ON THE RATIONAL TURÁN EXPONENTS CONJECTURE

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Abstract. The extremal number $\mathrm{ex}(n, F)$ of a graph F is the maximum number of edges in an n-vertex graph not containing F as a subgraph. A real number $r \in [1, 2]$ is realisable if there exists a graph F with $\mathrm{ex}(n, F) = \Theta(n^r)$. Several decades ago, Erdős and Simonovits conjectured that every rational number in $[1, 2]$ is realisable. Despite decades of effort, the only known realisable numbers are $0, 1, \frac{7}{5}, 2$, and the numbers of the form $1 + \frac{1}{m}$, $2 - \frac{1}{m}$, $2 - \frac{2}{m}$ for integers $m \geq 1$. In particular, it is not even known whether the set of all realisable numbers contains a single limit point other than two numbers 1 and 2.

In this paper, we make progress on the conjecture of Erdős and Simonovits. First, we show that $2 - \frac{a}{b}$ is realisable for any integers $a, b \geq 1$ with $b > a$ and $b \equiv \pm 1 \pmod a$. This includes all previously known ones, and gives infinitely many limit points $2 - \frac{1}{m}$ in the set of all realisable numbers as a consequence.

Secondly, we propose a conjecture on subdivisions of bipartite graphs. Apart from being interesting on its own, we show that, somewhat surprisingly, this subdivision conjecture in fact implies that every rational number between 1 and 2 is realisable.

1. Introduction

1.1. **History and previous results.** For a family of graphs $\mathcal{F}$, *the extremal number* $\mathrm{ex}(n, \mathcal{F})$ is the maximum number of edges in an n-vertex graph which does not contain any subgraph isomorphic to a graph in $\mathcal{F}$. If $\mathcal{F} = \{F\}$, then we write $\mathrm{ex}(n, F)$ instead of $\mathrm{ex}(n, \mathcal{F})$.

Since Mantel [21] determined the extremal number of a triangle in 1907, the study on the extremal number has been always at the core of extremal graph theory. The classical Erdős-Stone-Simonovits theorem [9, 11] showed that any k-chromatic graph F satisfies

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{k-1} + o(1)\right)\binom{n}{2}.$$

While this provides good estimates for the extremal numbers of non-bipartite graphs, it only shows $\mathrm{ex}(n, F) = o(n^2)$ for any bipartite graph F. Although there have been numerous attempts on finding better bounds of $\mathrm{ex}(n, F)$ for various bipartite graphs F, we know very little on the topic. One of the fundamental conjectures on the subject is the following conjecture proposed by Erdős and Simonovits.

Conjecture 1.1 (Erdős and Simonovits [7])**.** *For every rational number $r \in [1, 2]$, there exists a finite family $\mathcal{F}$ of graphs with $\mathrm{ex}(n, \mathcal{F}) = (c_{\mathcal{F}} + o(1))n^r$ for some real number $c_{\mathcal{F}} > 0$.*

Many authors (see [15, Conjecture 5.1] and [16, Conjecture 2.37]) stated a weaker version of Conjecture 1.1 that for every rational number $r \in [1, 2]$, there exists a finite family $\mathcal{F}$ of

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Date: September 26, 2018.

The first author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. NRF-2017R1A2B4005020) and also by TJ Park Science Fellowship of POSCO TJ Park Foundation. (D. Kang). The second author was supported by the Leverhulme Trust Early Career Fellowship ECF-2018-538 (J. Kim). The third author was supported by the Leverhulme Trust Early Career Fellowship ECF-2016-52 (H. Liu).

graphs with $\mathrm{ex}(n, \mathcal{F}) = \Theta(n^r)$. In a recent breakthrough, this has been verified by Bukh and Conlon [3] using the random algebraic construction introduced by Bukh [2].

Theorem 1.2 (Bukh and Conlon [3])**.** *For every rational number $r \in [1,2]$, there exists a finite collection $\mathcal{F}$ of graphs with $\mathrm{ex}(n, \mathcal{F}) = \Theta(n^r)$.*

Recently, Fitch [13] showed that for any integer $k \geq 2$ and rational number $1 \leq r \leq k$, there exists a finite family $\mathcal{F}_k$ of k-uniform hypergraphs with $\mathrm{ex}(n, \mathcal{F}_k) = \Theta(n^r)$, extending Theorem 1.2 to uniform hypergraphs.

As a strengthening of Conjecture 1.1, Erdős and Simonovits (see [8, Section 8]) also conjectured that for every rational number $r \in [1,2]$, there exists a graph F with $\mathrm{ex}(n, F) = (c_F + o(1))n^r$ for some real number $c_F > 0$. We state a slightly weaker version of their conjecture, which is our main interest.

Conjecture 1.3 (Rational exponents conjecture [8])**.** *For every rational number $r \in [1,2]$, there exists a graph F with $\mathrm{ex}(n, F) = \Theta(n^r)$.*

Let $r \in \mathbb{R}$ be *realisable* (by F) if there exists a graph F with $\mathrm{ex}(n, F) = \Theta(n^r)$. In contrast to the satisfying answer provided by Theorem 1.2, Conjecture 1.3 remains elusive. Until now, the only known realisable numbers are $0, 1, \frac{7}{5}, 2$, the numbers of the form $1 + \frac{1}{m}$, $2 - \frac{1}{m}$, $2 - \frac{2}{m}$ with $m \in \mathbb{N}$. Faudree and Simonovits [12], and Conlon [5] proved that the numbers $1 + \frac{1}{m}$ are realisable by Theta graphs $\theta_{m,\ell}$ for large ℓ, which are graphs consisting of ℓ internally vertex-disjoint paths of length m between two vertices (see [22, 4] for recent progress on the extremal number of Theta graphs). Kövari, Sós and Turán [20], and Alon, Rónyai and Szabó [1] proved that the numbers $2 - \frac{1}{m}$ are realisable by unbalanced complete bipartite graphs (see also [19, 2]). Recently, Jiang, Ma and Yepremyan [18] proved that $2 - \frac{2}{2m+1}$ is realisable by generalised cubes and $\frac{7}{5}$ is realisable by a so-called 3-comb-pasting graph. Note that it is not even known whether there is a single limit point on the set of realisable numbers in the interval $(1, 2)$.

1.2. **Our results.** One main contribution of this paper is the following theorem that provides infinitely many more realisable numbers, including all previously known realisable numbers.

Theorem 1.4. *For each $a, b \in \mathbb{N}$ with $a < b$ and $b \equiv \pm 1 \pmod{a}$, the number $2 - \frac{a}{b}$ is realisable.*

As a consequence, this theorem provides infinitely many limit points on the set of realisable numbers.

Corollary 1.5. *For each $m \in \mathbb{N}$, the number $2 - \frac{1}{m}$ is a limit point in the set of realisable real numbers.*

Secondly, we propose an approach to tackle Conjecture 1.3 via the following conjecture on subdivision of graphs. For a graph F, let $\mathrm{sub}(F)$ be the 1-*subdivision* of F, obtained from F by replacing all edges of F with pairwise internally disjoint paths of length two.

Conjecture 1.6 (Subdivision conjecture)**.** *Let F be a bipartite graph. If $\mathrm{ex}(n, F) = O(n^{1+\alpha})$ for some $\alpha > 0$, then*

$$\mathrm{ex}(n, \mathrm{sub}(F)) = O(n^{1+\frac{\alpha}{2}}).$$

Apart from being interesting on its own, somewhat surprisingly, we show that this seemingly unrelated conjecture implies Conjecture 1.3.

Theorem 1.7. *If Conjecture 1.6 holds, then for every rational number $r \in [1,2]$, there exists a graph F with $\mathrm{ex}(n, F) = \Theta(n^r)$.*

It is worth noticing that if one considers instead 1-subdivision of non-bipartite F in the Subdivision conjecture, then a stronger conclusion holds, as shown very recently by Conlon and Lee [6]. They proved that $\mathrm{ex}(n, \mathrm{sub}(K_t)) = O(n^{3/2-\delta})$ for some $\delta = \delta(t) > 0$. Nonetheless, the only known case for Conjecture 1.6 is when F is a Theta graph. Indeed, for any $m \geq 2$,

$$\mathrm{ex}(n, \theta_{m,\ell}) = O(n^{1+\frac{1}{m}}) \quad \text{and} \quad \mathrm{ex}(n, \mathrm{sub}(\theta_{m,\ell})) = \mathrm{ex}(n, \theta_{2m,\ell}) = O(n^{1+\frac{1}{2m}}).$$

We do not know whether Conjecture 1.6 is true for complete bipartite graphs. Conlon and Lee [6] proved that the extremal number of the 1-subdivision of $K_{s,t}$ is $O(n^{\frac{3}{2}-\frac{1}{12t}})$ when $t \geq s$. If Conjecture 1.6 is true, then this ought to be $O(n^{\frac{3}{2}-\frac{1}{2s}})$ for large t, where the exponent $\frac{3}{2} - \frac{1}{2s}$ only depends on the smaller number s rather than t. To suggest the conjecture is plausible, we provide a proof that $\mathrm{ex}(n, \mathrm{sub}(K_{s,t})) = O(n^{\frac{3}{2}-\frac{1}{4s-2}})$, see Theorem 5.3 in the concluding remark section. Independent of our work, Janzer [17] also proved the same bound for the 1-subdivision of complete bipartite graphs, and improved the upper bound of Conlon and Lee [6] for 1-subdivision of complete graphs.

1.3. Organisation of the paper.

The paper is organised as follows. In Section 2, we will define several graphs, discuss the concept of balanced rooted graphs and collect several lemmas. In Section 3, we will prove part of Theorem 1.4 that $2 - \frac{a}{b}$ is realisable by a certain graph when $b \equiv -1 \pmod{a}$. In Section 4, we will prove Theorem 1.7, and finish the proof of Theorem 1.4, i.e. $2 - \frac{a}{b}$ is realisable when $b \equiv 1 \pmod{a}$ by using a combination of the reduction theorem of Erdős and Simonovits [10] and the theorem of Bukh and Conlon [3]. Some concluding remarks are given in Section 5.

