Lecture 4

- vol Hamming balls $\sum_{i \le pn} \binom{n}{i} \le 2^{h(p)n}$
- Δ maximisation
- Δ-intersecting family

§ 1.4 Axioms determines $H(X)$ uniquely

Lem 1.16 If $X \sim A$, then $H(X) = \log |A|$.

Lem 1.17 If $H(X)$ satisfies all six axioms

then $H(X) = \sum_{a \in A} p_a \log \frac{1}{p_a}$, where $p_a = \Pr(X = a)$.

Pf: • By continuity, may assume $\forall a \in A$,

$$p_a = \frac{m_a}{n}, \quad m_a \in \mathbb{N}$$

• **Construction 1.4.** $U \sim [n]$, X' same dist. as X.

$$X' = a \text{ iff } U \in V_a, \quad [n] = \bigcup_{a \in A} V_a, \quad |V_a| = m_a = p_a \cdot n$$

• $(U \mid X' = a) \sim V_a$

$U \sim [n]$

$$\overset{\text{Lem 1.16}}{\Longrightarrow} \quad H(U) = \log n$$

$$\& \quad H(U \mid X' = a) = \log(p_a \cdot n)$$

$$\cdot \Rightarrow \quad H(U \mid X') = \sum_{a \in A} \Pr(X' = a) \, H(U \mid X' = a)$$

$$= \sum_{a \in A} p_a \cdot \log(p_a \cdot n) \quad \log p_a + \log n$$

So $\quad H(U|X') = \sum_{a\in A} p_a \log p_a + \sum_{a\in A} p_a \cdot \log n$

add. //

$H(U,X') - H(X') \quad = \cancel{\log n} - \sum_{a\in A} p_a \log \frac{1}{p_a}$

// monotonicity

$\cancel{H(U)} - H(X)$

$\blacksquare$

§ 2.5 Brégman's thm.

Brégman's thm bounds the permanent of 0,1-matrices with given row sums

Def. The permanent of $n\times n$ matrix A is

$$\operatorname{perm}(A) = \sum_{\sigma\in S_n} \prod_{i\in[n]} A_{i\sigma(i)}$$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Rmk. compared to determinant, permanent is a lot harder to eff. compute.

- Permanent of 0,1-matrix as # perfect matchings

$\operatorname{perm}(A) = \sum_{\sigma} \prod A_{i\sigma(i)}$ $\quad G$ bipartite $U \cup V$ (row: U, column: V)

$$A: \quad \begin{array}{ccc} \text{row sum} & & \\ 2 & u_1 & \\ 2 & u_2 & \\ 2 & u_3 & \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

$\sigma = id$

$\sigma \in S_n$ contributing 1 to perm(A)

$\Leftrightarrow$ perfect matching

sum row = deg of u_i

Thm 2.7 (Brégman) Let A be $n \times n$ 0,1-matrix with row sums $d_1, \dots, d_n$, then

$$\operatorname{perm}(A) \leq \prod_{i \in [n]} (d_i!)^{\frac{1}{d_i}}$$

$$\prod_{i \in [n]} (n!)^{\frac{1}{n}} = n!$$

Rmk: Tight bound:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

also

$$\begin{pmatrix} \boxed{1} & & 0 \\ & \boxed{1} & \\ 0 & & \boxed{1} \end{pmatrix} \Leftarrow \text{disj union of } K_{d,d}.$$

In terms of P.M. in bipartite graphs:

Thm (Brégman) $n \times n$ bipartite graph w/ deg seq $(d_1, \dots, d_n)$ on one side,

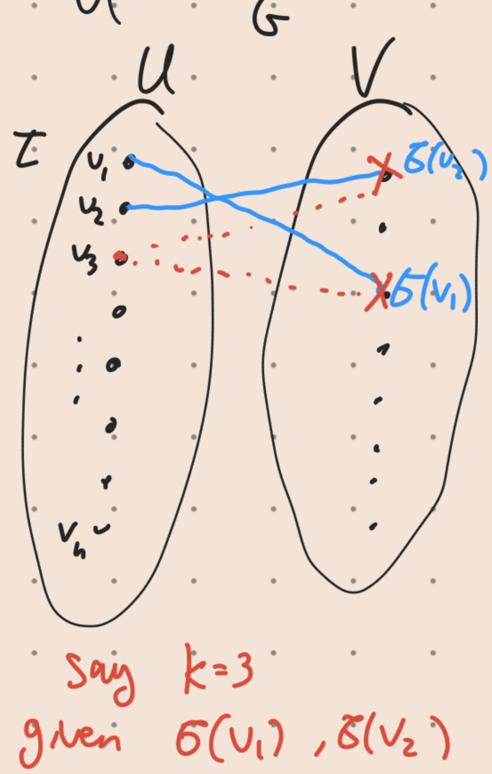
$$\Rightarrow \quad \# PM(G) \leq \prod_{i \in [n]} (d_i!)^{1/d_i}$$

Pf (Radhakrishnan) Let U, V be partite sets of G. $(d_1, \dots, d_n)$ deg seq for U.

