Lecture 5

Recap: 1. Brégman's thm $\quad \mathrm{perm}(A)$, A $0,1$-matrix $n \times n$

$$\mathrm{perm}(A) = \#\text{ perfect matchings}$$

$$\sigma \sim PM(G_A) \Rightarrow H(\sigma) = H(\sigma(v_1), \sigma(v_2), \cdots, \sigma(v_n))$$

$$\overset{\text{chain rule}}{=} H(\sigma(v_1)) + H(\sigma(v_2) \mid \sigma(v_1)) + \cdots + H(\sigma(v_n) \mid \sigma(v_1), \cdots, \sigma(v_{n-1}))$$

2. Sidorenko's conj.

H-density in G: $\quad t(H,G) = \dfrac{\hom(H,G)}{|V(G)|^{|V(H)|}}$

$$p = t(K_2, G) = \text{edge density} = \frac{\hom(K_2, G)}{n^2} = \frac{2e(G)}{n^2}$$

Conj (Sidorenko) bip H, H-density minimised by random graphs

$$t(H,G) \geq t(K_2, G)^{e(H)}$$

$\Leftrightarrow$ $|G| = n$ vxs

$$\hom(H,G) \geq \underbrace{n^{v(H)} \cdot p^{e(H)}}$$

= expected # hom from H to random graphs.

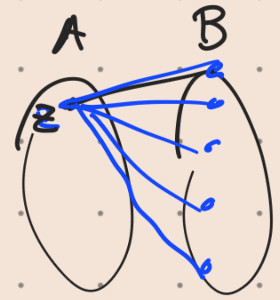
- Star of size 2, $K_{1,2}$

Idea
- sample $X \propto$ degree of vxs
- sample $Y, Z \sim N(X)$

(*) Y, Z indep. cond. on X

(*) both XY, $XZ \sim \hom(K_2, G)$. (labeled edges)

§ 2.6.2 Sidorenko for bip. w/ a dominating vx.



- Conlon-Fox-Sudakov proved Sidorenko's conj for such bip. H.

<u>Thm 2.9</u> Let H be a bip. graph on part. sets A, B w/ a vertex z in A adj to B. Let G be an n-vx graph w/ edge-density $p = t(K_2, G)$,

$$\Rightarrow \quad \hom(H, G) \geq n^{v(H)} \cdot p^{e(H)}.$$

- We need to bound entropy of a random star.

<u>Exer 2.10</u> Generate a random star of size b in an n-vertex graph G as follows.

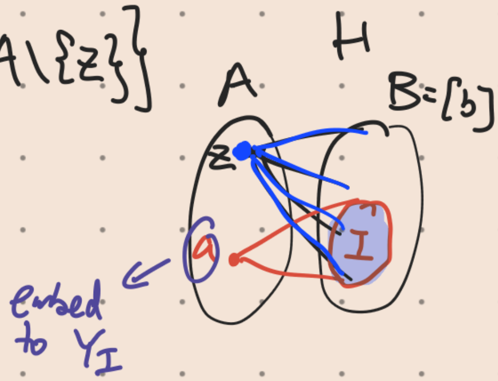
- Sample center $Y \propto$ degree $\left(\Pr(Y = v) = \dfrac{d_G(v)}{2e(G)} \right)$
- Sample $X_1, \dots, X_b \sim N(Y)$ as leaves

$\Rightarrow$ X_i's are cond. indep. given Y.

$$H(Y, X_1, \dots, X_b) \geq \log(n^{b+1} p^b)$$

Pf of Thm 2.9 (Goal: $\hom(H,G) \geq n^{v(H)} \cdot p^{e(H)}$)

Let $B=[b]$. Let $\mathcal{I} = \{ \overset{I\in}{N_H(a)} : a \in A \setminus \{z\} \}$



$\Rightarrow$ • $|\mathcal{I}| = |A \setminus \{z\}| = v(H) - b - 1$

• $\sum_{I \in \mathcal{I}} |I| = \sum_{a \in A \setminus \{z\}} d_H(a) = e(H) - b$

Sample a random homomorphism in $\mathrm{Hom}(H,G)$ as follows:

• (to embed z) Sample a random vertex $Z \propto$ degree

• (for B) sample vxs $X_1, \dots, X_b \sim N_G(Z)$

• (for $A \setminus \{z\}$) for each $I \in \mathcal{I}$, write $X_I = (X_i : i \in I)$

Sample $Y_I \in N_G(X_I)$ in such a way that (Y_I, X_I) is distributed as the random star in Exer 2.10