2. Preliminaries

2.1. Basic terminology and lemmas.

Let $\mathbb{N}$ be the set of natural numbers. For any $n \in \mathbb{N}$, denote $[n] := \{1, \ldots, n\}$. We only consider finite simple graphs in this paper. For a graph G and vertices $u, v \in V(G)$, we write $\mathrm{dist}(u, v)$ for the *distance* between u and v in G, i.e. the minimum number of edges in a path between u and v. For a set $A \subseteq V(G)$ and $i \in \mathbb{N} \cup \{0\}$, let

$$\mathrm{dist}(u, A) := \min_{v \in A}\{\mathrm{dist}(u, v)\} \quad \text{and} \quad \Gamma^i_G(A) := \{u \in V(G) \,:\, \mathrm{dist}(u, A) = i\}.$$

We denote *the external neighbourhood* of A to be $\Gamma_G(A) := \Gamma^1_G(A)$, *the common neighbourhood* of A to be $N_G(A) := \bigcap_{a \in A} \Gamma_G(a)$, and *the common degree* of A to be $d_G(A) := |N_G(A)|$. For vertex sets $A, B \subseteq V(G)$, we let $\Gamma_G(A, B) := \Gamma_G(A) \cap B$, $N_G(A, B) := N_G(A) \cap B$, $d_G(A, B) := |N_G(A, B)|$, $E(A, B) := \{\{a, b\} \,:\, a \in A, b \in B, ab \in E(G)\}$ and $e(A, B) := |E(A, B)|$. We also denote $E(A) := E(A, A)$ and $e(A) := e(A, A)$. For a set $A \subseteq V(G)$ and $s \in \mathbb{N}$, denote by $\binom{A}{s}$ all s-sets in A. We will omit the subscript G if it is clear from the context.

We claim a result holds for $x \gg y$ if there exists an increasing function $f : [1, \infty) \to [1, \infty)$ such that the claimed result holds for all $x, y \geq 1$ with $x \geq f(y)$. We will not explicitly compute this function. For convenience, we often omit the ceilings and floors and treats large number as integers if this does not affect the argument. We denote a star with k edges a *k-star* and the vertex of degree k its *centre*. If $k = 1$, then we choose any of two vertices to be the centre. The following lemma is an easy consequence of Hall's theorem.

Lemma 2.1. *Let $k \geq 1$ be an integer and G be a bipartite graph with a bipartition (A, B). If $k|S| \leq |\Gamma(S)|$ for any $S \subseteq A$, then G contains vertex-disjoint k-stars whose centres cover A.*

2.2. **Rooted blow-up of balanced bipartite graphs.** Bukh and Conlon [3] introduced the following concepts of rooted blow-ups and balanced rooted trees. Here, we slightly extend their definitions. Consider a tuple (F, R) of a graph F and a proper subset $R \subsetneq V(F)$ of vertices. We say that the tuple (F, R) is *rooted* on R and call R the set of *roots*. We simply write F instead of (F, R) if the roots are clear. For each non-empty set $S \subseteq V(F)$, let $\rho_F(S) := \frac{e_S}{|S|}$, where e_S is the number of edges in F incident with a vertex in S. Let $\rho(F) := \rho_F(V(F) \setminus R)$. Again, we omit the subscripts if it is clear. Note that $\rho(F)$ is well-defined as R is a proper subset of $V(F)$.

We say that (F, R) (or F if R is clear) is *balanced* if $\rho_F(S) \geq \rho(F)$ holds for any non-empty subset $S \subseteq V(F) \setminus R$. For $\ell \in \mathbb{N}$ and a bipartite graph F rooted on R, we let F^ℓ_R be the graph we obtain by taking disjoint union of copies of F and identifying the vertices corresponding to a vertex v into one vertex for each $v \in R$, see Figure 1. We omit the subscript R if it is clear from the context. If a graph F is rooted on some set R, we will treat its 1-subdivision $\mathrm{sub}(F)$ also as a rooted graph with the same set of roots R, see Figure 1.

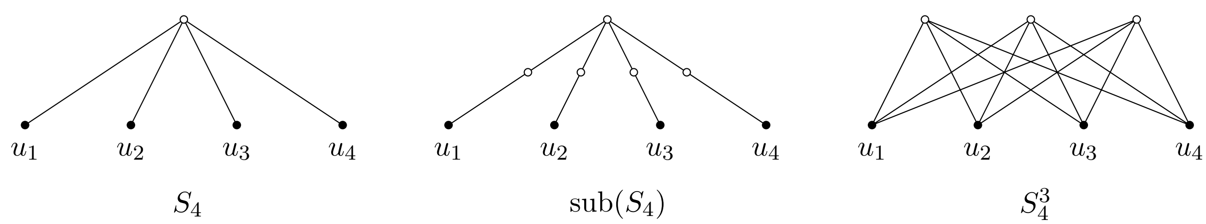

Figure 1: A 4-star S_4 rooted on its leaves, its 1-subdivision and its blow-up.

The following is a simple observation regarding balanced graphs. We omit its proof.

Observation 2.2. *Let (F, R) be a balanced graph F rooted on a non-empty set R. Then (F^ℓ_R, R) is balanced for all $\ell \in \mathbb{N}$. Moreover, if $F - E(R)$ is connected, then for any non-empty set $S \subseteq V(F) \setminus R$, we have $\rho_F(S) \geq 1$.*

For $a, b \in \mathbb{N}$ with $a - 1 \leq b \leq 2a - 2$, consider an a-vertex path with non-root vertices labelled $1, \ldots, a$ in order. Add $b - a + 1$ root leaves, each adjacent to the following vertices on the path, respectively:

$$1, \left\lfloor 1 + \frac{a}{b-a} \right\rfloor, \ldots, \left\lfloor 1 + (b-a-1)\frac{a}{b-a} \right\rfloor, a.$$

Denote the resulting rooted tree by $T_{a,b}$ and define recursively $T_{a,b}$ for $b \geq 2a - 1$ by adding one root leaf to each of the non-root vertices of $T_{a,b-a}$. It is proved in [3] that $T_{a,b}$ is a balanced tree with a non-root vertices and b edges.

Bukh and Conlon [3] proved the following result that provides the lower bound of the extremal number of balanced bipartite rooted graphs.

Lemma 2.3 (Bukh and Conlon [3])**.** *For every balanced bipartite rooted graph F with $\rho(F) > 0$, there exists a positive integer $\ell_0 = \ell_0(F)$ such that for all $\ell > \ell_0$, we have $\mathrm{ex}(n, F^\ell) = \Omega(n^{2 - \frac{1}{\rho(F)}})$.*

Indeed, they stated Lemma 2.3 only for balanced rooted trees F, but they did not use any assumption that F is a tree. They also use the assumption that the set of root vertices to be independent. Nevertheless, we can consider a subgraph F' of F by removing edges between root vertices in F, which results in balanced bipartite rooted graphs with $\rho(F) = \rho(F')$, and apply the lemma to F' to obtain the lower bound on $\mathrm{ex}(n, F^\ell)$.

Bukh and Conlon [3] also conjectured that for any balanced rooted tree T, there exists $\ell_0 = \ell_0(T)$ such that $\mathrm{ex}(n, T^\ell) = \Theta(n^{2-\frac{1}{\rho(T)}})$ for all $\ell \geq \ell_0$. As Lemma 2.3 shows that $\mathrm{ex}(n, T^\ell_{a,b}) = \Omega(n^{2-\frac{a}{b}})$ for large ℓ, their conjecture gives an approach to prove Conjecture 1.3. We remark that their conjecture cannot be generalised to balanced rooted bipartite graphs. Indeed, consider F obtained from $T_{3,5}$ by identifying two root vertices attached on a first non-root vertex and a third non-root vertex on the path. The resulting graph F contains C_4 as a subgraph, but $\mathrm{ex}(n, C_4) = \Theta(n^{3/2})$ while $\rho(F) = 5/3$.

For $s, t \in \mathbb{N}$, consider a t-star and attach s leaves to each one of $t + 1$ vertices of the t-stars. Let $D_{t,s}$ be the resulting tree rooted on its leaves. Note that $D_{1,s}$ and $D_{2,s}$ are isomorphic to $T_{2,2s+1}$ and $T_{3,3s+2}$, respectively. We call the centre of the original t-star the *centre* of $D_{t,s}$. The graph $D_{t,s}$ is a tree with $(t + 1)$ non-root vertices and $(s + 1)(t + 1) - 1$ edges, and $\rho(D_{t,s}) = \frac{(t+1)(s+1)-1}{t+1}$. Moreover, it is a balanced tree. Consider a blow-up $D^\ell_{t,s}$ of $D_{t,s}$. We call a vertex in $D^\ell_{t,s}$ a *centre vertex* if it is a centre of a copy of $D_{t,s}$ in $D^\ell_{t,s}$ and we call a vertex in $D^\ell_{t,s}$ a *core vertex* if it is a root vertex adjacent to all centre vertices. We call a set of s root vertices *a cluster* if they are all adjacent to a same vertex in $D^\ell_{t,s}$. Note that the root vertices of $D^\ell_{t,s}$ partition into $t + 1$ clusters, one of which contains all core vertices.

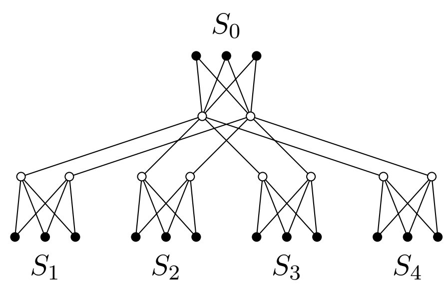

Figure 2: $D^2_{4,3}$ with core vertices S_0 and five clusters $S_0, \ldots, S_4$.

