STRUCTURE AND SUPERSATURATION FOR INTERSECTING FAMILIES

JÓZSEF BALOGH, SHAGNIK DAS, HONG LIU, MARYAM SHARIFZADEH AND TUAN TRAN

Abstract. The extremal problems regarding the maximum possible size of intersecting families of various combinatorial objects have been extensively studied. In this paper, we investigate supersaturation extensions, which in this context ask for the minimum number of disjoint pairs that must appear in families larger than the extremal threshold. We study the minimum number of disjoint pairs in families of permutations and in k-uniform set families, and determine the structure of the optimal families. Our main tool is a removal lemma for disjoint pairs. We also determine the typical structure of k-uniform set families without matchings of size s when $n \geq 2sk + 38s^4$, and show that almost all k-uniform intersecting families on vertex set $[n]$ are trivial when $n \geq (2 + o(1))k$.

1. Introduction

The extremal problem on the size of intersecting families of discrete objects has a long history, originating in extremal set theory. A set family is *intersecting* if any two of its sets share a common element. A classic result of Erdős, Ko and Rado [19] from 1961 states that when $n \geq 2k$, the size of the largest intersecting k-uniform set family is $\binom{n-1}{k-1}$. Furthermore, when $n \geq 2k+1$, the only extremal configurations are the *trivial* families, where all edges share a common element. This fundamental theorem has since inspired a great number of extensions and variations.

A recent trend in extremal combinatorics is to study the supersaturation extension of classic results. This problem, sometimes referred to as the Erdős–Rademacher problem, asks for the number of forbidden substructures that must appear in a configuration larger than the extremal threshold. We often observe an interesting phenomenon: while the extremal result only requires one forbidden substructure to appear, we usually find several. The first such line of research extended Mantel's Theorem [36], which states that an n-vertex triangle-free graph can have at most $\lfloor n^2/4 \rfloor$ edges. Rademacher (unpublished) showed that one additional edge would force the appearance of at least $\lfloor n/2 \rfloor$ triangles. Determining the number of triangles in larger graphs attracted a great deal of attention, starting with the works of Erdős [17, 18] and Lovász and Simonovits [35] and culminating in the asymptotic solution due to Razborov [41] and the recent exact solution determined by Liu, Pikhurko and Staden [32]. Supersaturation problems have since been studied in various contexts; examples include extremal graph theory [2, 29, 37, 38, 40, 42], extremal set theory [5, 9, 13, 30, 44], poset theory [4, 39, 45], and group theory [26, 46, 8].

The first result of our paper concerns supersaturation for the extension of the Erdős–Ko–Rado Theorem to families of permutations. A pair of permutations $\sigma, \pi \in S_n$ is said to be *intersecting* if $\{i \in [n] : \pi(i) = \sigma(i)\} \neq \emptyset$, and *disjoint* otherwise. A family $\mathcal{F} \subseteq S_n$

J.B. was partially supported by NSF Grant DMS-1500121 and by the Langan Scholar Fund (UIUC). H.L. was supported by the Leverhulme Trust Early Career Fellowship ECF-2016-523. M.S. was supported by ERC grant 306493 and the European Unions Horizon 2020 research and innovation programme under the Marie Curie grant agreement No 752426. T.T. was supported by the Alexander Humboldt Foundation, and by the GACR grant GJ16-07822Y, with institutional support RVO:67985807.

is *intersecting* if every pair of permutations in the family is. A natural construction of an intersecting family is to fix some pair $i, j \in [n]$, and take all permutations that map i to j; we call this a *coset*, and it has size $(n-1)!$. Deza and Frankl [12] showed that cosets are the largest intersecting families in S_n. In the corresponding supersaturation problem, we seek to determine how many disjoint pairs of permutations must appear in larger families. We write $\mathrm{dp}(\mathcal{F})$ for the number of disjoint pairs of permutations in a family $\mathcal{F} \subseteq S_n$. By the Deza–Frankl Theorem, when $|\mathcal{F}| \le (n-1)!$, we need not have any disjoint pairs in $\mathcal{F}$, while for $|\mathcal{F}| > (n-1)!$, $\mathrm{dp}(\mathcal{F})$ must be positive. Our first result determines, for certain ranges of family sizes s, the minimum value of $\mathrm{dp}(\mathcal{F})$ over all families $\mathcal{F} \subseteq S_n$ with $|\mathcal{F}| = s$. We denote by $\mathcal{T}(n, s)$ a subfamily of S_n of size s consisting of $\lfloor \frac{s}{(n-1)!} \rfloor$ pairwise disjoint cosets, with the remaining permutations coming from another disjoint coset.

Theorem 1.1. *There exists a constant $c > 0$ such that the following holds. Let n, k and s be positive integers such that $k \le cn^{1/2}$, and $s = (k + \varepsilon)(n-1)!$ for some real ε with $|\varepsilon| \le ck^{-3}$. Then any family $\mathcal{F} \subseteq S_n$ with $|\mathcal{F}| = s$ satisfies $\mathrm{dp}(\mathcal{F}) \ge \mathrm{dp}(\mathcal{T}(n, s))$.*

We next consider the supersaturation extension of the original Erdős–Ko–Rado Theorem, where one seeks to minimise the number of disjoint pairs of sets in a k-uniform family of s subsets of $[n]$. Bollobás and Leader [7] provided, for every s, a family of constructions known as the ℓ-balls, and conjectured that for some $1 \le \ell \le k$, an ℓ-ball is optimal for the supersaturation problem. In particular, when $\ell = 1$, the construction is an initial segment of the lexicographic ordering.

Letting $\mathcal{L}(n, k, s)$ be the initial segment of the first s sets in $\binom{[n]}{k}$, we write $\mathrm{dp}(n, k, s)$ for $\mathrm{dp}(\mathcal{L}(n, k, s))$, where again $\mathrm{dp}(\mathcal{F})$ is the number of disjoint pairs in a set family $\mathcal{F}$. Das, Gan and Sudakov [10] proved that if $n > 108(k^3 r + k^2 r^2)$ and $s \le \binom{n}{k} - \binom{n-r}{k}$, then for any family $\mathcal{F} \subseteq \binom{[n]}{k}$ of size s, $\mathrm{dp}(\mathcal{F}) \ge \mathrm{dp}(n, k, s)$. That is, when n is sufficiently large and the families are of small size, the initial segments of the lexicographic order minimise the number of disjoint pairs, confirming the Bollobás–Leader conjecture in this range.

Note that, for fixed r, the result in [10] requires $k = O(n^{1/3})$. Frankl, Kohayakawa and Rödl [22] showed that initial segments of the lexicographic order are *asymptotically* optimal even for larger uniformities k. In our next result, we extend the exact results to larger k as well, showing that the lexicographic initial segments are still optimal when $k = O(n^{1/2})$.

Theorem 1.2. *There is some absolute constant C such that if $n \ge Ck^2 r^3$ and $s \le \binom{n}{k} - \binom{n-r}{k}$, any family $\mathcal{F} \subseteq \binom{[n]}{k}$ with $|\mathcal{F}| = s$ satisfies $\mathrm{dp}(\mathcal{F}) \ge \mathrm{dp}(n, k, s)$; that is, $\mathcal{L}(n, k, s)$ minimises the number of disjoint pairs.*

With our next results, we address a different variation of classic extremal problems. Rather than considering the supersaturation phenomenon, we describe the typical structure of set families with a given property, showing that almost all such families are subfamilies of the trivial extremal constructions.

We first consider the famous Erdős Matching Conjecture concerning set families with no matching of size s. Frankl [21] showed that for $n \ge (2s-1)k - s + 1$, the extremal families are isomorphic to $\left\{ F \in \binom{[n]}{k} : F \cap [s-1] \ne \emptyset \right\}$; that is, they can be covered by $s - 1$ elements. Adapting the methods of Balogh et al. [3], we show that a slightly larger lower bound on n guarantees that almost all families without a matching of size s have a cover of size $s - 1$.

Theorem 1.3. *Let $n, k = k(n) \geq 3$ and $s = s(n) \geq 2$ be integers with $n \geq 2sk + 38s^4$. Then the number of subfamilies of $\binom{[n]}{k}$ with no matching of size s is $\left(\binom{n}{s-1} + o(1)\right) 2^{\binom{n}{k} - \binom{n-s+1}{k}}$.*

The $s = 2$ case corresponds to intersecting families. In this case, Balogh et al. [3] showed that when $n \geq (3 + o(1))k$, almost all intersecting families are trivial. Our final result improves the required bound on n to the asymptotically optimal $n \geq (2 + o(1))k$. Indeed, when $n = 2k$, then the number of intersecting families is $3^{\frac{1}{2}\binom{n}{k}} = 3^{\binom{n-1}{k-1}}$, since we can freely choose at most one set from each complementary pair of k-sets $\{A, [n] \setminus A\}$.