(Equivalently, Y_I is cond. indep. copy of Z given X_I)

• By maximality, entropy of this random homomorphism $\leq \log(\hom(H,G))$

It suffices to show

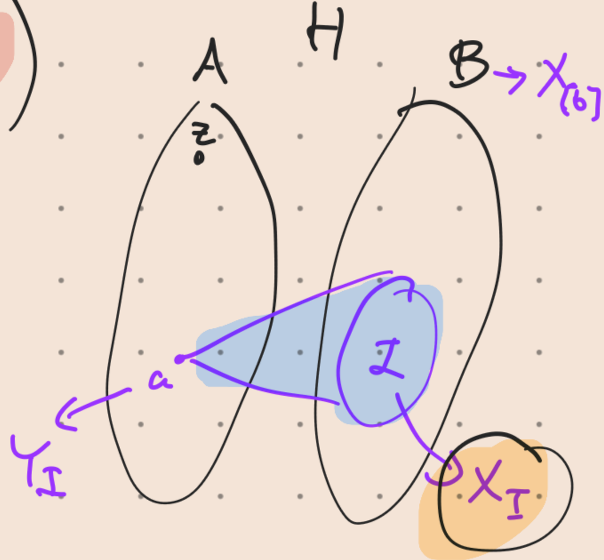
$$H(Z, (X_i : i \in [b]), (Y_I : I \in \mathcal{I})) \geq \log(n^{v(H)} p^{e(H)})$$

Chain rule

$$= \underbrace{H(Z, (X_i : i \in [b]))}_{\text{random } b\text{-star} \sim \text{Exer 2.10}} + H\left((Y_I : I \in \mathcal{I}) \middle| Z, (X_i : i \in [b])\right)$$

1st term $\overset{\text{Exer 2.10}}{\geq} \log(n^{b+1} \cdot p^b)$

$$H\Big((Y_I : I\in\mathcal{I}) \,\Big|\, Z, (X_i : i\in[b])\Big)$$



$$\overset{\text{cond. indep.}}{\underset{\text{drop } Z}{=}} H\Big(\underbrace{(Y_I : I\in\mathcal{I})} \,\Big|\, (X_i : i\in[b])\Big)$$

$$\overset{\text{cond. indep.}}{=} \sum_{\underbrace{I\in\mathcal{I}}} H\Big(Y_I \,\Big|\, (X_i : i\in[b])\Big)$$

$$= \sum_{I\in\mathcal{I}} H(Y_I \mid X_I)$$

subadd.: $H(X_I) \le \sum_{i\in I} \underbrace{H(X_i)}$

$$\overset{\text{add.}}{=} \sum_{I\in\mathcal{I}} \Big(H(Y_I, X_I) - H(X_I) \Big)$$

max.: $\le |I| \log n$

$$\overset{\text{Exer2.10}}{\geq} \sum_{I\in\mathcal{I}} \Big(\log\big(n^{|I|+1} p^{|I|}\big) - |I|\log n \Big)$$

$$= \sum_{I\in\mathcal{I}} \log\big(n \cdot p^{|I|}\big)$$

$$= \log\Big(n^{|\mathcal{I}|} \cdot p^{\sum_{I\in\mathcal{I}} |I|}\Big) = \underbrace{\log\Big(n^{v(H)-b-1} \cdot p^{e(H)-b}\Big)}$$

$$\Rightarrow H\big(Z, (Y_I : I\in\mathcal{I}), (X_i : i\in[b])\big) \geq \log\big(n^{b+1} p^{b}\big) + $$

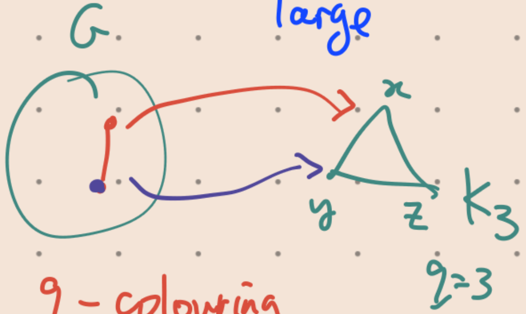
$$= \log\big(n^{v(H)} p^{e(H)}\big). \quad \blacksquare$$

§2.7 H-colourings

In Sidorenko's conj: $\underset{\text{(small)}}{\overset{\text{fixed}}{H}} \xrightarrow{\text{hom.}} \overset{\text{large}}{G}$