Proposition 2.4. *For any $s, t \in \mathbb{N}$, the rooted tree $D_{t,s}$ is balanced.*

Proof. Let R be the set of leaves of $D_{t,s}$ which is precisely the root set of $D_{t,s}$. Consider a non-empty set $S \subseteq V(D_{t,s}) \setminus R$. We have $1 \leq |S| \leq t+1$. If S contains the centre of $D_{t,s}$, then

$$\rho(S) = \frac{s|S| + t}{|S|} \geq \frac{s(t + 1) + t}{t + 1} = \rho(D_{t,s}).$$

If S does not contain the centre, then

$$\rho(S) = \frac{(s + 1)|S|}{|S|} = s + 1 \geq \rho(D_{t,s}).$$

Therefore, $D_{t,s}$ is balanced. □

2.3. **Dependent random choice and embedding $D^\ell_{t,s}$.** The following variation of dependent random choice (Lemma 2.5) together with the embedding lemma (Lemma 2.6) will be useful for estimating $\mathrm{ex}(n, D^\ell_{t,s})$. For an excellent survey of dependent random choice, see [14].

Lemma 2.5. *Let $d, t \in \mathbb{N}$ and G be a bipartite graph with a vertex partition (A, B). If each vertex in A has degree at least $d \geq \frac{2t|A|^{s-1}}{s!}$, then there exist a vertex $u \in B$ and a subset $A' \subseteq \Gamma_G(u, A)$ of size at least $\frac{d|A|}{2|B|}$ satisfying $d_G(S) \geq t$ for every $S \in \binom{A'}{s}$.*

Proof. Choose a vertex $u \in B$ uniformly at random, and consider a set $X := \Gamma_G(u) \subseteq A$. For each $v \in A$, the probability that $v \in X$ is $\mathbb{P}(v \in X) = \frac{d_G(v)}{|B|} \geq \frac{d}{|B|}$. Hence we obtain $\mathbb{E}[|X|] \geq \frac{d|A|}{|B|}$.

We say a set $S \in \binom{A}{s}$ of size s is *bad* if $d_G(S) < t$. Let Y be the random variable indicating the number of bad sets in $\binom{X}{S}$. As $\mathbb{P}(S \subseteq X) = \frac{d_G(S)}{|B|} < \frac{t}{|B|}$, we have

$$\mathbb{E}[Y] \leq \frac{t}{|B|} \cdot \left| \binom{X}{s} \right| \leq \frac{t|A|^s}{s!|B|}.$$

Let X' be the set obtained from X by deleting one element from each bad set $S \in \binom{X}{s}$, then $|X'| \geq |X| - Y$, and

$$\mathbb{E}[|X'|] \geq \mathbb{E}[|X|] - \mathbb{E}[Y] \geq \frac{d|A|}{|B|} - \frac{t|A|^s}{s!|B|} \geq \frac{d|A|}{2|B|}.$$

This implies that there exists a choice $A' \subseteq \Gamma_G(u)$ with $|A'| \geq \frac{d|A|}{2|B|}$ and $d_G(S) \geq t$ for each $S \in \binom{A'}{s}$, as desired. □

Lemma 2.6. *Let $s, t, \ell \in \mathbb{N}$ and G be a graph. Let $W, A \subseteq V(G)$ be sets satisfying $|W| = s$ and $A = N_G(W)$. If $|A| \geq st + \ell$ and each $S \in \binom{A}{s+1}$ satisfies $|N_G(S) \setminus (A \cup W)| \geq \ell t$, then G contains $D^\ell_{t,s}$ as a subgraph.*

Proof. Recall that $D^\ell_{t,s}$ is obtained from the disjoint unions of $D_{t,s}$ by identifying corresponding leaves which are root vertices. Map all s core vertices into W. Further, we injectively map the remaining st non-core root vertices and the ℓ centre vertices into A. This is possible as we have $|W| \geq s$ and $|A| \geq st + \ell$ with $W \cap A = \emptyset$. Let ψ be the injective function we have defined, which embeds all but ℓt vertices of $D^\ell_{t,s}$ into $W \cup A$.

Each vertex $v \in D^\ell_{t,s}$, with $\psi(v)$ not yet defined, is adjacent to s root vertices and one centre vertex in $D^\ell_{t,s}$. As these $s+1$ neighbours of v are injectively embedded in A, the set S_v of their ψ-images is in $\binom{A}{s+1}$. Hence we have $|S'_v| \geq \ell t$ where $S'_v := N_G(S_v) \setminus (A \cup W)$. As there are ℓt vertices v for which $\psi(v)$ is not yet defined and $|S'_v| \geq \ell t$ holds for all such vertices v, we can choose $\psi(v) \in S'_v$ for all these vertices so that ψ is still injective. By the construction of ψ, it is easy to see that $\psi(D^\ell_{t,s}) \subseteq G$. Hence G contains $D^\ell_{t,s}$ as a subgraph. □

3. The extremal number of $D^\ell_{t,s}$.

In this section, we prove the following theorem. Here, we write $D^\ell_{t-1,s-1}$ instead of $D^\ell_{t,s}$ only to make the formulas simpler.

Theorem 3.1. *Let $s, t \in \mathbb{N} \setminus \{1\}$. Then there exists ℓ_0 such that for all $\ell \geq \ell_0$, we have* $\mathrm{ex}(n, D^\ell_{t-1,s-1}) = \Theta(n^{2-\frac{t}{st-1}})$.

As $D_{t-1,s-1}$ is balanced rooted graphs due to Observation 2.2 and Proposition 2.4, the following Lemma 3.2 together with Lemma 2.3 implies Theorem 3.1.

Lemma 3.2. *For all $\ell, s, t \in \mathbb{N} \setminus \{1\}$, we have* $\mathrm{ex}(n, D^\ell_{t-1,s-1}) = O(n^{2-\frac{t}{st-1}})$.

Proof. As proved in [10], any m-vertex graph with average degree m^α contains an $\widetilde{m}$-vertex graph with minimum degree at least $\frac{1}{10\cdot 2^{\alpha^{-2}}}\widetilde{m}^\alpha$ with $\widetilde{m} \geq m^{\frac{\alpha(1-\alpha)}{1+\alpha}}$. Also any graph contains a spanning bipartite subgraph with the minimum degree at least the half of the original graph. Hence, it suffices to prove that for given ℓ, t and s there exist $Q, n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, any n-vertex bipartite graph with minimum degree at least $Qn^{1-\frac{t}{st-1}}$ contains $D^\ell_{t-1,s-1}$ as a subgraph.

Choose $n, q \in \mathbb{N}$ with $n \gg q \gg \ell, s, t$ and let $d := n^{1-\frac{t}{st-1}}$. Then we have

$$\frac{d^{s-1}}{n^{s-2}} = n^{\frac{t-1}{st-1}} \text{ and } \frac{d^s}{n^{s-1}} = n^{-\frac{1}{st-1}}. \tag{3.1}$$

To derive a contradiction, assume that G is an n-vertex bipartite graph satisfying $\delta(G) \geq 4qd$ that does not contain $D^\ell_{t-1,s-1}$ as a subgraph. Let $V := V(G)$.

Recall that $D^\ell_{t-1,s-1}$ consists of $s-1$ core vertices, ℓ centre vertices, $(s-1)(t-1)$ non-core root vertices, and remaining $\ell(t-1)$ vertices that are neither roots nor centre. Also recall that the root vertices of $D^\ell_{t-1,s-1}$ partition into t clusters each of which contains $s-1$ vertices.

Proof strategy. We first choose pairwise disjoint vertex sets L_0, L_1, L_2 and L_3 with $L_{i+1} \subseteq \Gamma_G(L_i)$ and $|L_{i+1}|$ is sufficiently larger than $|L_i|$. We aim to embed the core vertices into L_0, the centre vertices into L_1, non-root neighbours of centre vertices into L_2 and the non-core root vertices to L_3. We let $S_1, \dots, S_{t-1}$ be the non-core clusters.

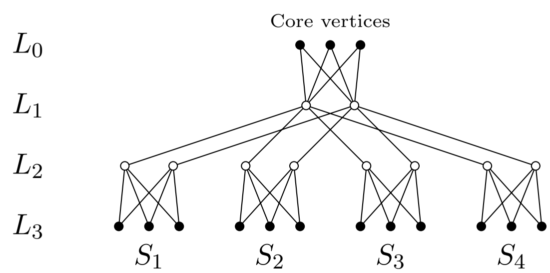

Figure 3: An embedding of $D^2_{4,3}$ with respect to the levels L_0, L_1, L_2 and L_3.

We will embed S_1 into $C_1 \subseteq L_3$ in a nice manner that we can find $A_1 \subseteq L_1$ and $B_1 \subseteq L_2$ such that A_1 is a set of candidates for the images of centre vertices and B_1 is a set of candidates for the images of neighbours of S_1. By repeatedly embedding $S_1, \dots, S_i$ into $C_1, \dots, C_i \subseteq L_3$ in an injective manner, we will find candidate sets $A_1 \supseteq \cdots \supseteq A_i$ for the images of the centre vertices, and the pairwise disjoint candidate sets $B_1, \dots, B_i \subseteq L_2$ for the neighbours of $S_1, \dots, S_i$. After embedding all $t-1$ clusters, if $|A_{t-1}| \geq \ell$ then this will give us a copy of $D^\ell_{t-1,s-1}$.

Stage 1. We first choose a set L_0 of $s-1$ vertices which will be the images of the $s-1$ core vertices of $D^\ell_{t-1,s-1}$, and a set L_1 of vertices which are candidates for the images of the centre vertices of $D^\ell_{t-1,s-1}$. As $\delta(G) \geq 4qd$, we have

$$\sum_{L_0 \in \binom{V}{s-1}} |N_G(L_0)| = \sum_{v \in V} \binom{d_G(v)}{s-1} \geq n \cdot \binom{4qd}{s-1}.$$

Hence, by averaging, there exists a vertex set $L_0 \in \binom{V}{s-1}$ with $d_G(L_0) \geq \binom{n}{s-1}^{-1} n \cdot \binom{4qd}{s-1} \geq \frac{qd^{s-1}}{n^{s-2}} \overset{(3.1)}{\geq} qn^{\frac{t-1}{st-1}}$. As $q \gg \ell$, we can choose a set $L_1 \subseteq N_G(L_0)$ with $|L_1| = \ell n^{\frac{t-1}{st-1}}$.