Take $\sigma \sim$ PMs in G

$$\log |\# PM \text{ in } G| = H(\sigma) \overset{\text{NTS}}{\leq} \sum_{i \in [n]} \frac{\log(d_i!)}{d_i}$$

Fix an ordering $\tau : v_1, \dots, v_n$ of U



- think of σ as the random vector $(\sigma(v_1), \sigma(v_2), \dots, \sigma(v_n))$

Chain rule

$$\Rightarrow H(\sigma) = H(\sigma(v_1)) + H(\sigma(v_2) \mid \sigma(v_1)) + \dots + H(\sigma(v_n) \mid \sigma(v_1), \dots, \sigma(v_{n-1}))$$

- Fix v_k, consider

$$H(\sigma(v_k) \mid \underline{\sigma(v_1), \dots, \sigma(v_{k-1})})$$

After revealing $\sigma(v_1), \dots, \sigma(v_{k-1})$, $\sigma(v_k)$ has to take a neighbour that is not one of $\sigma(v_j), j<k$.

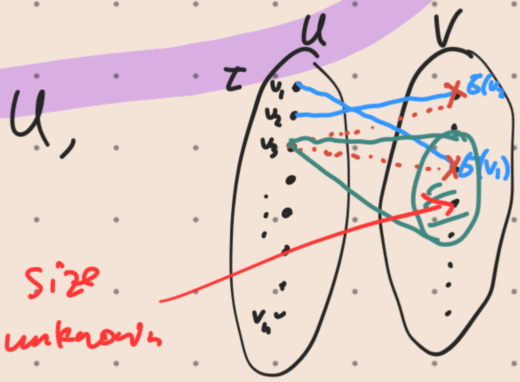
Denote $d_{\tau,k-1}(v_k) = \#$ such free neighbours.

- maximality

$$\Rightarrow H(\sigma(v_k) \mid \sigma(v_1), \dots, \sigma(v_{k-1})) \leq \mathbb{E}_{\sigma} \log(d_{\tau,k-1}(v_k))$$

$$\Rightarrow H(\sigma) \leq \mathbb{E}_{\sigma} \sum_{k \in [n]} \log(d_{\tau,k-1}(v_k))$$

Idea For a fixed ordering τ of U, it's extremely complicated to determine $d_{\tau,k-1}(v_k)$.



Rather take a unif random ordering τ of $\mathcal{U}$

$\Rightarrow$ we know how $d_{\tau,k-1}(v_k)$ behaves in expectation.

- Take now $\tau: v_1, \dots, v_n$ a unif random ordering of $\mathcal{U}$.
- Fix σ, see its contribution to

$$\sum_{k\in[n]} \log\left(d_{\tau,k-1}(v_k)\right)$$

As τ is unif, # used neighbours of v_k at time deciding $\sigma(v_k)$ $\sim \{0, 1, \dots, d(v_k)-1\}$

$$\Leftrightarrow \quad d_{\tau,k-1}(v_k) \sim [d(v_k)]$$

$\log(d(v)!)$

$$\Rightarrow \mathbb{E}_{\tau} \sum_{k\in[n]} \log\left(d_{\tau,k-1}(v_k)\right) = \sum_{v\in\mathcal{U}} \frac{1}{d(v)} \sum_{i\in[d(v)]} \log i$$

$$= \sum_{i\in[n]} \frac{\log(d_i!)}{d_i}$$

for any fixed σ.

$$\mathbb{E}_{\tau}\, \mathbb{E}_{\sigma} \log\left(d_{\tau,k-1}(v_k)\right) = \sum_{i\in[n]} \frac{\log(d_i!)}{d_i}$$

$\Rightarrow \exists$ a choice of τ s.t.

$$H(\sigma) \leq \mathbb{E}_{\sigma} \log\left(d_{\tau,k-1}(v_k)\right) \leq \sum_{i\in[n]} \frac{\log(d_i!)}{d_i} \qquad \blacksquare$$

§ 2.6 Sidorenko's conjecture.