Theorem 1.4. *There exists a positive constant C such that for $k \geq 2$ and $n \geq 2k + C\sqrt{k \ln k}$, almost all intersecting families in $\binom{[n]}{k}$ are trivial. In particular, the number of intersecting families in $\binom{[n]}{k}$ is $(n + o(1))2^{\binom{n-1}{k-1}}$.*

Remark: During the preparation of this paper, Theorem 1.4 (with a superior constant $C = 2$) was proven independently by Frankl and Kupavskii [24] using different methods.

Outline and notation. The rest of the paper is organised as follows. We discuss families of permutations in Section 2, in particular proving the supersaturation result of Theorem 1.1. Section 3 is devoted to supersaturation for set families and the proof of Theorem 1.2. In Section 4, we address the typical structure of families, proving Theorems 1.3 and 1.4. Section 5 contains some concluding remarks, including a counterexample to the Bollobás–Leader conjecture.

We use standard set-theoretic and asymptotic notation. We write $\binom{X}{k}$ for the family of all k-element subsets of a set X. Given two functions f and g of some underlining parameter n, if $\lim_{n\to\infty} f(n)/g(n) = 0$, we write $f = o(g)$. For $a, b, c \in \mathbb{R}_+$, we write $a = b \pm c$ if $b - c \leq a \leq b + c$.

2. Supersaturation for families of permutations

In this section, we study the supersaturation problem concerning the number of disjoint pairs in a family of permutations. Our main tool is a removal lemma for disjoint pairs of permutations, showing that families with relatively few disjoint pairs are close to unions of cosets. We start by collecting some basic facts.

2.1. The derangement graph.

Let S_n be the symmetric group on $[n]$. A permutation $\tau \in S_n$ is called *derangement* if $\tau(i) \neq i$ for every $i \in [n]$. Let $\mathcal{D}_n$ be the set of all derangements in S_n. Denote by Γ_n the *derangement graph* on S_n, that is, $\sigma \sim \pi$ if $\sigma \cdot \tau = \pi$ for some $\tau \in \mathcal{D}_n$. In other words, σ and τ are adjacent in Γ_n if and only if they are disjoint.

For $i, j \in [n]$, denote by $\mathcal{T}_{(i,j)}$ the coset consisting of those permutations $\sigma \in S_n$ with $\sigma(i) = j$. Note that for $(i, j) \neq (i', j')$, $\mathcal{T}_{(i,j)} \cap \mathcal{T}_{(i',j')} \neq \emptyset$ if and only if $i \neq i'$ and $j \neq j'$, and any two such cosets have exactly $(n-2)!$ permutations in common.

We denote by d_n the number of derangements in S_n. By a standard application of the inclusion–exclusion principle, we have

$$d_n = |\mathcal{D}_n| = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} \sim \frac{n!}{e}.$$

We also introduce the following notation, which we will use to keep track of disjoint pairs in certain subgraphs of the derangement graph:

$$D_n = d_n + d_{n-1} \qquad \text{and} \qquad D'_n = d_n + 2d_{n-1}.$$

Note that the derangement numbers satisfy the recurrence

$$d_n = (n-1)(d_{n-1} + d_{n-2}) = (n-1)D_{n-1}. \tag{1}$$

Furthermore, consider the bipartite graph $\Gamma_n[\mathcal{T}_{(i_1,j)}, \mathcal{T}_{(i_2,j)}]$ induced by two disjoint cosets $\mathcal{T}_{(i_1,j)}$ and $\mathcal{T}_{(i_2,j)}$. For any $\sigma \in \mathcal{T}_{(i_1,j)}$ and any neighbour $\pi = \sigma \cdot \tau$, where $\tau \in \mathcal{D}_n$, we have $\pi \in \mathcal{T}_{(i_2,j)}$ if and only if $\tau(i_2) = i_1$. It is straightforward to see that there are d_{n-2} such derangements τ with $\tau(i_1) = i_2$ and d_{n-1} derangements with $\tau(i_1) \neq i_2$. As a result, every vertex of the bipartite graph has the same degree $d_{n-2} + d_{n-1} = D_{n-1}$.

For our investigation we shall need some information on the spectrum of the derangement graph Γ_n. Note that Γ_n is d_n-regular, and thus its largest eigenvalue is $\lambda_0 := d_n$ with constant eigenvector $\vec{\mathbf{1}} := (1, \dots, 1)$. We shall order the eigenvalues $\{\lambda_0, \lambda_1, \dots, \lambda_{n!-1}\}$ in a (perhaps non-standard) way, so that $d_n = |\lambda_0| \geq |\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_{n!-1}|$. Rentner [43] showed $\lambda_1 = -d_n/(n-1)$, while Ellis [14] proved there is some positive constant K such that

$$|\lambda_2| \leq K d_n / n^2. \tag{2}$$

Furthermore, as shown by Ellis, Friedgut and Pilpel [16], the span of the λ_0- and λ_1-eigenspaces is $U_1 := \operatorname{span}\{\mathbb{1}_{\mathcal{T}_{(i,j)}} : i, j \in [n]\}$, the span of the characteristic vectors of the cosets.

2.2. **A removal lemma.** For any integer $s > \frac{1}{2}(n-1)!$, there are unique $k \in \mathbb{N}$ and $\varepsilon \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ such that $s = (k+\varepsilon)(n-1)!$. Then $\mathcal{T}(n,s)$ is a subfamily of S_n consisting of $\lfloor k+\varepsilon \rfloor$ pairwise disjoint cosets and $(k+\varepsilon-\lfloor k+\varepsilon \rfloor)(n-1)!$ permutations from another disjoint coset. Hence,

$$\begin{aligned} \operatorname{dp}(\mathcal{T}(n,s)) &= \left(\binom{\lfloor k+\varepsilon \rfloor}{2} + (k+\varepsilon-\lfloor k+\varepsilon \rfloor)\right)(n-1)!D_{n-1} \\ &= \left(\binom{k}{2} + \left(k - \frac{1}{2}\right)\varepsilon + \frac{1}{2}|\varepsilon|\right)(n-1)!D_{n-1}, \end{aligned} \tag{3}$$

as $\operatorname{dp}(\mathcal{F}) = e(\Gamma_n[\mathcal{F}])$, and the bipartite subgraphs of Γ_n induced by disjoint cosets are D_{n-1}-regular.

We will now prove a removal lemma for disjoint pairs of permutations, which states that any family $\mathcal{F} \subseteq S_n$ of size $s \approx k(n-1)!$ with $\operatorname{dp}(\mathcal{F}) \approx \operatorname{dp}(\mathcal{T}(n,s))$ must be 'close' to a union of k cosets.

Lemma 2.1. *There exist positive constants C and c such that the following holds for sufficiently large n. Let $1 \leq k < n/2$ be an integer, and let $\varepsilon \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$ be such that $\max\{|\varepsilon|, \beta\} \leq ck$. If $\mathcal{F} \subseteq S_n$ is a family of size $s = (k+\varepsilon)(n-1)!$ and $\operatorname{dp}(\mathcal{F}) \leq \operatorname{dp}(\mathcal{T}(n,s)) + \beta(n-1)!D_{n-1}$, then there is some union $\mathcal{G}$ of k cosets with the property that*

$$|\mathcal{F} \Delta \mathcal{G}| \leq Ck^2 \left(\frac{1}{n} + \sqrt{\frac{6(|\varepsilon| + \beta)}{k}}\right)(n-1)!.$$

In the proof of Lemma 2.1 we shall use a stability result due to Ellis, Filmus and Friedgut [15, Theorem 1]. To state their theorem we need some additional notation. We equip S_n with the uniform distribution. Then, for any function $f : S_n \to \mathbb{R}$, the expected value of f is defined by $\mathbb{E}[f] = \frac{1}{n!}\sum_{\sigma \in S_n} f(\sigma)$. The inner product of two functions $f, g : S_n \to \mathbb{R}$ is defined as $\langle f, g \rangle := \mathbb{E}[fg]$; this induces the norm $\|f\| := \sqrt{\langle f, f \rangle}$. Given $c > 0$, let $\operatorname{round}(c)$ denote the nearest integer to c.

Theorem 2.2 (Ellis, Filmus and Friedgut)**.** *There exist positive constants C_0 and ε_0 such that the following holds. Let $\mathcal{F}$ be a subfamily of S_n with $|\mathcal{F}| = \alpha(n-1)!$ for some $\alpha \le n/2$. Let $f = \mathbb{1}_{\mathcal{F}}$ be the characteristic function of $\mathcal{F}$ and let f_{U_1} be the orthogonal projection of f onto U_1. If $\mathbb{E}[(f - f_{U_1})^2] = \varepsilon\mathbb{E}[f]$ for some $\varepsilon \le \varepsilon_0$, then*

$$\mathbb{E}[(f-g)^2] \le C_0\alpha^2(1/n + \varepsilon^{1/2})/n,$$

where g is the characteristic function of a union of $\mathrm{round}(\alpha)$ *cosets of S_n.*

We now derive the removal lemma from Theorem 2.2.