Claim 3.3. *There exists a collection* $\{\Gamma_u \subseteq N_G(u) \setminus (L_0 \cup L_1) : u \in L_1\}$ *of pairwise disjoint vertex sets with* $|\Gamma_u| = 2d$.

Proof. Note that $N_G(u) \cap L_1 = \emptyset$ for each $u \in L_1$ as G is bipartite. We first show that L_1 expands: for each $A \subseteq L_1$, we have

$$|\Gamma_G(A) \setminus L_0| \geq 2d|A|. \tag{3.2}$$

Suppose that $A \subseteq L_1$ satisfies $|B| < 2d|A|$ where $B := \Gamma_G(A) \setminus L_0$. Let H be a bipartite graph with vertex partition (A, B). As $q \gg \ell, s, t \geq 2$, for any $v \in A$, we have

$$d_H(v, B) \geq \delta(G) - |L_0| \geq 4qn^{\frac{(s-1)t-1}{st-1}} - s + 1 \geq qn^{\frac{(t-1)(s-1)}{st-1}} \geq \frac{2|L_1|^{s-1}\ell t}{s!} \geq \frac{2|A|^{s-1}\ell t}{s!}.$$

Hence, we can apply Lemma 2.5 to the bipartite graph H with $4qd - s + 1$ and ℓt playing the roles of d and t, respectively to obtain $A' \subseteq A$ with

$$|A'| \geq \frac{|A|(4qd - s + 1)}{2|B|} \geq \frac{|A|(4qd - s + 1)}{2 \cdot 2d|A|} \geq \ell + st,$$

such that any $S \in \binom{A'}{s}$ satisfies $d_G(S) \geq \ell t$. We can then apply Lemma 2.6 to G with A', L_0 and $s - 1$ playing the role of A, W and s, respectively to show that G contains $D^{\ell}_{t-1,s-1}$ as a subgraph, a contradiction. Hence, (3.2) holds. Thus Lemma 2.1 implies the existence of desired collection. This proves the claim. □

Let $L_2 := \bigcup_{u \in L_1} \Gamma_u$ and $L_3 := \Gamma_G(L_2) \setminus L_1$. As G is bipartite, L_0, L_1, L_2, L_3 are pairwise disjoint vertex sets and the vertices in L_2 has no edges to L_0. Note that since $|L_1| = \ell n^{\frac{t-1}{st-1}} \leq qd$, each vertex $v \in L_2$ satisfies

$$d_G(v, L_3) \geq 4qd - (qd + s - 1) \geq 2qd. \tag{3.3}$$

Stage 2. Let $S_1, \ldots, S_{t-1}$ be the sets of non-core clusters of $D^{\ell}_{t-1,s-1}$. We will embed these sets into sets $C_1, \ldots, C_{t-1}$ in L_3. The following claim is useful for choosing the set C_i so that we obtain candidate sets A_i and B_i of the correct sizes once we embedded S_i into C_i.

Claim 3.4. *Let* $A^{\#} \subseteq L_1$, $B^* \subseteq L_2$ *and* $C^* \subseteq L_3$. *Suppose that* $|C^*| \leq (s-1)(t-1)$ *and for each* $u \in A^{\#}$, *we have* $|\Gamma_u \cap B^*| \leq t - 1$. *Then there exist sets* $A \subseteq A^{\#}$, $B \subseteq L_2 \setminus B^*$, $C \subseteq L_3 \setminus C^*$ *and a bijective function* $f : A \to B$ *satisfying the following.*

(a) $|A| = |B| \geq n^{-\frac{1}{st-1}}|A^{\#}|$ *and* $|C| = s - 1$.
(b) $B \subseteq N_G(C)$.
(c) $f(a) \in \Gamma_a$ *for all* $a \in A$.

Proof. For each $u \in A^{\#}$, we consider the collection of $(s-1)$-tuples

$$\mathcal{C}_u := \{S \subseteq L_3 \setminus C^* : |S| = s - 1 \text{ and } N_G(S) \cap (\Gamma_u \setminus B^*) \neq \emptyset\}.$$

We claim that for each $u \in A^{\#}$, we have

$$|\mathcal{C}_u| \geq d^{s-1}|\Gamma_u \setminus B^*|. \tag{3.4}$$

Suppose $u \in A^{\#}$ and $|\mathcal{C}_u| < d^{s-1}|\Gamma_u \setminus B^*|$. Let $X := \Gamma_u \setminus B^*$. Let H_u be an auxiliary bipartite graph with a vertex partition $(X, \mathcal{C}_u)$ and

$$E(H_u) = \{wS \in X \times \mathcal{C}_u : w \in N_G(S)\}.$$

For each $w \in X$, by (3.3), we have $d_G(w, L_3 \setminus C^*) = 2qd - |C^*| \geq 2qd - st \geq qd$. Since $|X| \leq |\Gamma_u| \leq 2d$, we have

$$d_{H_u}(w) \geq \binom{qd}{s-1} \geq \frac{qd^{s-1}}{s^s} \geq \frac{2|X|^{s-1}(\ell t)^s}{s!}.$$

Hence, we can apply Lemma 2.5 to H_u with X, $\mathcal{C}_u$ $\frac{qd^{s-1}}{s^s}$ and $(\ell t)^s$ playing the roles of A, B, d and t, respectively. Then we obtain $X' \subseteq \Gamma_{H_u}(S, X) \subseteq X$, where $S \in \mathcal{C}_u$ and

$$|X'| \geq \frac{qd^{s-1}|X|}{2s^s|\mathcal{C}_u|} \geq \frac{qd^{s-1}|X|}{2s^s d^{s-1}|\Gamma_u \setminus B^*|} \geq (s-1)(t-1) + \ell$$

such that the following holds.

(3.5) *For any $U \in \binom{X'}{s}$, we have $d_{H_u}(U) \geq (\ell t)^s$.*

Note that an $(s-1)$-set $S' \in N_{H_u}(U)$ if and only if all vertices $z \in S'$ are in $N_G(U)$. Thus,

$$d_{H_u}(U) = \binom{|N_G(U, L_3 \setminus C^*)|}{s-1} \geq (\ell t)^s,$$

implying that $d_G(U, L_3 \setminus C^*) \geq \ell t$. Hence, we can apply Lemma 2.6 to G with $X', S, s-1$ and $t-1$ playing the roles of A, W, s and t to obtain a copy of $D^{\ell}_{t-1,s-1}$ in G, a contradiction. So (3.4) holds.

Now we aim to choose an appropriate $(s-1)$-set $C \subseteq L_3 \setminus C^*$. Let H be an auxiliary bipartite graph with a vertex partition $(A^{\#}, \binom{L_3 \setminus C^*}{s-1})$ and

$$E(H) := \left\{ uS \in A^{\#} \times \binom{L_3 \setminus C^*}{s-1} : N_G(S) \cap (\Gamma_u \setminus B^*) \neq \emptyset \right\}.$$

In other words, $uS \in E(H)$ if $S \in \mathcal{C}_u$. Claim 3.3 and (3.4) imply that

$$e(H) \geq \sum_{u \in A^{\#}} d^{s-1}|\Gamma_u \setminus B^*| \geq |A^{\#}| \cdot d^{s-1} \cdot (2d - t) \geq |A^{\#}| \cdot d^s.$$

Hence, by average, there exists a set $C \in \binom{|L_3 \setminus C^*|}{s-1}$ with

$$d_H(C) \geq \binom{n}{s-1}^{-1} e(H) \geq \binom{n}{s-1}^{-1} |A^{\#}|d^s \geq \frac{d^s}{n^{s-1}}|A^{\#}| \overset{(3.1)}{=} n^{-\frac{1}{st-1}}|A^{\#}|.$$

Let $A := N_H(C)$. By the definition of H, for each $a \in A$, there exists a vertex $f(a) \in \Gamma_u \setminus B^*$ such that $f(a) \in N_G(C)$. Let $B := f(A)$. As $\Gamma_u \cap \Gamma_{u'} = \emptyset$ for distinct $u, u' \in A$, such a function f is bijective between A and B. It is easy to see that A, B, C and f satisfy properties (a)–(c). $\square$

Let $A_0 := L_1$. For each $i = 1, 2, \ldots, t-1$ in order, we apply Claim 3.4 with $A_{i-1}, \bigcup_{j=1}^{i-1} B_j$ and $\bigcup_{j=1}^{i-1} C_j$ playing the roles of $A^{\#}, B^*$ and C^*, respectively to obtain sets A_i, B_i, C_i and f_i. This repetition is possible as the properties (a) and (c) ensures that $\bigcup_{j=1}^{i-1} B_j = \bigcup_{j=1}^{i-1} f_j(A_j)$ contains at most $i-1$ vertices in Γ_u for each $u \in A_{i-1}$, as well as $|\bigcup_{j=1}^{i-1} C_j| \leq (t-1)(s-1)$ holds.

Then we obtain sets $A_1 \supseteq \cdots \supseteq A_{t-1}$ and pairwise disjoint sets $B_1, \ldots, B_{t-1}, C_1, \ldots, C_{t-1}$ and bijective functions $f_1, \ldots, f_{t-1}$ with $f_i : A_i \to B_i$. Furthermore, for all $i \in [t-1]$ and $a \in A_i$, we have $|A_i| \geq n^{-\frac{i}{st-1}}|A_0| \geq \ell n^{\frac{t-1-i}{st-1}}$ and $f_i(a) \in B_i \subseteq N_G(C_i)$. Moreover, as $A_1 \supseteq \cdots \supseteq A_{t-1}$, for each $i \in [t-1]$, the function f_i is defined on each of the sets $A_{i+1}, \ldots, A_{t-1}$.

