_k$ s.t.

$$\mathcal{B} = \left\{ \{e, e'\} : e \in E_i,\ e' \in E_j,\ ij \in \binom{[k]}{2} \right\}$$



Pf: Take $E_0 \subseteq E$ to be the set of singletons that are dependent sets.

$$E_0 = \{ e \in E : e \notin B,\ \forall B \in \mathcal{B} \}$$
$$= E \setminus \left(\bigcup_{B \in \mathcal{B}} B \right)$$

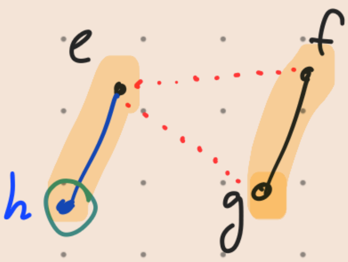
- It suffices to show that $\forall\ e, f, g \in E \setminus E_0$

$$ef,\ eg \notin \mathcal{B} \implies fg \notin \mathcal{B}$$

Supp. not, say $fg \in \mathcal{B}$.

$e \notin E_0$

$\Rightarrow \exists h$ s.t. $eh \in \mathcal{B}$



$\Rightarrow$ Base exchange fails for $B' = fg$, $B = eh$ &

$h \in B' \setminus B$ ▨

Lem 2.15 provides an injective map from all rank-2 matroids to partitions of E

$$\Rightarrow \quad m_{n,2} \le n^n.$$

Lem 2.16 Let $0 \le t \le r \le n$, then

$$\boxed{\frac{1}{\binom{n}{r}} \cdot \log(m_{n,r}+1) \le \frac{1}{\binom{n-t}{r-t}} \log(m_{n-t,r-t}+1)}.$$

Taking $t = r-2$, we have in particular that

$$\frac{1}{\binom{n}{r}} \log(m_{n,r}+1) \le \frac{1}{\binom{n-r+2}{2}} \log(m_{n-r+2,2}+1).$$

Pf (Thm 2.14 using Lem 2.16)

$$\le m_{n,2}+1 \le n^n+1 \le (n+1)^{n+1}$$

$$\log(m_{n,r}+1) \le \frac{\log(\overbrace{m_{n-r+2,2}+1})}{\binom{n-r+2}{2}} \cdot \binom{n}{r}$$

$$\le \frac{(n+1)\log(n+1)}{\binom{n-r+2}{2}} \binom{n}{r} = \frac{2\log(n+1)}{n+2} \underbrace{\binom{n+2}{r}}_{\le \binom{n+2}{(n+2)/2}}$$

- $m_n = \sum_{r=0}^{n} m_{n,r} \le (n+1) \cdot \max_r m_{n,r}$

$$\Rightarrow \log m_n \le \log(n+1) + \max_r \log m_{n,r}$$

$$\le \log(n+1) + \frac{2\log(n+1)}{n+2} \cdot \binom{n+2}{(n+2)/2}$$

$$= O(2^n \cdot n^{-3/2} \cdot \log n)$$

$$\log\log m_n \le n - \frac{3}{2}\log n + \log\log n + O(1). \quad \blacksquare$$

$\boxed{\text{Idea: (Lem 2.16) use contractions on matroids to do random projections in Shearer's lem.}}$

For a set E, define

$$\mathcal{M}_{E,r} = \left\{ \mathcal{B} \subseteq \binom{E}{r} : \mathcal{B} \text{ satisfies base exchange} \right\}$$

Rmk. if $|E| = n$, $|\mathcal{M}_{E,r}| = m_{n,r} + 1$ as $\mathcal{M}_{E,r}$ contains also empty set.

- (Contraction) Let $M = (E, \mathcal{B})$ be a matroid.

If $T \subseteq E$ is contained in some basis of M ($T \subseteq B$, for some $B \in \mathcal{B}$)

then contracting $T \Rightarrow$ another matroid

$$M/T = (E \setminus T, \mathcal{B}/T)$$

where $\mathcal{B}/T = \{ B \setminus T : B \in \mathcal{B}, T \subseteq B)$

("link" of T)

- If T is not contained in any $B \in \mathcal{B}$

$\Rightarrow \mathcal{B}/T = \phi$

So $\forall\ \mathcal{B} \in \mathcal{M}_{E,r}$ and $\forall\ T \subseteq E$

$$\mathcal{B}/T \in \mathcal{M}_{E \setminus T,\, r-|T|}$$

<u>Pf (Lem 2.16)</u> Let $E = [n]$, draw $X \sim \mathcal{M}_{E,r}$.

$$\overset{\text{max}}{\Rightarrow} H(X) = \log(m_{n,r} + 1)$$

- View X as the random vector

set of coordinates

$$(X_R : R \in \tbinom{E}{r}), \text{ where}$$

$$X_R = \mathbb{1}_{\{R \in X\}} \Rightarrow H(X) = H\left(X_R : R \in \tbinom{E}{r}\right)$$

- Fix $T \subseteq E$ of size t, the contraction

X/T takes values in $\mathcal{M}_{E \setminus T,\, r-t}$

- $\overset{\text{max}}{\Longrightarrow}$ $\boxed{H(X/T) \leq \log(m_{n-t,\, r-t} + 1)}$

- X/T is the projection of X to the set of coordinates $F(T) := \{R \in \tbinom{E}{r} : T \subseteq R\}$

i.e. $X/T = X_{F(T)}$

- Now, let $T \sim \tbinom{E}{t}$ & let $F(T)$ as above.

$$F(T) = \{R \in \tbinom{E}{r} : R \supseteq T\}.$$

$\Rightarrow \forall R \in \binom{E}{r}$.

$$\Pr(R \in F(T)) = \Pr(R \supseteq T) = \mu = \frac{\binom{r}{t}}{\binom{n}{t}} = \frac{\binom{n-t}{r-t}}{\binom{n}{r}}$$

- By Shearer's lem:

$$\log(m_{n,r}+1) = H(X) = H(X_R : R \in \binom{E}{r})$$

X/T

$$\le \frac{\binom{n}{r}}{\binom{n-t}{r-t}} \mathbb{E}_{F(T)} H(X_{F(T)})$$

$$= \frac{\binom{n}{r}}{\binom{n-t}{r-t}} \mathbb{E}_T H(X/T)$$

$$\le \frac{\binom{n}{r}}{\binom{n-t}{r-t}} \log(m_{n-t,r-t}+1).$$