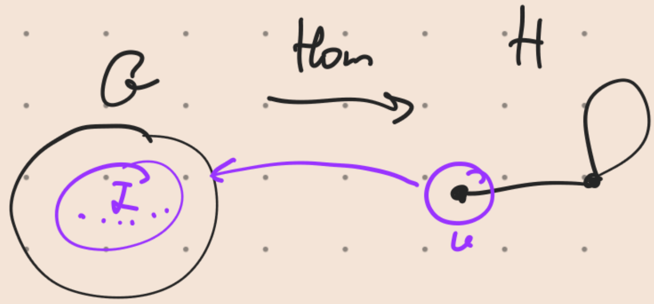
Now, "reverse" count: $\underset{\text{large}}{G} \xrightarrow{\text{hom}} \underset{\text{fixed}}{H}$

1) $H = K_q$



$\mathrm{hom}(G, K_q) = \#$ proper q-colouring of G

2) $H =$



$\forall$ hom $\in \mathrm{Hom}(G,$  $)$

pre-image of u $\longleftrightarrow$ indep sets in G

So $\mathrm{hom}(G,$  $) = \#$ indep sets in G.

- Galvin-Tetali: found maximiser for H-colourings among all bip. regular graphs G.

Thm 2.11 Let H be a graph w./ possibly loops but no multiple edges, then $\forall$ n-vertex d-reg bipartite G

$$\mathrm{hom}(G,H) \le \mathrm{hom}(K_{d,d}, H)^{n/2d}$$

Rmk n-vx d-reg 

$K_{d,d}$ $\cdots\cdots$ $K_{d,d}$

$\underbrace{\qquad\qquad}_{n/2d \text{ copies}}$

disj G_1 G_2 $\Rightarrow \hom(G_1 \dot\cup G_2, H) = \hom(G_1, H) \cdot \hom(G_2, H)$

- Applying Thm 2.11 to $H = \begin{cases} K_q & \Rightarrow \text{proper colouring} \\ \text{(graph below)} & \Rightarrow \text{indep sets} \end{cases}$

Cor 2.12 $\forall$ n-vx d-reg bip. G

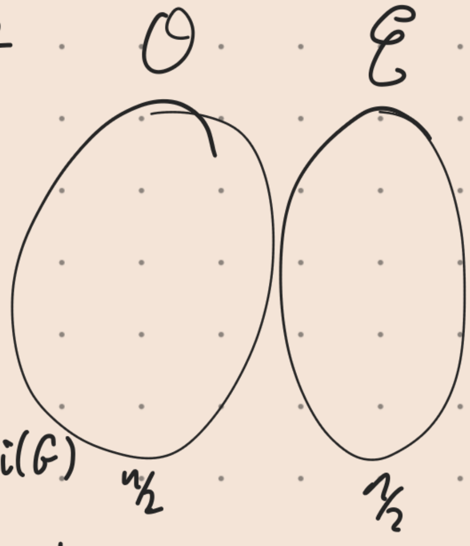
$$c_q(G) \le c_q(K_{d,d})^{n/2d}, \qquad c_q(\cdot) = \#\text{ proper } q\text{-col.}$$

Thm 2.13 $\forall$ n-vx d-reg bip. G

$$i(G) \le i(K_{d,d})^{n/2d}, \qquad i(\cdot) = \#\text{ indep sets}$$

Pf: • Let $V(G) = \mathcal{O} \cup \mathcal{E}$ be bipartition of G.
(odd) (even)

regular $\Rightarrow |\mathcal{O}| = |\mathcal{E}| = n/2$



- Write $\mathcal{I}(G)$ = set of all indep sets in G

$i(G) = |\mathcal{I}(G)|$

- $X \sim \mathcal{I}(G)$, $Y \sim \mathcal{I}(K_{d,d})$ $\overset{\text{max.}}{\Longrightarrow}$ $H(X) = \log i(G)$, $H(Y) = \log i(K_{d,d})$

Suffices to prove $H(X) \le \frac{n}{2d} \cdot H(Y)$

- Writing $X_v = \mathbb{1}_{\{v \in X\}}$, we can think of X as the random vector $X = (X_{\mathcal{O}}, X_{\mathcal{E}})$, where

$$X_{\mathcal{O}} = (X_v : v \in \mathcal{O}), \quad X_{\mathcal{E}} = (X_v : v \in \mathcal{E})$$

- Chain rule

$$\implies H(X) = H(X_{\mathcal{O}}, X_{\mathcal{E}})$$
$$= H(X_{\mathcal{E}}) + H(X_{\mathcal{O}} \mid X_{\mathcal{E}})$$