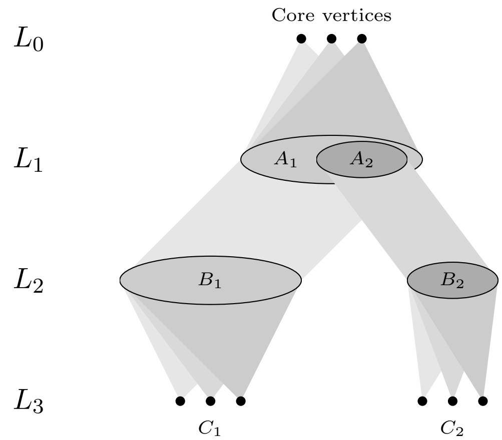

Figure 4: Embedding process of $D^{\ell}_{2,3}$ using Claim 3.4

As $|A_{t-1}| \geq \ell$, we can choose a set A of ℓ vertices in A_{t-1}. Note that for each $i \in [t-1]$, the bipartite graph $G[A, f_i(A)]$ contains a perfect matching as we have $f_i(a) \in \Gamma_a$, and $G[C_i, f_i(A)]$ induces a complete bipartite graph $K_{s-1,\ell}$ as $f_i(A) \subseteq B_i \subseteq N_G(C_i)$. Since the sets $f_1(A) \subseteq B_1, \dots, f_{t-1}(A) \subseteq B_{t-1}$ are pairwise disjoint, the sets $L_0, A, f_1(A), \dots, f_{t-1}(A), C_1, \dots, C_{t-1}$ form a copy of $D^{\ell}_{t-1,s-1}$. More precisely, we can embed a copy of $D^{\ell}_{t-1,s-1}$ in such a way that the core vertices embed into L_0, centre vertices embed into A and non-core root vertices embed into $C_1, \dots, C_{t-1}$. This proves the Lemma. □

4. Reduction theorems

In this section, we will prove that in a certain class of bipartite graphs, the extremal number of a graph can be deduced from the extremal number of another simpler graph.

4.1. **Densification.** For $t \in \mathbb{N}$ and a connected bipartite graph F, let (A, B) be its unique bipartition. We consider two disjoint set R'_1 and R'_2 of t vertices disjoint from $V(F)$; and make the vertices of R'_2 adjacent to all vertices in A and the vertices in R'_1 adjacent to all vertices in B; and add all possible edges between R'_1 and R'_2. Let $F(t)$ denote the resulting graph. If F is a connected bipartite graph rooted on R, then we consider $F(t)$ as rooted on $R \cup R'_1 \cup R'_2$ and let $F_*(t)$ denote the rooted graph we obtain from $F(t)$ by deleting all edges inside $R \cup R'_1 \cup R'_2$, see Figure 5.

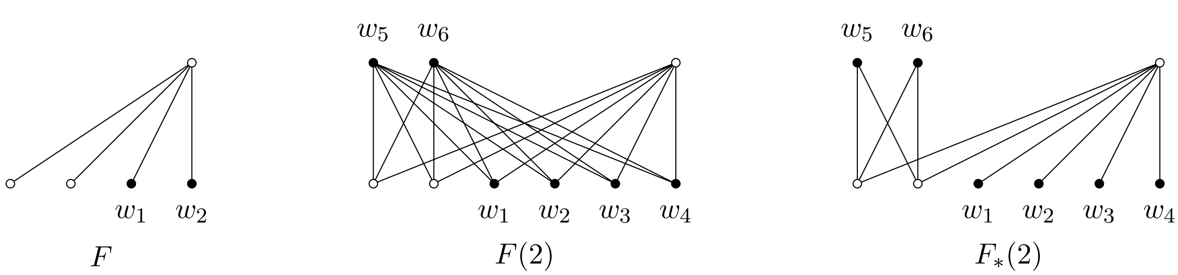

Figure 5: A connected bipartite graph F with root set $R := \{w_1, w_2\}$; and connected bipartite graphs $F(2), F_*(2)$ with root sets $R \cup R'_1 \cup R'_2$, where $R'_1 := \{w_3, w_4\}$ and $R'_2 := \{w_5, w_6\}$.

The following reduction theorem by Erdős and Simonovits relates the extremal number of bipartite graphs F and $F(t)$.

Theorem 4.1 (Erdős and Simonovits [10]). *Let $t \in \mathbb{N}$ and F be a connected bipartite graph with $\mathrm{ex}(n, F) = O(n^{2-\alpha})$. Then $\mathrm{ex}(n, F(t)) = O(n^{2-\beta})$ where $\beta^{-1} = \alpha^{-1} + t$.*

Another important tools we use is Lemma 2.3 by Bukh and Conlon. To be able to use Theorem 4.1 and Lemma 2.3 in the same framework, we need the following proposition.

Proposition 4.2. *Let $t \in \mathbb{N}$ and F be a balanced rooted bipartite graph. Then both $F(t)$ and $F_*(t)$ are balanced rooted bipartite graph.*

Proof. As the edges between roots do not affect the definition of balancedness, it suffices to prove it for $F(t)$. As every non-root vertices are adjacent to t more vertices in $F(t)$ than F, it is easy to check that for any non-empty set $S \subseteq V(F) \setminus R$ we have $\rho_{F(t)}(S) = t + \rho_F(S)$. Hence, for every non-empty set S of non-root vertices of $F(t)$, we have $\rho(F(t)) = \rho(F) + t \leq \rho_F(S) + t = \rho_{F(t)}(S)$ Hence, $F(t)$ and $F_*(t)$ are balanced. □

We say that a number $r \in [1, 2)$ is *balancedly realisable by a graph F* if there exist a balanced connected rooted bipartite graph F and a positive integer ℓ_0 satisfying $\rho(F) = \frac{1}{2-r}$ and for every $\ell \geq \ell_0$, we have $\mathrm{ex}(n, F^\ell) = \Theta(n^r)$. By combining Theorem 4.1 and Lemma 2.3, we can prove the following lemma.

Lemma 4.3. *For $a, b \in \mathbb{N}$ with $b > a$, if $2 - \frac{a}{b}$ is balancedly realisable, then $2 - \frac{a}{a+b}$ is also balancedly realisable.*

Proof. By the assumption, there exist a balanced connected rooted bipartite graph F and ℓ_0 such that $\mathrm{ex}(n, F^\ell) = \Theta(n^{2-\frac{a}{b}})$ for any $\ell \geq \ell_0$ and $\rho(F) = \frac{b}{a}$.

By Proposition 4.2, $F_*(1)$ is also a balanced connected rooted bipartite graph with $\rho(F_*(1)) = \rho(F) + 1 = \frac{a+b}{a}$. Hence, Lemma 2.3 implies that there exists $\ell_0' \in \mathbb{N}$ such that for all $\ell \geq \ell_0'$, we have

$$\mathrm{ex}(n, F_*(1)^\ell) = \Omega(n^{2-\frac{1}{\rho(F_*(1))}}) = \Omega(n^{2-\frac{a}{a+b}}).$$

On the other hand, as $F(1)^\ell = F^\ell(1)$ and $(\frac{a}{a+b})^{-1} = (\frac{a}{b})^{-1} + 1$, Theorem 4.1 with the definition of ℓ_0 implies that for all $\ell \geq \ell_0$,

$$\mathrm{ex}(n, F(1)^\ell) = \mathrm{ex}(n, F^\ell(1)) = O(n^{2-\frac{a}{a+b}}).$$

As $F_*(1)$ is a subgraph of $F(1)$, we have $\mathrm{ex}(n, F_*(1)^\ell) \leq \mathrm{ex}(n, F(1)^\ell)$. Thus, for any $\ell \geq \max\{\ell_0, \ell_0'\}$, we have

$$\mathrm{ex}(n, F_*(1)^\ell) \leq \mathrm{ex}(n, F(1)^\ell) = \Theta(n^{2-\frac{a}{a+b}}).$$

Therefore, $2 - \frac{a}{a+b}$ is balancedly realisable. □

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $a \in \mathbb{N}$, it is known that the number $2 - \frac{a}{a+1}$ is realisable by any large Theta graphs [12, 5], which is a blow-up of path rooted on the two end points. Hence $2 - \frac{a}{a+1}$ is balancedly realisable. This with Lemma 4.3 implies that $2 - \frac{a}{b}$ is realisable if $b > a$ and $b \equiv 1 \pmod{a}$. By Theorem 3.1, it also follows that $2 - \frac{a}{b}$ is realisable if $b > a$ and $b \equiv -1 \pmod{a}$, completing the proof. □

4.2. **Subdivision conjecture.** To see the motivation behind Conjecture 1.6, suppose that $\mathrm{ex}(n,F) = O(n^{1+\alpha})$. Suppose that we have an n-vertex bipartite graph G having no $\mathrm{sub}(F)$ as a subgraph with $e(G) = Cn^{1+\alpha/2}$. Consider an auxiliary graph G^* with $V(G^*) = V(G)$ and $uv \in E(G^*)$ if and only if there exists a path of length two between u and v. By using a dependent random choice to the bipartite graph $G[\Gamma_G(v), \Gamma^2_G(v)]$ for each $v \in V(G)$, it is easy to see that G^* contains at least $\Omega(n^{1+\alpha})$ edges, hence contains a copy of F. Note that this copy of F will correspond to a (possibly degenerate) copy of $\mathrm{sub}(F)$ in G. Indeed, as G^* contains many cliques of size $\Omega(n^{\alpha/2})$, namely $N_G(w)$ for each $w \in V(G)$, there is no guarantee that the copies of F is non-degenerate. However, it is plausible that a non-degenerate copy of $\mathrm{sub}(F)$ exists if C is sufficiently large.

Conjecture 1.3 and Conjecture 1.6 seem unrelated. However, much to our surprise, Conjecture 1.6 implies Conjecture 1.3. The rest of this section is devoted to show how two conjectures are connected.

Proposition 4.4. *Given a balanced bipartite graph F rooted on an independent set R with $\rho(F) \geq 1$, the 1-subdivision* $\mathrm{sub}(F)$ *is also balanced rooted bipartite graph.*

Proof. Let R be the set of root vertices of F. Let $b := |E(F)|$ and $a := |V(F) \setminus R|$. As R is an independent set, we have $b \geq a$ as $\rho(F) = \frac{b}{a} \geq 1$. For $S \subseteq V(F)$, let e_S be the number of edges incident with a vertex in S in the graph F and let $e(S)$ be the number of edges whose both endpoints lie in S in the graph F.