Proof of Lemma 2.1. Set $c = \min\{\frac{\varepsilon_0}{12}, \frac{1}{2}\}$ and $C = 3C_0$, where ε_0 and C_0 are the positive constants from Theorem 2.2. Let f be the characteristic vector of $\mathcal{F}$. Write $f = f_0 + f_1 + f_2$, where f_i is the projection of f onto the λ_i-eigenspace for $i = 0, 1$. By the orthogonality of the eigenspaces,

$$\|f\|^2 = \|f_0\|^2 + \|f_1\|^2 + \|f_2\|^2. \tag{4}$$

Since f is Boolean,

$$\|f\|^2 = \mathbb{E}[f^2] = \mathbb{E}[f] = \frac{|\mathcal{F}|}{n!} = \frac{k+\varepsilon}{n}, \quad \text{and} \quad \|f_0\|^2 = \langle f, \vec{\mathbf{1}}\rangle^2 = \mathbb{E}[f]^2 = \left(\frac{k+\varepsilon}{n}\right)^2. \tag{5}$$

Let A be the adjacency matrix of the derangement graph Γ_n. Then

$$2\mathrm{dp}(\mathcal{F}) = 2e(\Gamma_n[\mathcal{F}]) = f^T A f = \sum_{i=0}^{2} f_i^T A f_i \overset{(2)}{\ge} \lambda_0 f_0^T f_0 + \lambda_1 f_1^T f_1 - \frac{Kd_n}{n^2} f_2^T f_2.$$

Dividing both sides by $n!$, we obtain the following inequalities when $n \ge 4K$:

$$\begin{aligned}
\frac{2\mathrm{dp}(\mathcal{F})}{n!} &\ge \lambda_0\|f_0\|^2 + \lambda_1\|f_1\|^2 - \frac{Kd_n}{n^2}\|f_2\|^2 \\
&\overset{(4)}{=} \lambda_0\|f_0\|^2 + \lambda_1(\|f\|^2 - \|f_0\|^2 - \|f_2\|^2) - \frac{Kd_n}{n^2}\|f_2\|^2 \\
&= (\lambda_0 - \lambda_1)\|f_0\|^2 + \lambda_1\|f\|^2 + \left(|\lambda_1| - \frac{Kd_n}{n^2}\right)\|f_2\|^2 \\
&\overset{(5)}{\ge} \frac{nd_n}{n-1}\cdot\left(\frac{k+\varepsilon}{n}\right)^2 - \frac{d_n}{n-1}\cdot\left(\frac{k+\varepsilon}{n}\right) + \frac{3d_n}{4(n-1)}\cdot\|f_2\|^2 \\
&= \frac{d_n}{n(n-1)}\cdot(k+\varepsilon)(k+\varepsilon-1) + \frac{3d_n}{4(n-1)}\cdot\|f_2\|^2 \\
&\overset{(1)}{=} \frac{2D_{n-1}}{n}\cdot\left(\binom{k}{2} + \left(k - \frac{1}{2}\right)\varepsilon + \frac{\varepsilon^2}{2}\right) + \frac{3}{4}D_{n-1}\cdot\|f_2\|^2.
\end{aligned} \tag{6}$$

On the other hand, by assumption we have

$$\mathrm{dp}(\mathcal{F}) \le \mathrm{dp}(\mathcal{T}(n,s)) + \beta(n-1)!D_{n-1} \overset{(3)}{=} \left(\binom{k}{2} + \left(k - \frac{1}{2}\right)\varepsilon + \frac{1}{2}|\varepsilon| + \beta\right)(n-1)!D_{n-1}.$$

Combined with (6), we get

$$\|f_2\|^2 \le \frac{4}{3}\cdot\frac{|\varepsilon| + 2\beta - \varepsilon^2}{n} \le \frac{3(|\varepsilon| + \beta)}{n}. \tag{7}$$

Moreover, $\mathbb{E}[f] = \frac{k+\varepsilon}{n} \ge \frac{k}{2n}$, as $|\varepsilon| \le ck \le \frac{k}{2}$. Therefore,

$$\mathbb{E}[(f - f_{U_1})^2] = \mathbb{E}[(f - f_0 - f_1)^2] = \|f_2\|^2 \le \frac{6(|\varepsilon| + \beta)}{k}\cdot\mathbb{E}[f].$$

Since $\frac{6(|\varepsilon|+\beta)}{k} \le 12c \le \varepsilon_0$, we may apply Theorem 2.2 to conclude that there exists a union $\mathcal{G}$ of k cosets in S_n such that

$$\mathbb{E}[(f-\mathbb{1}_{\mathcal{G}})^2] \le C_0 \cdot \frac{(k+\varepsilon)^2}{n} \cdot \left(\frac{1}{n} + \sqrt{\frac{6(|\varepsilon|+\beta)}{k}}\right) \le C \cdot \frac{k^2}{n} \cdot \left(\frac{1}{n} + \sqrt{\frac{6(|\varepsilon|+\beta)}{k}}\right).$$

This gives $|\mathcal{F}\Delta\mathcal{G}| = \mathbb{E}[(f-\mathbb{1}_{\mathcal{G}})^2] \cdot n! \le Ck^2\left(\frac{1}{n} + \sqrt{\frac{6(|\varepsilon|+\beta)}{k}}\right)(n-1)!$, completing our proof. □

We will use this removal lemma to prove Theorem 1.1 in Subsection 2.4, a supersaturation result for disjoint pairs in S_n. However, from the proof above we can immediately deduce that for any[1] $1 \le k \le n$, the union of k pairwise disjoint cosets minimises the number of disjoint pairs among all families of $k(n-1)!$ permutations.

Proposition 2.3. *For any positive integers $1 \le k \le n$, the family $\mathcal{T} = \cup_{j=1}^k \mathcal{T}_{(1,j)}$ minimises the number of disjoint pairs over all families $\mathcal{F} \subseteq S_n$ of size $k(n-1)!$.*

Proof. Let $\mathcal{F} \subseteq S_n$ be an extremal family of size $k(n-1)!$, and let $f = \mathbb{1}_{\mathcal{F}}$. By (3), we must have $\mathrm{dp}(\mathcal{F}) \le \mathrm{dp}(\mathcal{T}) = \binom{k}{2}(n-1)!D_{n-1}$. Hence, as in the proof of Lemma 2.1, we can use (7) with $\varepsilon = \beta = 0$, and so $\|f_2\|^2 = 0$. It follows from (6) that $\mathrm{dp}(\mathcal{F}) \ge \binom{k}{2}(n-1)!D_{n-1} = \mathrm{dp}(\mathcal{T})$, showing that $\mathcal{T}$ minimises the number of disjoint pairs. □

2.3. **Intersection graphs.** The removal lemma guarantees that families with relatively few disjoint pairs are close to unions of cosets. However, for Theorem 1.1, we shall want to conclude that the cosets are pairwise disjoint. We shall thus collect some facts about unions of cosets.

Let $\mathcal{G}$ be a union of 'relatively few' cosets in S_n. In Lemma 2.4 we will provide a formula for $\mathrm{dp}(\mathcal{G})$ in terms of the so-called intersection graph of $\mathcal{G}$. Using this formula in Corollary 2.11, we shall show that $\mathrm{dp}(\mathcal{G}) \ge \mathrm{dp}(\mathcal{T}(n,s))$ for $s = |\mathcal{G}|$. Lemma 2.4 and Corollary 2.11 will then be used in the proof of Theorem 1.1.

Given a union $\mathcal{G} = \mathcal{T}_{(i_1,j_1)} \cup \ldots \cup \mathcal{T}_{(i_k,j_k)}$ of k different cosets in S_n, its *intersection graph*, is the graph with vertex set $\{(i_1,j_1),\ldots,(i_k,j_k)\}$, where vertices (i,j) and (i',j') are adjacent if the corresponding cosets have non-empty intersection; that is, $(i,j) \sim (i',j')$ whenever $i \ne i'$ and $j \ne j'$. In other words, two cosets are adjacent in the intersection graph if and only if the corresponding vertices are not on any axis-aligned line in $\mathbb{Z}^2$. We say that $\mathcal{G}$ is *canonical* if at least $k-1$ of its cosets are pairwise disjoint; in terms of its intersection graph G, this means that there is an axis-aligned line containing at least $v(G)-1$ vertices of G.