This conj. relates subgraph density to edge density.

Def: A homomorphism from H to G:

$f : V(H) \longrightarrow V(G)$ preserving adjacencies

i.e. $\forall\, uv \in E(H) \Rightarrow f(u)f(v) \in E(G)$

- $\mathrm{Hom}(H,G)$ = set of all homomorphisms $H \to G$.
- $\hom(H,G) = |\mathrm{Hom}(H,G)|$

The homomorphism density of H in G

is $$t(H,G) = \frac{\hom(H,G)}{|V(G)|^{|V(H)|}}$$

$t(K_2, G)$ = edge density.

Conj (Sidorenko's conj) Let H be a bipartite graph, then $\forall\, G$,

$$\frac{\hom(H,G)}{|V(G)|^{|V(H)|}} = \underbrace{t(H,G)}_{}^{\text{H-density}} \geq t(K_2,G)^{e(H)} \quad \cdots\cdots \quad (*)$$

Equivalently .....

Say G has n vertices, write

$$p = t(K_2, G) = \frac{2e(G)}{n^2}$$

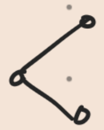
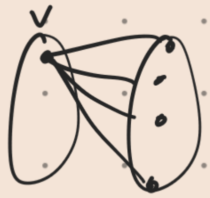
$$(*) \Longleftrightarrow \boxed{\hom(H,G) \geqslant n^{v(H)} \cdot \underline{p}^{e(H)}}$$

Rmk: The right-hand-side is $\mathbb{E}\left(\begin{array}{l}\#\text{ homomorphism from} \\ H \text{ to random graph}\end{array}\right)$

So Sidorenko's conj states

given the edge-density p, the random graph

minimises H-density.

- Two examples.
  1. $H =$  star of size 2
  2. $H =$ 

w/ a dominating vertex.

- § 2.6.1 Star of size 2 : $K_{1,2}$

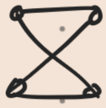
Sidorenko $(K_{1,2})$: $p = t(K_2, G) = \frac{2e(G)}{n^2}$

$$\boxed{\hom(K_{1,2}, G) \geqslant n^3 p^2} \quad \ldots\ldots (\heartsuit)$$

- We will prove $(\heartsuit)$ using entropy.

It can be easily extended to handled

$$\begin{cases} H: \text{ trees} \\ H: \text{ complete bipartite.} \end{cases}$$

<u>Ex</u> H: $K_{1,b}$, P_4, $K_{2,2} =$  $= C_4$.

<u>Pf of (♡)</u> $\hom(K_{1,2}, G) \geq n^3 p^2$

- Take a random homomorphism of $K_{1,2}$ (X, Y, Z) from $\mathrm{Hom}(K_{1,2}, G)$
  - Sample XY edge $\sim$ all (labeled) edges

(equivalently, choose a vertex X proportional to its degree $\Pr(X=v) = \dfrac{d(v)}{2e(G)}$)
& then unif choose Y in $N(X)$)

  - Sample $Z \sim N(X)$ unif random neighbour.

The pt is that Y, Z are independent conditioning on X.

$$\Rightarrow H(Y, Z \mid X) = H(Y \mid X) + H(Z \mid X)$$

- Both XY, XZ edges are $\sim$ all edges

maximality $\Rightarrow$ $H(X,Y) = H(X,Z) = \log(\hom(K_2, G))$

$$\hom(K_2, G) = 2e(G) = n^2 p, \quad p = t(K_2, G)$$

$$H(X,Y) = H(X,Z) = \log(n^2 p)$$

- $$\log(\hom(K_{1,2}, G)) \overset{\text{maximality}}{\geq} H(X,Y,Z)$$

$$\overset{\text{additivity}}{=} H(Y,Z \mid X) + H(X)$$

$$\overset{\text{cond. indep}}{=} H(Y \mid X) + H(Z \mid X) + H(X)$$

$$\overset{\text{add.}}{=} \underline{H(X,Y)} \underbrace{- H(X)}_{\substack{\leq \log n \\ \text{max.}}} + \underline{H(X,Z)} - \cancel{H(X)} + \cancel{H(X)}$$

$$\geq 2\log(n^2 p) - \log n$$

$$= \log(n^3 p^2) \qquad \square$$

Key idea: Y, Z cond. indep. given X.