As F is balanced, for each non-empty set $S \subseteq V(F) \setminus R$, we have

$$\rho_F(S) = \frac{e_S}{|S|} \geq \frac{b}{a} = \rho(F) \geq 1. \tag{4.1}$$

Let $S \subseteq V(\mathrm{sub}(F)) \setminus R$. We aim to show $\rho_{\mathrm{sub}(F)}(S) \geq \frac{2b}{a+b} = \rho(\mathrm{sub}(F))$. For each $i \in \{0,1,2\}$, let

$$S^* := S \cap V(F) \text{ and } S_i := \{v \in S \setminus S^* : |N_{\mathrm{sub}(F)}(v) \cap S^*| = i\}.$$

From these definitions, it is easy to see that the number of edges incident to S in the graph $\mathrm{sub}(F)$ is $e_{S^*} + e(S^*) + |S_1| + 2|S_0|$.

If $S^* = \emptyset$, then S is an independent set with each vertex having degree two, hence $\rho(S) = 2 \geq \frac{2b}{a+b} = \rho(\mathrm{sub}(F))$. Now we may assume $S^* \neq \emptyset$. Note that S_1 corresponds to a set of edges of F incident with only one vertex of S^*, thus we have $e_{S^*} \geq e(S^*) + |S_1|$. Also as S_2 corresponds to a set of edges whose both endpoints are in S^*, we have $|S_2| \leq e(S^*)$. Thus, $|S_1| + |S_2| \leq e_{S^*}$. Together with $\frac{e_{S^*}}{|S^*|} \geq \rho(F) \geq 1$, we have

$$\frac{e_{S^*} + e(S^*) + |S_1|}{|S^*| + |S_1| + |S_2|} \geq \frac{e_{S^*} + |S_2| + |S_1|}{|S^*| + |S_1| + |S_2|} \geq \frac{e_{S^*} + e_{S^*}}{|S^*| + e_{S^*}} \overset{(4.1)}{\geq} \frac{2b}{a+b}. \tag{4.2}$$

Then, we have

$$\begin{aligned} \rho_{\mathrm{sub}(F)}(S) &= \frac{e_{S^*} + e(S^*) + |S_1| + 2|S_0|}{|S^*| + |S_1| + |S_2| + |S_0|} \geq \frac{e_{S^*} + e(S^*) + |S_1| + \frac{2b}{a+b}|S_0|}{|S^*| + |S_1| + |S_2| + |S_0|} \\ &\overset{(4.2)}{\geq} \frac{\frac{2b}{a+b}|S \setminus S_0| + \frac{2b}{a+b}|S_0|}{|S \setminus S_0| + |S_0|} = \frac{2b}{a+b} = \rho(\mathrm{sub}(F)). \end{aligned}$$

This proves the proposition. □

Let $\mathcal{F}_0$ be a collection of balanced connected rooted bipartite graphs defined as follows.

- $\mathcal{F}_0$ includes all stars rooted on the leaves;
- $\mathcal{F}_0$ is closed under taking 1-subdivision, i.e. if $F \in \mathcal{F}_0$, then $\mathrm{sub}(F) \in \mathcal{F}_0$;

- If $F \in \mathcal{F}_0$, then $F_*(1) \in \mathcal{F}_0$.

Note that Observation 2.2, Propositions 4.2 and 4.4 guarantee that every $F \in \mathcal{F}_0$ is bipartite, balanced and connected, and $\rho(F) \geq 1$. Moreover, for every rooted bipartite graph $(F, R) \in \mathcal{F}_0$, the root set R is always an independent set.

Lemma 4.5. *Suppose for any $F \in \mathcal{F}_0$, there exists $\ell_0 = \ell_0(F)$ such that Conjecture 1.6 holds for F^ℓ for all $\ell \geq \ell_0$. If $a, b \in \mathbb{N}$, $b > a$, are such that $2 - \frac{a}{b}$ is balancedly realisable by a graph in $\mathcal{F}_0$, then $2 - \frac{a+b}{2b}$ is also balancedly realisable by a graph in $\mathcal{F}_0$.*

Proof. By the assumption, there exist a balanced connected rooted bipartite graph $F \in \mathcal{F}_0$ and ℓ_0 such that $\rho(F) = \frac{b}{a}$ and $\mathrm{ex}(n, F^\ell) = \Theta(n^{2-\frac{a}{b}}) = \Theta(n^{1+\frac{b-a}{b}})$ for all $\ell \geq \ell_0$.

By Proposition 4.4, $\mathrm{sub}(F) \in \mathcal{F}_0$ is also a balanced connected rooted bipartite graph with $\rho(\mathrm{sub}(F)) = \frac{2b}{a+b}$. Hence, Lemma 2.3 implies that there exists some $\ell_1 \in \mathbb{N}$ such that, for all $\ell \geq \ell_1$ we have

$$\mathrm{ex}(n, \mathrm{sub}(F)^\ell) = \Omega(n^{2-\frac{1}{\rho(\mathrm{sub}(F))}}) = \Omega(n^{2-\frac{a+b}{2b}}).$$

On the other hand, by assumption, Conjecture 1.6 holds for F^ℓ for all $\ell \geq \ell_0$, i.e.

$$\mathrm{ex}(n, \mathrm{sub}(F^\ell)) = O(n^{1+\frac{b-a}{2b}}) = O(n^{2-\frac{a+b}{2b}}).$$

Note that the root set of $F \in \mathcal{F}_0$ is an independent set, so taking 1-subdivision of a rooted blow-up of F is the same as taking a rooted blow-up of the 1-subdivision of F, that is, $\mathrm{sub}(F^\ell) = \mathrm{sub}(F)^\ell$. Thus, for any $\ell \geq \max\{\ell_0, \ell_1\}$, $\mathrm{ex}(n, \mathrm{sub}(F)^\ell) = \Theta(n^{2-\frac{a+b}{2b}})$. Consequently, $2 - \frac{a+b}{2b}$ is balancedly realisable by the graph $\mathrm{sub}(F) \in \mathcal{F}_0$. $\square$

Now we are ready to prove Theorem 1.7. In fact, a weaker version of Conjecture 1.6 already implies Conjecture 1.3 as follows.

Theorem 4.6. *Suppose that for each $F \in \mathcal{F}_0$, there exists $\ell_0 = \ell_0(F)$ such that Conjecture 1.6 holds for F^ℓ for all $\ell \geq \ell_0$. Then Conjecture 1.3 holds.*

Proof. We will show that for all $a, b \in \mathbb{N}$ with $a < b$, the number $2 - \frac{a}{b}$ is balancedly realisable under the assumption of theorem. Note that unbalanced complete bipartite graph (which is a blow-up of a star rooted on its leaves) shows that the number $2 - \frac{a}{b}$ is balancedly realisable by a graph in $\mathcal{F}_0$ for $a = 1$. We use induction on $a + b$. Assume that (a, b) is a minimum counterexample.

If $b > 2a$, then $b - a > a$ and $a + (b - a) < a + b$. Hence, by the induction hypothesis, $2 - \frac{a}{b-a}$ is balancedly realisable by a graph $F \in \mathcal{F}_0$ and Lemma 4.3 implies that $2 - \frac{a}{b}$ is also balancedly realisable by $F_*(1) \in \mathcal{F}_0$.

If $a < b < 2a$, then $(2a - b) + b < a + b$ and $2a - b \geq 1$. Hence, by the induction hypothesis, $2 - \frac{2a-b}{b}$ is balancedly realisable by a graph in $\mathcal{F}_0$ and Lemma 4.5 implies that $2 - \frac{2a-b+b}{2b} = 2 - \frac{a}{b}$ is balancedly realisable by a graph in $\mathcal{F}_0$. Hence, $2 - \frac{a}{b}$ is balancedly realisable for all natural numbers $b > a$. As 1 and 2 are trivially realisable, this shows that every rational number $r \in [1, 2]$ is realisable if the assumption of Theorem 4.6 is true. $\square$

5. Concluding Remarks

5.1. **Bipartite graphs with large radius.** Our results provide infintely many realisable numbers most of which are somewhat closer to 2 than 1. The reason for this is that the graph $D^\ell_{t,s}$ we considered has radius two and gets denser as we increase the parameter t and s, and Lemma 4.3 also produces new realisable number which is bigger than the original. Hence, to attack Conjecture 1.3, we need a method to deal with sparse graphs.

One obvious way to is to prove Conjecture 1.6 for blow-ups of balanced rooted bapartite graphs. As Theorem 1.7 suggests, this implies Conjecture 1.3. Another natural way to pursue is to consider a balanced tree with large radius, and study its blow-up. Towards this direction, we are only able to extend our method slightly to obtain the following result, regarding a blow-up of a balanced tree with radius three. Note that $\frac{10}{7}$ does not provide new realisable sequence as Theorem 3.1 shows this is also realisable by $D^{\ell}_{3,1}$. We include its proof in the appendix.

Theorem 5.1. *There exists $\ell_0 \in \mathbb{N}$ such that for all $\ell > \ell_0$, we have* $\mathrm{ex}(n, T^{\ell}_{4,7}) = \Theta(n^{10/7})$.

It would be interesting to generalise Theorem 3.1 as follows. For $s, t \in \mathbb{N}$, consider the following balanced tree with large radius. Let $H_{t,s}$ be the rooted tree obtained from a t-star by subdividing each edge s times and by attaching a leaf to the centre of the t-star; the root set of $H_{t,s}$ is its leaf-set. It is easy to check that $H_{t,s}$ is a balanced tree with $\rho(H_{t,s}) = \frac{1+(s+1)t}{1+st}$. The following seems plausible.