To analyse the intersection graph, we need further notation. For $i \ge 1$ and $X \subseteq V(G)$, denote by $k_{i,X}(G)$ the number of cliques of size i in G containing X. We write $K_i(G)$ for the set of i-cliques in G, $\{Y : G[Y] \cong K_i\}$. In particular, $k_i(G) := k_{i,\emptyset}(G) = |K_i(G)|$ is the number of cliques of size i in G. If the graph G is clear from the context, we use $k_{i,X}$ and k_i instead of $k_{i,X}(G)$ and $k_i(G)$. Let $\mathbb{1}_{\bar{P}_3} \colon V(G) \times \binom{V(G)}{2} \to \{0,1\}$ be a function defined by setting $\mathbb{1}_{\bar{P}_3}(x,\{y,z\}) = 1$ if and only if yz is the only edge of the induced subgraph $G[\{x,y,z\}]$. Then $i(\bar{P}_3, G) := \sum_{x,y,z \in \binom{V(G)}{3}} \mathbb{1}_{\bar{P}_3}(x,\{y,z\})$ is the number of induced copies of $\bar{P}_3$ in G.

Lemma 2.4. *Let G be the intersection graph of a union $\mathcal{G}$ of cosets in S_n. Then the following formulas hold for every sufficiently large n.*

[1]In the proof of Lemma 2.1, we used that n was sufficiently large to bound $\frac{Kd_n}{n^2}\|f_2\|^2$. However, in Proposition 2.3, we will have $\|f_2\|^2 = 0$, and so will not require n to be large.

(i) $|\mathcal{G}| = v(G)(n-1)! - e(G)(n-2)! + k_3(G)(n-3)! \pm k_4(G)(n-4)!$.

(ii) $\mathrm{dp}(\pi, \mathcal{G}) = v(G)D_{n-1} - e(G)D'_{n-2} \pm 4v(G)e(G)(n-3)!$ *for every permutation* $\pi \in S_n \setminus \mathcal{G}$.

(iii) $\mathrm{dp}(\mathcal{G}) = \sum_{i=1}^{3} a_i(n-i)!D_{n-1} \pm 1000v(G)^3e(G)(n-1)!(n-4)!$ *for* $v(G) \le n$, *where* $a_1 = \binom{v(G)}{2}$, $a_2 = -(v(G)-1)e(G)$, *and* $a_3 = \frac{1}{2}\big((2v(G)-3)k_3(G) + e(G)(e(G)-v(G)+1) + i(\bar{P}_3, G)\big)$.

For the proof of Lemma 2.4, we make the following crucial observations.

Observation 2.5. The following properties hold.

(i) For every subset $X \subset V(G)$, the intersection $\cap_{x\in X}\mathcal{T}_x$ is non-empty if and only if $G[X]$ is a clique. Furthermore, if $G[X]$ is a clique, then $|\cap_{x\in X}\mathcal{T}_x| = (n-|X|)!$.

(ii) If $(i_1, j_1), \ldots, (i_\ell, j_\ell)$ form an ℓ-clique in G and $\pi \in S_n \setminus \bigcup_{s=1}^{\ell} \mathcal{T}_{(i_s,j_s)}$, then

$$\mathrm{dp}(\pi, \bigcap_{s=1}^{\ell} \mathcal{T}_{(i_s,j_s)}) = d_{n-\ell} + \big(\ell - \left|\{i_1,\ldots,i_\ell\} \cap \{\pi^{-1}(j_1),\ldots,\pi^{-1}(j_\ell)\}\right|\big)\, d_{n-\ell-1} \pm 51\ell^2(n-\ell-2)!,$$

provided $n \ge 10\ell^2$.

Proof. (i) Since $\mathcal{T}_{(i,j)} \cap \mathcal{T}_{(i',j')} = \emptyset$ in the case when (i,j) and (i',j') do not form an edge of G, we must have $\cap_{x\in X}\mathcal{T}_x = \emptyset$ whenever $G[X]$ is not a clique. Suppose that $X = \{(i_1,j_1),\ldots,(i_\ell,j_\ell)\}$ spans a clique in G. Then $|\{i_1,\ldots,i_\ell\}| = |\{j_1,\ldots,j_\ell\}| = \ell$. Hence $|\cap_{x\in X}\mathcal{T}_x|$ is the number of bijections from $[n]\setminus\{i_1,\ldots,i_\ell\}$ to $[n]\setminus\{j_1,\ldots,j_\ell\}$, which is $(n-\ell)!$.

(ii) Fix a permutation $\pi \in S_n \setminus \big(\mathcal{T}_{(i_1,j_1)} \cup \ldots \cup \mathcal{T}_{(i_\ell,j_\ell)}\big)$, that is, $\pi(i_s) \ne j_s$ for all $s \in [\ell]$. Observe that for any permutation $\sigma \in \cap_{s=1}^{\ell}\mathcal{T}_{(i_s,j_s)}$ and any index $x \in \{i_1,\ldots,i_\ell\} \cup \{\pi^{-1}(j_1),\ldots,\pi^{-1}(j_\ell)\}$, we have $\sigma(x) \ne \pi(x)$. For simplicity of notation, set

$$c = \ell - \left|\{i_1,\ldots,i_\ell\} \cap \{\pi^{-1}(j_1),\ldots,\pi^{-1}(j_\ell)\}\right| \text{ and } I = [n]\setminus\big(\{i_1,\ldots,i_\ell\} \cup \{\pi^{-1}(j_1),\ldots,\pi^{-1}(j_\ell)\}\big).$$

We can see that $|I| = n-\ell-c$. For each $x \in I$, we write $\mathcal{A}_x$ for the family of all permutations $\sigma \in \cap_{s=1}^{\ell}\mathcal{T}_{(i_s,j_s)}$ with $\sigma(x) = \pi(x)$. Then $\cup_{x\in I}\mathcal{A}_x$ is the family of all permutations in $\cap_{s=1}^{\ell}\mathcal{T}_{(i_s,j_s)}$ which intersect π. By the definition, $(i_1,j_1),\ldots,(i_\ell,j_\ell),(x,\pi(x))$ form a clique in G. Applying the inclusion-exclusion formula and part (i), we thus obtain

$$\mathrm{dp}(\pi, \cap_{s=1}^{\ell}\mathcal{T}_{(i_s,j_s)}) = \left|\cap_{s=1}^{\ell}\mathcal{T}_{(i_s,j_s)} \setminus \cup_{x\in I}\mathcal{A}_x\right| = \sum_{i=0}^{n-\ell-c} (-1)^i \binom{n-\ell-c}{i}(n-\ell-i)!. \tag{8}$$

Let α_i be the real number such that $\binom{n-\ell-c}{i}(n-\ell-i)! = \alpha_i \frac{(n-\ell)!}{i!}$. We shall approximate α_i by a simple function, and then use it to compute $\mathrm{dp}(\pi, \cap_{s=1}^{\ell}\mathcal{T}_{(i_s,j_s)})$. By definition,

$$\alpha_i = \frac{(n-\ell-c)!(n-\ell-i)!}{(n-\ell)!(n-\ell-c-i)!} = \frac{(n-\ell-i)\cdots(n-\ell-i-c+1)}{(n-\ell)\cdots(n-\ell-c+1)} = \prod_{j=0}^{c-1}\left(1 - \frac{i}{n-\ell-j}\right).$$

Hence $\alpha_i \ge \left(1 - \frac{i}{n-\ell-c+1}\right)^c \ge 1 - \frac{ci}{n-\ell-c+1} = 1 - \frac{ci}{n-\ell} - \frac{c(c-1)i}{(n-\ell)(n-\ell-c+1)}$. On the other hand, $\alpha_i \le \left(1 - \frac{i}{n-\ell}\right)^c \le \exp\left\{-\frac{ci}{n-\ell}\right\} \le 1 - \frac{ci}{n-\ell} + \left(\frac{ci}{n-\ell}\right)^2$. Since $c \le \ell$ and $n \ge 10\ell^2$, we may write $\alpha_i = 1 - \frac{ci}{n-\ell} + \varepsilon_i$ for some $|\varepsilon_i| \le \frac{c^2i^2}{(n-\ell)^2}$, providing an effective estimate when i is small.

Since $\left|1 - \frac{ci}{n-\ell}\right| \le c$ and $\frac{(n-\ell)!}{i!} \le n^{c-1}$ for $n-\ell-c+1 \le i \le n-\ell$, we have

$$\sum_{i=n-\ell-c+1}^{n-\ell} \frac{(n-\ell)!}{i!}\left|1 - \frac{ci}{n-\ell}\right| \le c^2n^{c-1} \le \ell^2n^{c-1}.$$

Furthermore, $\sum_{0\le i\le n-\ell}\frac{(n-\ell)!}{i!}|\varepsilon_i| \le (n-\ell-2)!\sum_{i\ge 0}\frac{c^2 i^2}{i!} \le 50\ell^2(n-\ell-2)!$. Substituting these inequalities into (8), we find that

$$\begin{aligned}
\mathrm{dp}(\pi, \cap_{s=1}^{\ell}\mathcal{T}_{(i_s,j_s)}) &= \sum_{i=0}^{n-\ell}(-1)^i\frac{(n-\ell)!}{i!}\left(1-\frac{ci}{n-\ell}\right) \pm \ell^2 n^{c-1} \pm 50\ell^2(n-\ell-2)! \\
&= \sum_{i=0}^{n-\ell}(-1)^i\frac{(n-\ell)!}{i!} + c\sum_{i=0}^{n-\ell-1}(-1)^i\frac{(n-\ell-1)!}{i!} \pm 51\ell^2(n-\ell-2)! \\
&= d_{n-\ell} + c\cdot d_{n-\ell-1} \pm 51\ell^2(n-\ell-2)!,
\end{aligned}$$

assuming $n \ge 10\ell^2$. $\square$

With this observation, we can now proceed to prove Lemma 2.4.

Proof of Lemma 2.4. (i) Using the Bonferroni inequalities, we obtain

$$\begin{aligned}
|\mathcal{G}| &= \left|\bigcup_{x\in V(G)}\mathcal{T}_x\right| = \sum_{1\le i\le 3}(-1)^{i-1}\left(\sum_{X\in K_i(G)}\left|\bigcap_{x\in X}\mathcal{T}_x\right|\right) \pm \sum_{X\in K_4(G)}\left|\bigcap_{x\in X}\mathcal{T}_x\right| \\
\text{(by Observation 2.5 (i))} &= \sum_{i=1}^{3}(-1)^{i-1}k_i(n-i)! \pm k_4(n-4)!,
\end{aligned}$$

as desired.

(ii) Let π be an arbitrary permutation in $S_n \setminus \mathcal{G}$. It follows from the Bonferroni inequalities and Observation 2.5 (i) that

$$\mathrm{dp}(\pi,\mathcal{G}) = \sum_{i=1}^{3}(-1)^{i-1}\left(\sum_{X\in K_i(G)}\mathrm{dp}(\pi,\bigcap_{x\in X}\mathcal{T}_x)\right) \pm \sum_{X\in K_4(G)}\mathrm{dp}(\pi,\bigcap_{x\in X}\mathcal{T}_x).$$

Recall that $\mathrm{dp}(\pi,\mathcal{T}_x) = D_{n-1}$ for every $x\in V(G)$. For $xy\in E(G)$, say $x=(x_1,x_2)$ and $y=(y_1,y_2)$, Observation 2.5 (ii) implies

$$\mathrm{dp}(\pi,\mathcal{T}_x\cap T_y) = D'_{n-2} - \left|\{x_1,y_1\}\cap\{\pi^{-1}(x_2),\pi^{-1}(y_2)\}\right| d_{n-3} \pm 250(n-4)!$$

as $D'_{n-2} = d_{n-2}+2d_{n-3}$. Again using Observation 2.5 (ii) gives $\mathrm{dp}(\pi,\cap_{x\in X}\mathcal{T}_x) = d_{n-3}\pm 4(n-4)!$ for every $X\in K_3(G)$. For each $X\in K_4(G)$, we may deduce from Observation 2.5 (i) that $\mathrm{dp}(\pi,\cap_{x\in X}\mathcal{T}_x)\le|\cap_{x\in X}\mathcal{T}_x| = (n-4)!$. Combining these bounds, we obtain

$$\begin{aligned}
\mathrm{dp}(\pi,\mathcal{G}) = k_1(G)D_{n-1} - k_2(G)D'_{n-2} &+ \left(k_3(G) + \sum_{xy\in E(G)}\left|\{x_1,y_1\}\cap\{\pi^{-1}(x_2),\pi^{-1}(y_2)\}\right|\right)d_{n-3} \\
&\pm (250k_2(G)+4k_3(G)+k_4(G))(n-4)!.
\end{aligned} \tag{9}$$

The statement then follows immediately from this formula.

(iii) Our main tools are once again the Bonferroni inequalities. For each vertex set $X\subseteq V(G)$, let $\mathcal{M}_X$ be the family of all permutations $\pi\in\mathcal{G}$ which satisfy $\{x\in V(G):\pi\in\mathcal{T}_x\} = X$. From Observation 2.5 (i) we find $\mathcal{M}_i := \dot\cup_{|X|=i}\mathcal{M}_X = \dot\cup_{X\in K_i(G)}\mathcal{M}_X$, resulting in $\mathrm{dp}(\mathcal{M}_i,\mathcal{G}) = \sum_{X\in K_i(G)}\mathrm{dp}(\mathcal{M}_X,\mathcal{G})$. Since $\mathcal{G} = \dot\cup_{i\ge 1}\mathcal{M}_i$, one has $\mathrm{dp}(\mathcal{G}) = \frac{1}{2}\sum_{1\le i\le k}\mathrm{dp}(\mathcal{M}_i,\mathcal{G})$. We shall use Observation 2.5 to bound the summands in this formula.

Claim 2.6. $\sum_{i\ge 4}\mathrm{dp}(\mathcal{M}_i,\mathcal{G}) \le k_1k_4(n-1)!(n-4)!$.

Proof. One has $|\mathcal{G}| \le k_1(n-1)!$, and $|\cup_{i\ge 4}\mathcal{M}_i| \le \sum_{X\in K_4(G)} |\cap_{x\in X}\mathcal{T}_x| \le k_4(n-4)!$ due to Observation 2.5 (i). Hence $\sum_{i\ge 4} \mathrm{dp}(\mathcal{M}_i,\mathcal{G}) \le |\cup_{i\ge 4}\mathcal{M}_i|\,|\mathcal{G}| \le k_1k_4(n-1)!(n-4)!$. $\square$

Our next claim determines the term $\mathrm{dp}(\mathcal{M}_3,\mathcal{G})$ up to an error term of $O((n-1)!(n-4)!)$.

Claim 2.7. $\mathrm{dp}(\mathcal{M}_3,\mathcal{G}) = (k_1-3)k_3\cdot(n-3)!D_{n-1} \pm 3k_1^2k_3\cdot(n-1)!(n-4)!$.

Proof. For $X \in K_3(G)$ and $\pi\in\mathcal{M}_X$, by applying (9) to $G-X$, we see that

$$\mathrm{dp}(\pi,\mathcal{G}) = \mathrm{dp}(\pi,\cup_{x\in V(G)-X}\mathcal{T}_x) = (k_1-3)D_{n-1} \pm 2k_2\cdot(n-2)!.$$

On the other hand, it follows from the inclusion-exclusion formula and Observation 2.5 (i) that $|\mathcal{M}_X| = (n-3)! \pm k_{4,X}\cdot(n-4)!$ for every triangle $X\in K_3(G)$. Combining with the trivial bound $k_{4,X}\le k_1$, this gives

$$|\mathcal{M}_3| = \sum_{X\in K_3(G)} |\mathcal{M}_X| = k_3\cdot(n-3)! \pm k_1k_3\cdot(n-4)!.$$

Summing $\mathrm{dp}(\pi,\mathcal{G})$ over all permutations $\pi\in\mathcal{M}_3$ and using the estimate $k_2\le k_1^2$, we obtain

$$\mathrm{dp}(\mathcal{M}_3,\mathcal{G}) = (k_1-3)k_3\cdot(n-3)!D_{n-1} \pm 3k_1^2k_3\cdot(n-1)!(n-4)!. \qquad \square$$

We shall use a similar counting argument to estimate $\mathrm{dp}(\mathcal{M}_2,G)$.

Claim 2.8. $\mathrm{dp}(\mathcal{M}_2,\mathcal{G}) = \sum_{i=2}^{3} c_i\cdot(n-i)!D_{n-1} \pm 35k_1k_2^2\cdot(n-1)!(n-4)!$, *where* $c_2=(k_1-2)k_2$ *and* $c_3 = -3(k_1-2)k_3 - \sum_{X\in K_2(G)} k_2(G-X)$.

Proof. Applying (9) to $G-X$, we find that

$$\mathrm{dp}(\pi,\mathcal{G}) = (k_1-2)D_{n-1} - k_2(G-X)D'_{n-2} \pm (2k_3+7k_2)\cdot(n-3)!$$

for every edge $X\in K_2(G)$ and every permutation $\pi\in\mathcal{M}_X$. On the other hand, we see that $|\mathcal{M}_X| = (n-2)! - k_{3,X}\cdot(n-3)! \pm k_{4,X}\cdot(n-4)!$, by appealing to the inclusion-exclusion formula and Observation 2.5 (i). These two bounds together imply that

$$\begin{aligned}\mathrm{dp}(\mathcal{M}_X,\mathcal{G}) = {}&(k_1-2)(n-2)!D_{n-1} - (k_1-2)k_{3,X}\cdot(n-3)!D_{n-1}\\ &- k_2(G-X)\cdot(n-2)!D'_{n-2} \pm 32k_1k_2\cdot(n-1)!(n-4)!.\end{aligned}$$