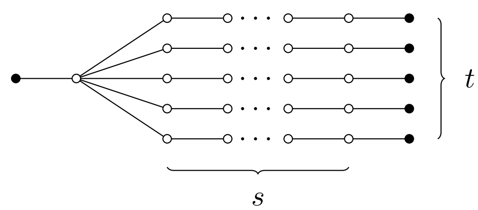

Figure 6: $H_{t,s}$

Problem 5.2. *For any positive integers s and t, there exists $\ell_0 = \ell_0(s,t)$ such that*

$$\mathrm{ex}(n, H^{\ell}_{t,s}) = \Theta(n^{1+\frac{t}{1+(s+1)t}})$$

for any integer $\ell \geq \ell_0$.

If this is true, then it would provide infinitely many new limit points $1 + \frac{1}{m}$ in the set of realisable number and this together with Lemma 4.3 would provide more realisable numbers. The method we used in Lemma 3.2 cannot be directly generalised to this problem. In particular, we need to prove that i-th neighbourhood of a vertex has size proportional to the i-th power of the average degree of G. This seems difficult to prove without a major improvement of the method.

5.2. **The 1-subdivision of complete bipartite graphs.** We may consider the 1-subdivision of complete bipartite graphs as an example of Conjecture 1.6. If Conjecture 1.6 is true, then

$$\mathrm{ex}(n, \mathrm{sub}(K_{s,t})) = \Theta(n^{\frac{3}{2}-\frac{1}{2s}})$$

must hold for large t, where the lower bound is obtained from Lemma 2.3. The best known upper bound is by Conlon and Lee [6] $\mathrm{ex}(n, \mathrm{sub}(K_{s,t})) \leq O(n^{\frac{3}{2}-\frac{1}{12t}})$. Improving their result, we are able to prove the following proposition with an exponent depending only on s. We remark that Janzer [17] independently obtained the same result.

Proposition 5.3. *For $t, s \in \mathbb{N}$ with $t \geq s$, we have* $\mathrm{ex}(n, \mathrm{sub}(K_{s,t})) \leq O(n^{\frac{3}{2}-\frac{1}{4s-2}})$.

Proof sketch. Note that $\mathrm{sub}(K_{s,t})$ is a subgraph of $D^{t}_{s,1}$. Hence a direct application of Theorem 3.1 implies $\mathrm{ex}(n, \mathrm{sub}(K_{s,t})) \leq O(n^{\frac{3}{2}-\frac{1}{4s+2}})$. To improve the number $\frac{1}{4s+2}$ to $\frac{1}{4s-2}$, we can follow the proof of Theorem 3.1 with the following minor modifications.

Let us choose the numbers $n \gg q \gg s,t$. We may assume that G is a bipartite graph with n vertices and the minimum degree $4qd$ with $d := n^{1-\frac{1}{4s-2}}$. We choose an arbitrary vertex $u_0 \in V(G)$ and $L_{-1} := \{u_0\}$. Let $L_0 \subseteq \Gamma_G(u_0)$ with $|L_0| = qd$. By using Hall's theorem, we can find a set $L_1 \subseteq \Gamma_G(L_0) \setminus \{r_0\}$ and a perfect matching on $G[L_0, L_1]$. For each $v \in L_0$, let $g(v) \in L_1$ be the vertex adjacent to v in the matching.

For each $w \in L_1$, we can find a collection of pairwise disjoint sets $\{\Gamma_w \subseteq N_G(w) \setminus (L_0 \cup L_1) : w \in L_1\}$ with $|\Gamma_w| \geq 2d$. Let $L_2 := \bigcup_{w \in L_1} \Gamma_w$ and $L_3 := \Gamma_G(L_2) \setminus (L_0 \cup L_1)$. Now the rest of the proof follows Stage 2 in the proof of Theorem 3.1 with $s = 2$ and $\ell = t$, except that we have to use vertices of $g^{-1}(A)$ and u_0 at the end to obtain a copy of $\mathrm{sub}(K_{s,t})$. □

5.3. **Phase transition with respect to the number of blown-up copies.** It would be interesting to determine for what ℓ the order of magnitude changes in Theorem 3.1. Note that the upper bound

$$\mathrm{ex}(n, D^{\ell}_{t-1,s-1}) = O(n^{2-\frac{t}{st-1}})$$

is not tight when ℓ is small. Indeed, it is known that

$$\mathrm{ex}(n, D^2_{2,1}) = \Theta(n^{4/3}) \quad \text{and} \quad \mathrm{ex}(n, D^{\ell}_{2,1}) = \Theta(n^{7/5})$$

for ℓ sufficiently large. Indeed, for $D^2_{2,1}$, the lower bound follows from the fact that $D^2_{2,1}$ contains C_6 as a subgraph, and the upper bound follows from a reduction theorem of Faudree and Simonovits [12].

Thus, for the blow-up of the graph $D_{2,1}$, the transition happens when the number of copies ℓ is larger than 2. This is in contrast to the well-known conjecture for even-cycles, stating that $\mathrm{ex}(n, C_{2k}) = \Theta(n^{1+1/k})$. Indeed, even-cycles are theta graphs with two disjoint paths, and $\mathrm{ex}(n, \theta_{k,\ell}) = \Theta(n^{1+1/k})$ for large ℓ. So the even-cycles conjecture suggests that for paths rooted at leaves, the transition happens already at $\ell = 2$. Recently, Verstraëte and Williford [22] showed that $\mathrm{ex}(n, \theta_{4,3}) = \Theta(n^{5/4})$, giving an evidence to the even-cycle conjecture for C_8.

References

[1] N. Alon, L. Rónyai, and T. Szabó. Norm-graphs: variations and applications. *J. Combin. Theory Ser. B*, 76(2):280–290, 1999.

[2] B. Bukh. Random algebraic construction of extremal graphs. *Bull. Lond. Math. Soc.*, 47(6):939–945, 2015.

[3] B. Bukh and D. Conlon. Rational exponents in extremal graph theory. *J. Eur. Math. Soc.*, 20:1747–1757, 2018.

[4] B. Bukh and M. Tait. Turán number of theta graphs. *arXiv preprint arXiv:1804.10014*, Apr. 2018.

[5] D. Conlon. Graphs with few paths of prescribed length between any two vertices. *Bull. Lond. Math. Soc.*, 2014. to appear.

[6] D. Conlon and J. Lee. On the extremal number of subdivisions. *arXiv preprint arXiv:1807.05008*, July 2018.

[7] P. Erdős. On the combinatorial problems which I would most like to see solved. *Combinatorica*, 1(1):25–42, 1981.

[8] P. Erdős. Problems and results in combinatorial analysis and graph theory. In *Proceedings of the First Japan Conference on Graph Theory and Applications (Hakone, 1986)*, volume 72, pages 81–92, 1988.

[9] P. Erdős and M. Simonovits. A limit theorem in graph theory. *Studia Sci. Math. Hungar.*, 1:51–57, 1966.

[10] P. Erdős and M. Simonovits. Some extremal problems in graph theory. pages 377–390, 1970.

[11] P. Erdős and A. H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc.*, 52:1087–1091, 1946.

[12] R. J. Faudree and M. Simonovits. On a class of degenerate extremal graph problems. *Combinatorica*, 3(1):83–93, 1983.

[13] M. Fitch. Rational exponents for hypergraph Turan problems. *arXiv preprint arXiv:1607.05788*, July 2016.

[14] J. Fox and B. Sudakov. Dependent random choice. *Random Structures Algorithms*, 38(1-2):68–99, 2011.

[15] P. Frankl. All rationals occur as exponents. *J. Combin. Theory Ser. A*, 42(2):200–206, 1986.

[16] Z. Füredi and M. Simonovits. The history of degenerate (bipartite) extremal graph problems. In *Erdös centennial*, volume 25 of *Bolyai Soc. Math. Stud.*, pages 169–264. János Bolyai Math. Soc., Budapest, 2013.

[17] O. Janzer. Improved bounds for the extremal number of subdivision. *ArXiv e-prints*, 2018.
[18] T. Jiang, J. Ma, and L. Yepremyan. On Turán exponents of bipartite graphs. *arXiv preprint arXiv:1806.02838*, June 2018.
[19] J. Kollár, L. Rónyai, and T. Szabó. Norm-graphs and bipartite Turán numbers. *Combinatorica*, 16(3):399–406, 1996.
[20] T. Kövari, V. T. Sós, and P. Turán. On a problem of K. Zarankiewicz. *Colloquium Math.*, 3:50–57, 1954.
[21] W. Mantel. Problem 28. *Wiskundige Opgaven*, 10(60-61):320, 1907.
[22] J. Verstraëte and J. Williford. Graphs without Theta subgraphs. *Journal of Combinatorial Theory, Series B*, 2018.

Appendix A. Blow-up of a balanced rooted tree with radius three

In this appendix, we present the proof of Theorem 5.1. As $T_{4,7}$ is a balanced rooted tree, by Lemma 2.3, it suffices to prove the following proposition.

Proposition A.1. *For each $\ell \in \mathbb{N}$, we have* $\mathrm{ex}(n, T^{\ell}_{4,7}) = O(n^{10/7})$.

To prove Proposition A.1, we need a variant of dependent random choice.

Lemma A.2. *Let $\ell \in \mathbb{N}$ and G be a graph. Let $u_1 \in V(G)$, $A \subseteq \Gamma_G(u_1)$ and $B \subseteq \Gamma_G(A) \setminus \{u_1\}$. If $e(A,B) \geq 4\ell|A|^2$ and $|B| \leq \frac{e(A,B)}{10\ell}$, then G contains $T^{\ell}_{4,7}$ as a subgraph.*

Proof. Let r_1, r_2, r_3, r_4 be the four root vertices of $T^{\ell}_{4,7}$ in such a way that distance between r_i and r_{i+1} is three for each $i \in [3]$. For each $i \in [4]$, let $Z_i := \Gamma_{T^{\ell}_{4,7}}(r_i)$.