Here we used the estimates $k_{3,X}\le k_1$, $k_{4,X}\le k_2$ and the assumption that $k_1\le n$ to bound the error term. As $(n-2)!D'_{n-2} = (n-3)!D_{n-1} \pm (n-1)!(n-4)!$, this expression can be simplified further as

$$\begin{aligned}\mathrm{dp}(\mathcal{M}_X,\mathcal{G}) = {}&(k_1-2)(n-2)!D_{n-1} - ((k_1-2)k_{3,X} + k_2(G-X))\cdot(n-3)!D_{n-1}\\ &\pm 35k_1k_2\cdot(n-1)!(n-4)!.\end{aligned}$$

Summing $\mathrm{dp}(\mathcal{M}_X,\mathcal{G})$ over all $X\in K_2(G)$, and using the identity $\sum_{X\in K_2(G)} k_{3,X} = 3k_3$, we get

$$\begin{aligned}\mathrm{dp}(\mathcal{M}_2,\mathcal{G}) = {}&(k_1-2)k_2\cdot(n-2)!D_{n-1} - \Big(3(k_1-2)k_3 + \sum_{X\in K_2(G)} e(G-X)\Big)\cdot(n-3)!D_{n-1}\\ &\pm 35k_1k_2^2\cdot(n-1)!(n-4)!. \qquad \square\end{aligned}$$

Claim 2.9. $\mathrm{dp}(\mathcal{M}_1,\mathcal{G}) = \sum_{i=1}^{3} b_i\cdot(n-i)!D_{n-1} \pm 610(k_1^2k_2 + k_1^2k_3 + k_1k_2^2)\cdot(n-1)!(n-4)!$, *in which* $b_1 = k_1(k_1-1)$, $b_2 = (-3k_1+4)k_2$, $b_3 = (4k_1-6)k_3 - (k_1-2)k_2 + i(\bar{P}_3,G) + \sum_{x\in V(G)} k_{2,\{x\}}\cdot k_2(G-x)$.

Proof. Finally we come to what is, in some sense, the trickiest part of our proof. Fix a vertex $x \in V(G)$ and let $\pi \in \mathcal{T}_x$ be an arbitrary permutation. Applying (9) to $G-x$, we find

$$\begin{aligned}
\mathrm{dp}(\pi,\mathcal{G}) &= (k_1-1)D_{n-1} - k_2(G-x)D'_{n-2} \\
&+ \Big(k_3(G-x) + \sum_{(y_1,y_2)(z_1,z_2)\in E(G-x)} \big|\{y_1,z_1\}\cap\{\pi^{-1}(y_2),\pi^{-1}(z_2)\}\big|\Big)d_{n-3} \\
&\pm (250k_2+4k_3+k_4)\cdot(n-4)!.
\end{aligned}$$

On the other hand, from the inclusion-exclusion we have

$$\big|\mathcal{M}_{\{x\}}\big| = \sum_{1\le i\le 3}(-1)^{i-1}k_{i,\{x\}}\cdot(n-i)! \pm k_{4,\{x\}}\cdot(n-4)!.$$

For each edge $(y_1,y_2)(z_1,z_2)\in E(G-x)$, it is not difficult to see that

(α) If $x\notin\{(y_1,z_2),(z_1,y_2)\}$, then $|\{y_1,z_1\}\cap\{\pi^{-1}(y_2),\pi^{-1}(z_2)\}| = 0 = \mathbb{1}_{\bar{P}_3}(x,\{y,z\})$ for all permutations $\pi\in\mathcal{T}_x$;

(β) If $x\in\{(y_1,z_2),(z_1,y_2)\}$, then $|\{y_1,z_1\}\cap\{\pi^{-1}(y_2),\pi^{-1}(z_2)\}| = 1 = \mathbb{1}_{\bar{P}_3}(x,\{y,z\})$ for all permutations $\pi\in\mathcal{T}_x$, with at most $2(n-2)!$ exceptions.

Putting these facts together, and summing $\mathrm{dp}(\pi,\mathcal{G})$ over all $\pi\in\mathcal{M}_{\{x\}}$ then gives

$$\begin{aligned}
\mathrm{dp}(\mathcal{M}_{\{x\}},\mathcal{G}) = (k_1-1)\cdot(n-1)!D_{n-1} - k_2(G-x)\cdot(n-1)!D'_{n-2} - (k_1-1)k_{2,\{x\}}\cdot(n-2)!D_{n-1} \\
+(n-1)!d_{n-3}\textstyle\sum_{yz\in E(G-x)}\mathbb{1}_{\bar{P}_3}(x,\{y,z\}) \pm 2k_2\cdot 2(n-2)!d_{n-3} \\
+k_3(G-x)\cdot(n-1)!d_{n-3} + k_{2,\{x\}}\cdot k_2(G-x)\cdot(n-2)!D'_{n-2} \\
+(k_1-1)k_{3,\{x\}}\cdot(n-3)!D_{n-1} \pm 600(k_1k_3+k_2^2)\cdot(n-1)!(n-4)!.
\end{aligned}$$

Moreover, we have the identities $(n-1)!D'_{n-2} = (n-2)!D_{n-1} + (n-3)!D_{n-1} \pm (n-1)!(n-4)!$, $(n-1)!d_{n-3} = (n-3)!D_{n-1} \pm (n-1)!(n-4)!$, and $(n-2)!D'_{n-2} = (n-3)!D_{n-1} \pm (n-1)!(n-4)!$. Hence

$$\mathrm{dp}(\mathcal{M}_{\{x\}},\mathcal{G}) = \sum_{i=1}^{3} b'_i\cdot(n-i)!D_{n-1} \pm 610(k_1k_2+k_1k_3+k_2^2)\cdot(n-1)!(n-4)!,$$

where

$$\begin{aligned}
b'_1 &= k_1-1, \\
b'_2 &= -k_2(G-x)-(k_1-1)k_{2,\{x\}}, \\
b'_3 &= -k_2(G-x)+k_3(G-x)+(k_1-1)k_{3,\{x\}}+k_{2,\{x\}}\cdot k_2(G-x) + \sum_{yz\in E(G-x)}\mathbb{1}_{\bar{P}_3}(x,\{y,z\}).
\end{aligned}$$

Noting that $\sum_x k_2(G-x) = (k_1-2)k_2$, $\sum_x k_{2,\{x\}} = 2k_2$, $\sum_x k_3(G-x) = (k_1-3)k_3$, and $\sum_x k_{3,\{x\}} = 3k_3$, and summing the above estimate for $\mathrm{dp}(\mathcal{M}_{\{x\}},\mathcal{G})$ over all $x\in V(G)$, we get the desired formula for $\mathrm{dp}(\mathcal{M}_1,\mathcal{G})$. $\square$

Finally the result follows from Claims 2.6 – 2.9 by noting that $\mathrm{dp}(\mathcal{G}) = \frac{1}{2}\sum_{i\ge 1}\mathrm{dp}(\mathcal{M}_i,\mathcal{G})$, and $\sum_x k_{2,\{x\}}\cdot k_2(G-x) - \sum_{X\in K_2(G)} k_2(G-X) = k_2(k_2-1)$. $\square$

The rest of this section is devoted to showing that if $\mathcal{G}$ is a union of few cosets in S_n, then $\mathcal{G}$ has at least as many disjoint pairs as $\mathcal{T}(n,s)$, where $s=|\mathcal{G}|$. To bound the gap $\mathrm{dp}(\mathcal{G}) - \mathrm{dp}(\mathcal{T}(n,s))$, we shall use part (3) of the following lemma.

Lemma 2.10. *Let G be the intersection graph of a union $\mathcal{G}$ of cosets in S_n. Then the following properties hold.*
(1) $e(G) \geq \max\{v(G), 2v(G)-6\}$ unless one of the following cases occurs:
(i) *$\mathcal{G}$ is canonical;*
(ii) *G is isomorphic to $2K_2, P_4, P_5$ or $C_4 \cup K_1$.*
(2) If $e(G) \geq v(G)$, then

$$e(G)\,(e(G)-v(G)+1) \geq 2k_3(G)+1.$$

(3) If $\mathcal{G}$ is not canonical, then

$$e(G)\,(e(G)-v(G)+1)+i(\bar{P}_3,G)-k_3(G) \geq \tfrac{1}{50}v(G)e(G).$$

We note that parts (1) and (2) are solely used to prove part (3).

Proof of Lemma 2.10. (1) It is not difficult to verify the result for $v(G) \leq 5$. It remains to deal with the case that $v(G) \geq 6$ and $e(G) < \max\{v(G), 2v(G)-6\} = 2v(G)-6$. We wish to show that $\mathcal{G}$ is canonical. Let ℓ be an axis-aligned line that maximises $d := |\ell \cap V(G)|$. If $d \geq v(G)-1$, then $\mathcal{G}$ is canonical, as desired. If $d \leq 2$, then $d(x) \geq v(G)-3$ for every $x \in V(G)$. Hence

$$e(G) \geq \frac{1}{2}v(G)(v(G)-3) \geq 3(v(G)-3) > 2v(G)-6$$

as $v(G) \geq 6$, a contradiction. Suppose, then, that $d \in \{3,\ldots,v(G)-2\}$. Since each vertex $x \in V(G) \setminus \ell$ is incident to all but at most one vertex in ℓ, we must have

$$e(G) \geq (v(G)-d)(d-1) \geq 2(v(G)-3),$$

which contradicts the assumption that $e(G) < 2v(G)-6$.

(2) We shall use induction on $v(G)$. It is not hard to verify the statement for $v(G) \leq 6$ using the Kruskal-Katona theorem (Theorem 4.5). Now suppose $v(G) \geq 7$. Since $e(G) \geq v(G) \geq 7$, it follows from part (1) that

$$e(G) \geq 2v(G)-6 \geq v(G)+1. \tag{10}$$

(Note that $e(G) \in \{v(G)-2, v(G)-1\}$ if $\mathcal{G}$ is canonical.) Let x be a vertex of G of minimum degree. We distinguish two cases.

Case 1: x is an isolated vertex of G. In this case, vertices of G must lie entirely in the two axis-aligned lines ℓ_1 and ℓ_2 passing through x, and so $k_3(G)=0$. As a consequence,

$$e(G)\,(e(G)-v(G)+1) \overset{(10)}{\geq} (v(G)+1)\cdot 2 \geq 16 > 2k_3(G)+1.$$

Case 2: $d_G(x) \geq 1$. Let $G' := G-\{x\}$. Then as x is of minimum degree in G,

$$e(G') \geq e(G) - \frac{2e(G)}{v(G)} \overset{(10)}{>} v(G)-2.$$

Thus $e(G') \geq v(G)-1 = v(G')$, and so $e(G')(e(G')-v(G')+1) \geq 2k_3(G')+1$, by the induction hypothesis. Note that

$$\begin{aligned} & e(G)\,(e(G)-v(G)+1) = (e(G')+d_G(x))(e(G')-v(G')+1+d_G(x)-1) \\ = \; & e(G')(e(G')-v(G')+1)+e(G')(d_G(x)-1)+d_G(x)(e(G')-v(G')+1)+d_G(x)(d_G(x)-1). \end{aligned}$$

Since $d_G(x) \geq 1, e(G') \geq v(G')$ and $e(G')(e(G')-v(G')+1) \geq 2k_3(G')+1$, the right hand side of the above expression is at least $2k_3(G')+1+d_G(x)(d_G(x)-1) \geq 2k_3(G)+1$, as there are at most $\binom{d_G(x)}{2}$ triangles in G containing x.

(3) If $e(G) < v(G)$, then by part (1), G is isomorphic to $2K_2, P_4, P_5$ or $C_4 \cup K_1$. We can easily check that $e(G)\left(e(G) - v(G) + 1\right) + i(\bar{P}_3, G) - k_3(G) \geq \frac{1}{50} v(G)e(G)$ in these cases. Suppose, then, that $e(G) \geq v(G)$. If $v(G) \leq 5$, then by part (2) we have

$$e(G)\left(e(G) - v(G) + 1\right) + i(\bar{P}_3, G) - k_3(G) \geq 1 \geq \tfrac{1}{50} v(G)e(G),$$

as desired. It remains to handle the case $v(G) \geq 6$. Since $e(G) \geq v(G) \geq 6$, part (1) implies $e(G) \geq 2v(G) - 6$, and so $e(G) - v(G) + 1 \geq \frac{1}{6} v(G)$. Combined this estimate with part (2), we find

$$e(G)\left(e(G) - v(G) + 1\right) + i(\bar{P}_3, G) - k_3(G) \geq \tfrac{1}{2} e(G)(e(G) - v(G) + 1) \geq \tfrac{1}{12} v(G)e(G),$$

finishing the proof. $\square$

We shall deduce from Lemma 2.10 that if $\mathcal{G}$ is a union of $o(n^{1/2})$ cosets, then $\mathrm{dp}(\mathcal{G}) \geq \mathrm{dp}(\mathcal{T}(n, s))$ where $s := |\mathcal{G}|$.

Corollary 2.11. *There is a positive constant c such that the following holds. Let n and k be positive integers with $2 \leq k \leq cn^{1/2}$, and let $\mathcal{G} = \mathcal{T}_{(i_1,j_1)} \cup \ldots \cup \mathcal{T}_{(i_k,j_k)}$ be a union of k cosets in S_n. Then $\mathrm{dp}(\mathcal{G}) \geq \mathrm{dp}(\mathcal{T}(n, s))$, where $s := |\mathcal{G}|$. Equality holds if and only if $\mathcal{G}$ is canonical.*

Proof. Let $c > 0$ be a sufficiently small constant. We denote by G the intersection graph of $\mathcal{G}$. If $\mathcal{G}$ is canonical, then $\mathcal{G}$ is a union of $k-1$ pairwise disjoint cosets and an intersecting family, and so $\mathrm{dp}(\mathcal{G}) = \mathrm{dp}(\mathcal{T}(n, s))$. Now suppose that $\mathcal{G}$ is not canonical. By appealing to Lemma 2.4 (i), we get

$$s = |\mathcal{G}| = v(G) \cdot (n-1)! - e(G) \cdot (n-2)! + k_3(G) \cdot (n-3)! \pm v(G)^2 e(G) \cdot (n-4)!.$$

So if we write $s =: (k + \varepsilon)(n-1)!$, then

$$\varepsilon(n-1)! = s - k(n-1)! = -e(G)(n-2)! + k_3(G)(n-3)! \pm k^2 e(G)(n-4)!.$$

Using (3) with $\varepsilon < 0$ yields

$$\mathrm{dp}(\mathcal{T}(n, s)) = \sum_{1 \leq i \leq 3} b_i \cdot (n-i)! D_{n-1} \pm v(G)^3 e(G) \cdot (n-1)!(n-4)!,$$

where $b_1 = \binom{v(G)}{2}, b_2 = -(v(G)-1)e(G)$ and $b_3 = (v(G)-1)k_3(G)$. We can easily derive from this and Lemma 2.4 (iii) that

$$\begin{aligned}\mathrm{dp}(\mathcal{G}) - \mathrm{dp}(\mathcal{T}(n, s)) = \tfrac{1}{2} \cdot &\left[e(G)(e(G) - v(G) + 1) + i(\bar{P}_3, G) - k_3(G)\right](n-3)! D_{n-1} \\ &\pm 1001 v(G)^3 e(G) \cdot (n-1)!(n-4)!.\end{aligned}$$

Furthermore, we have $e(G)(e(G) - v(G) + 1) + i(\bar{P}_3, G) - k_3(G) \geq \frac{1}{50} v(G)e(G)$ due to Lemma 2.10. Therefore,

$$\mathrm{dp}(\mathcal{G}) - \mathrm{dp}(\mathcal{T}(n, s)) \geq \tfrac{1}{100} v(G)e(G) \cdot (n-3)! D_{n-1} - 1001 v(G)^3 e(G) \cdot (n-1)!(n-4)! > 0,$$

as $D_{n-1} = (e^{-1} + o(1))(n-1)!$, $e(G) \geq 1$ and $1 \leq v(G) \leq cn^{1/2}$. $\square$

2.4. **Supersaturation.** Here we shall use Lemma 2.4 and Corollary 2.11 to prove Theorem 1.1. Our strategy is to reduce the statement to the case when $\mathcal{F}$ is a union of some cosets in S_n.

Proof of Theorem 1.1. Let $c_{2.1}$ and $C_{2.1}$ be the positive constants from Lemma 2.1, and set

$$c = \min\left\{c_{2.1},\ 10^{-5} C_{2.1}^{-2},\ 10^{-2}\right\}.$$

Now letting n, k and ε be as in the statement of the theorem, let $\mathcal{F} \subseteq S_n$ be an extremal family of $s = (k + \varepsilon)(n-1)!$ permutations. Our proof splits into two parts. We first use the removal lemma to prove Claim 2.12, a rough structural result for $\mathcal{F}$. We will then bound $\mathrm{dp}(\mathcal{F})$ from below with the aid of intersection graphs.

Claim 2.12. *Either $\mathcal{F}$ contains k cosets or $\mathcal{F}$ is contained in a union of k cosets.*

Proof. Since $|\varepsilon| \le ck^{-3} \le c_{2.1}$ and $\mathrm{dp}(\mathcal{F}) \le \mathrm{dp}(\mathcal{T}(n,s))$ by the extremality of $\mathcal{F}$, we may apply Lemma 2.1 to $\mathcal{F}$ with $\beta_{2.1} = 0$ to find a union $\mathcal{G} = \bigcup_{i=1}^{k} \mathcal{T}_i$ of k cosets in S_n such that

$$|\mathcal{F}\Delta\mathcal{G}| \le C_{2.1}k^2\left(\frac{1}{n} + \sqrt{\frac{6|\varepsilon|}{k}}\right)(n-1)!. \tag{11}$$

Let $\mathcal{A} = \mathcal{F} \setminus \mathcal{G}$ and $\mathcal{B} = \mathcal{G} \setminus \mathcal{F}$. We may assume that $\mathcal{A} \neq \emptyset$ and $\mathcal{B} \neq \emptyset$, otherwise either $\mathcal{F} \subseteq \mathcal{G}$ or $\mathcal{G} \subseteq \mathcal{F}$ as claimed. We shall show that if the permutations in $\mathcal{A}$ are replaced by those in $\mathcal{B}$, the number of disjoint pairs decreases, which then contradicts the extremality of $\mathcal{F}$. Fix two arbitrary permutations $\sigma \in \mathcal{A}$ and $\pi \in \mathcal{B}$. It suffices to show that $\mathrm{dp}(\sigma,\mathcal{F}) > \mathrm{dp}(\pi,\mathcal{F})$.