Let $B' := \{b \in B : d_G(b, A) \geq 2\ell + 2\}$. Then we have

$$e(A, B') \geq e(A,B) - (2\ell+1)|B| \geq e(A,B)/2.$$

We choose $u \in B'$ uniformly at random and let $X := \Gamma_G(u, A)$. We say a pair $P \in \binom{A}{2}$ is *bad* if $|N_{B'}(P)| \leq 2\ell$. Let Y be the expected number of bad pairs in X. Then

$$\mathbb{E}[Y] \leq \sum_{P \text{ a bad pair}} \mathbb{P}[P \subseteq X] \leq \frac{2\ell}{|B'|}\binom{|A|}{2} \leq \frac{\ell|A|^2}{|B'|}.$$

Let X' be a subset of X obtained by deleting an element from each bad pair in X. Since $|X'| \geq |X| - Y$ and $|B'| \leq |B|$, we have

$$\mathbb{E}[|X'|] \geq \mathbb{E}[|X|] - \mathbb{E}[Y] \geq \frac{e(A,B')}{|B'|} - \frac{\ell|A|^2}{|B'|} \geq \frac{e(A,B)}{4|B'|} \geq 2\ell + 2.$$

Hence, there exist a vertex $u_3 \in B'$ and a set $X' \subseteq \Gamma_G(u_3)$ with $|X'| \geq 2\ell+2$ such that every pair $P \in \binom{X'}{2}$ has at least $2\ell + 1$ common neighbors in B'.

Now, we construct an embedding ϕ of $T^{\ell}_{4,7}$ into G. We arbitrarily choose two vertices $u_2, u_4, \in X'$ and a subset U_3 of $X' \setminus \{u_2, u_4\}$ with $|U_3| = \ell$. Let $\phi(r_i) = u_i$ for each $i \in [4]$ and assign ϕ in an arbitrary way that $\phi(Z_3) = U_3 \subseteq \Gamma_G(u_3)$. Note that ϕ is injective as $|Z_3| = |U_3|$ and $u_2, u_4 \notin U_3$.

For each $i \in \{2, 4\}$ and each $x \in Z_i$, let $z_x \in Z_3$ be the unique neighbor of x in Z_3. As $\phi(z_x) \in U_3 \subseteq X'$, we have $d_G(\{u_i, \phi(z_x)\}, B) \geq 2\ell + 1$, we can define $\phi(x)$ in such a way that $\phi(x) \in N_G(\{u_1, \phi(z_x)\}) \setminus \{u_3\}$ and ϕ is still injective. This is possible since the number of neighbours of r_2 or r_4 is 2ℓ.

For each $x \in Z_1$, let z'_x be the unique neighbour of x in Z_2. We choose $\phi(x)$ from $\Gamma_G(\phi(z'_x), A) \setminus (U_3 \cup \{u_2, u_4\})$ in such a way that ϕ is injective on Z_1. Since $|Z_1| = \ell$, it is possible by the definition of B' as we have $\phi(z'_x) \in B'$. Since every vertex in A is adjacent to $u_1 = \phi(r_1)$, this ϕ embeds a copy of $T^{\ell}_{4,7}$ into G. □

Proof of Proposition A.1. Consider the numbers n, q such that

$$n \gg q \gg \ell.$$

Let $d := n^{3/7}$. As before, it suffices to prove that an n-vertex graph G with $\delta(G) \geq qd$ contains $T^{\ell}_{4,7}$ as a subgraph. The following claim will be useful for us.

Claim A.3. *Suppose that we are given a vertex $u_0 \in V(G)$ and subsets $A^{\#} \subseteq \Gamma_G(u_0)$ with $|A^{\#}| \leq \ell d$ and $C \subseteq V(G) \setminus A^{\#}$ with $|C| \leq 10\ell d$. Then, there exist a vertex $u \in V(G) \setminus (A^{\#} \cup C \cup \{u_0\})$, sets $A \subseteq A^{\#}$, $B \subseteq \Gamma_G(u) \setminus (A^{\#} \cup C \cup \{u_0\})$ and a bijective function $f : B \to A$ satisfying the following. For each $a \in A$, we have $f(a)a \in E(G)$ and $|A| = |B| \geq n^{-1/7}|A^{\#}|$.*

Proof. Let $B^{\#} := \Gamma_G(A^{\#}) \setminus (C \cup \{u_0\})$. As $\delta(G) \geq qd$, for each $v \in V(G)$, we have

$$|\Gamma_G(v) \setminus (A^{\#} \cup C \cup \{u_0\})| \geq qd - 10\ell d - |A^{\#}| - 1 \geq qd/2. \tag{A.1}$$

For each $S \subseteq A^{\#}$, let $B_S := \Gamma_G(S, B^{\#})$. We claim that for each $S \subseteq A^{\#}$, we have

$$|B_S| \geq d|S|. \tag{A.2}$$

To show this, assume that we have a non-empty set $S \subseteq A^{\#}$ with $|B_S| < d|S|$. Since $|S| \leq |A^{\#}| \leq \ell d$ and $q \gg \ell$, by (A.1), we have

$$e(S, B_S) \geq \sum_{v \in S} d_G(v, B^{\#}) \geq qd|S|/2 \geq 4\ell|S|^2,$$

and $|B_S| < d|S| \leq \frac{qd|S|}{20\ell} \leq \frac{e(S,B_S)}{10\ell}$. Hence, we can apply Lemma A.2 to G with u_0, S, B_S and ℓ playing the roles of u_1, A, B and ℓ respectively to obtain a copy of $T^{\ell}_{4,7}$ in G, a contradiction. Hence (A.2) holds for all non-empty subset S of $A^{\#}$.

Thus Lemma 2.1 implies that there exists a collection $\{\Gamma_a \subseteq \Gamma_G(a, B^{\#}) : a \in A^{\#}\}$ of pairwise disjoint sets such that $|\Gamma_a| = d$ for all $a \in A^{\#}$.

For each $a \in A^{\#}$, let $U_a := \Gamma_G(\Gamma_a) \setminus (A^{\#} \cup C \cup \{u_0\})$. We claim that for each $a \in A^{\#}$, we have

$$|U_a| \geq d|\Gamma_a|. \tag{A.3}$$

Suppose there exists a vertex $a \in A^{\#}$ with $|U_a| < d|\Gamma_a|$. By (A.1), for each $v \in \Gamma_a$, we have $d_G(v, U_a) \geq qd/2$, hence $e_G(\Gamma_a, U_a) \geq qd|\Gamma_a|/2 \geq 4\ell|\Gamma_a|^2$. Moreover, we have

$$|U_a| < d|\Gamma_a| \leq \frac{qd|\Gamma_a|}{20\ell} \leq \frac{e_G(\Gamma_a, U_a)}{10\ell}.$$

Hence, we can apply Lemma A.2 to G with a, Γ_a, U_a, and ℓ playing the roles of u_1, A, B, and ℓ, respectively to obtain a copy of $T^{\ell}_{4,7}$ in G, a contradiction. Thus we obtain (A.3).

Let $U := \bigcup_{a \in A^{\#}} U_a$ and consider an auxiliary biparitte graph H with a vertex partition $(A^{\#}, U)$ with

$$E(H) = \{aw \in A^{\#} \times U : w \in U_a\}.$$

For each $a \in A^{\#}$, (A.3) implies that $d_H(a) = |U_a| \geq d|\Gamma_a| = d^2$. Thus, by averaging, there exists a vertex $u \in U$ with

$$d_H(u) \geq \frac{|E(H)|}{|U|} \geq \frac{d^2|A^{\#}|}{n} \geq n^{-1/7}|A^{\#}|.$$

Let $A := \Gamma_H(u) \subseteq A^{\#}$. For each $a \in A$, choose a vertex $f(a) \in \Gamma_a \cap \Gamma_G(u)$ which is a non-empty set by the definition of H and choice of A. As $\{\Gamma_a : a \in A^{\#}\}$ is a collection of pairwise disjoint sets, the function f is injective. Let $B := f(A)$. From the construction, it is obvious that

$A, B, C, \{u\}$ are pairwise disjoint. Hence the set A, B and a function f are as desired. This proves the claim. □

Let us choose an arbitrary vertex $u_0 \in V(G)$ and a subset $A_0 \subseteq \Gamma_G(u_0)$ with $|A_0| = \ell d$. Let $B_0 := A_0$ and $C_0 := \emptyset$.

For each $1 \leq i \leq 3$ in order, let us apply Claim A.3 with u_{i-1}, B_{i-1} and C_{i-1} where

$$C_{i-1} := \bigcup_{j=0}^{i-2} (B_j \cup \{u_j\})$$

playing the roles of $u_0, A^{\#}$ and C to obtain a vertex u_i, disjoint sets $A_i \subseteq B_{i-1}, B_i \subseteq \Gamma_G(u_i) \setminus C_{i-1}$ with $|A_i| = |B_i| \geq n^{-1/7}|B_{i-1}|$. and a bijective function $f_i : A_i \to B_i$ satisfying the following

$$B_i \subseteq \Gamma_G(u_i) \text{ and } af_i(a) \in E(G) \text{ for each } a \in A_i.$$

These repetitive applications of Claim A.3 are possible, since we have $|C_i| \leq 3|A_0| \leq 3\ell d$.

Let $B'_3 := B_3$ and for each $i = 2, 1, 0$ in order, we let $B'_i := f^{-1}_{i+1}(B'_{i+1})$. As $B'_i \subseteq A_i = B_{i-1}$, it follows that, for each $i \in [3]$,

- there exists a perfect matching between B'_{i-1} and B'_i via f_i,
- $|B'_0| = \cdots = |B'_3| \geq n^{-3/7}|A_0|$.

Now the sets $\{u_0\}, \ldots, \{u_3\}, B'_0, \ldots, B'_3$ are all pairwise disjoint, and we claim that they induce a copy of $T^{\ell}_{4,7}$. Indeed, $|B'_0| = \cdots = |B'_3| \geq n^{-3/7}|A_0| = \ell$ and for each $j \in [3]$, there is a perfect matching between B'_{j-1} and B'_j as well as $B'_i \subseteq B_i \subseteq \Gamma_G(u_i)$. Hence, we obtain a copy of $T^{\ell}_{4,7}$, completing the proof. □

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