First, using (11), $|\varepsilon| \le ck^{-3}$, $k \le cn^{1/2}$ and $c \le 10^{-5}C_{2.1}^{-2}$, we see that

$$|\mathcal{A}| + |\mathcal{B}| = |\mathcal{F}\Delta\mathcal{G}| \le 0.02(n-1)!. \tag{12}$$

Recall that any two cosets in S_n have at most $(n-2)!$ elements in common, and that a permutation is disjoint from D_{n-1} other permutations in any coset not containing it. Since $D_{n-1} = d_{n-1} + d_{n-2} = (e^{-1} + o(1))(n-1)!$ and $k \le cn^{1/2} \le 10^{-2}n^{1/2}$, we have

$$\begin{aligned}\mathrm{dp}(\sigma,\mathcal{F}) &\ge \mathrm{dp}(\sigma,\mathcal{G}\setminus\mathcal{B}) \ge \mathrm{dp}(\sigma,\mathcal{G}) - |\mathcal{B}| \ge \sum_{i=1}^{k}\mathrm{dp}(\sigma,\mathcal{T}_i) - \sum_{i<j}|\mathcal{T}_i \cap \mathcal{T}_j| - |\mathcal{B}| \\ &\overset{(12)}{\ge} kD_{n-1} - \binom{k}{2}(n-2)! - 0.02(n-1)! > (k-0.1)D_{n-1}.\end{aligned}$$

On the other hand, π is contained in $\mathcal{G}$, and thus can have disjoint pairs to at most $k-1$ of the cosets in $\mathcal{G}$. Hence,

$$\begin{aligned}\mathrm{dp}(\pi,\mathcal{F}) &= \mathrm{dp}(\pi,\mathcal{F}\cap\mathcal{G}) + \mathrm{dp}(\pi,\mathcal{F}\setminus\mathcal{G}) = \mathrm{dp}(\pi,\mathcal{G}\setminus\mathcal{B}) + \mathrm{dp}(\pi,\mathcal{A}) \\ &\le (k-1)D_{n-1} + |\mathcal{A}| \overset{(12)}{<} (k-0.2)D_{n-1} < \mathrm{dp}(\sigma,\mathcal{F}).\end{aligned}$$ □

We now combine this claim with Lemma 2.4 and Corollary 2.11 to finish the proof. We consider two cases, depending on the sign of ε.

Case 1: $\varepsilon \le 0$. We have shown in Claim 2.12 that either $\mathcal{F} \supseteq \mathcal{G}$ or $\mathcal{F} \subseteq \mathcal{G}$, where $\mathcal{G}$ is a union of some k cosets. Let $t = |\mathcal{G}|$.

We first treat the case $\mathcal{F} \supseteq \mathcal{G}$. From Corollary 2.11 we find $\mathrm{dp}(\mathcal{G}) \ge \mathrm{dp}(\mathcal{T}(n,t))$. We can choose $\mathcal{T}(n,t)$ to be a subfamily of $\mathcal{T}(n,s)$. Since $s = (k+\varepsilon)(n-1)! \le k(n-1)!$, the family $\mathcal{T}(n,s)$ is contained in a union of k disjoint cosets in S_n. Hence $\mathrm{dp}(\pi,\mathcal{T}(n,s)) \le (k-1)D_{n-1}$ for every $\pi \in \mathcal{T}(n,s)\setminus\mathcal{T}(n,t)$, and there are $|\mathcal{F}\setminus\mathcal{G}|$ such permutations π.

Moreover, as $\mathcal{G}$ is a union of k cosets, we have

$$\mathrm{dp}(\sigma,\mathcal{G}) \ge kD_{n-1} - \binom{k}{2}(n-2)! > (k-0.5)D_{n-1}$$

for each $\sigma \in \mathcal{F}\setminus\mathcal{G}$, and there are again $|\mathcal{F}\setminus\mathcal{G}|$ such permutations σ. Altogether, we deduce that, as required,

$$\mathrm{dp}(\mathcal{F}) \ge \mathrm{dp}(\mathcal{G}) + \sum_{\sigma\in\mathcal{F}\setminus\mathcal{G}}\mathrm{dp}(\sigma,\mathcal{G}) > \mathrm{dp}(\mathcal{T}(n,t)) + \sum_{\pi\in\mathcal{T}(n,s)\setminus\mathcal{T}(n,t)}\mathrm{dp}(\pi,\mathcal{T}(n,s)) \ge \mathrm{dp}(\mathcal{T}(n,s)).$$

We next deal with the case $\mathcal{F} \subseteq \mathcal{G}$. It is convenient to think of $\mathcal{F}$ as a family obtained by removing permutations in $\mathcal{G}$ one by one. Since $\mathcal{G}$ is a union of k cosets in S_n, the number of disjoint pairs is decreased by at most $(k-1)D_{n-1}$ each time. Following the same process for the family $\mathcal{T}(n,t)$, we see that the number of disjoint pairs is decreased by exactly $(k-1)D_{n-1}$ each time we remove a permutation from the last coset in $\mathcal{T}(n,t)$. Moreover, at the beginning of the process, $\mathrm{dp}(\mathcal{G}) \geq \mathrm{dp}(\mathcal{T}(n,t))$ by Corollary 2.11. Thus

$$\mathrm{dp}(\mathcal{F}) \geq \mathrm{dp}(\mathcal{G}) - (t-s)(k-1)D_{n-1} \geq \mathrm{dp}(\mathcal{T}(n,t)) - (t-s)(k-1)D_{n-1} = \mathrm{dp}(\mathcal{T}(n,s)),$$

completing the proof in this case.

Case 2: $\varepsilon > 0$. This case will be handled rather differently. Since $\varepsilon > 0$, formula (3) gives

$$\mathrm{dp}(\mathcal{T}(n,s)) = \left(\binom{k}{2} + k\varepsilon\right)(n-1)!D_{n-1}. \tag{13}$$

Also, as $|\mathcal{F}| = (k+\varepsilon)(n-1)! > k(n-1)!$, Claim 2.12 shows that $\mathcal{F} = \mathcal{G} \sqcup \mathcal{H}$, where $\mathcal{G}$ is a union of k (not necessarily disjoint) cosets in S_n.

If $\mathcal{G}$ is a union of k disjoint cosets, then

$$\mathrm{dp}(\mathcal{F}) = \mathrm{dp}(\mathcal{G}) + \mathrm{dp}(\mathcal{H},\mathcal{G}) + \mathrm{dp}(\mathcal{H}) \geq \mathrm{dp}(\mathcal{G}) + \mathrm{dp}(\mathcal{H},\mathcal{G}) = \mathrm{dp}(\mathcal{T}(n,s)),$$

where equality holds if and only if $\mathrm{dp}(\mathcal{H}) = 0$, that is, $\mathcal{H}$ is intersecting.

It remains to verify that $\mathrm{dp}(\mathcal{F}) \geq \mathrm{dp}(\mathcal{T}(n,s))$ when the k cosets of $\mathcal{G}$ are not pairwise disjoint. In this scenario we in fact have a strict inequality. Indeed, let G be the intersection graph of $\mathcal{G}$. We shall use the inequality $\mathrm{dp}(\mathcal{F}) \geq \mathrm{dp}(\mathcal{G}) + \mathrm{dp}(\mathcal{H},\mathcal{G})$ to lower bound $\mathrm{dp}(\mathcal{F})$. From Lemma 2.4 (iii) we have

$$\mathrm{dp}(\mathcal{G}) = \binom{k}{2}(n-1)!D_{n-1} - (k-1)k_2(n-2)!D_{n-1} \pm 3k^2k_2(n-1)!(n-3)!. \tag{14}$$

We next estimate the number of disjoint pairs between $\mathcal{H}$ and $\mathcal{G}$. By Lemma 2.4 (ii),

$$\mathrm{dp}(\pi,\mathcal{G}) = kD_{n-1} - k_2D'_{n-2} \pm 4kk_2(n-3)!$$

for every $\pi \in \mathcal{H}$. Furthermore, Lemma 2.4 (i) gives $|\mathcal{G}| = k(n-1)! - k_2(n-2)! \pm 2kk_2(n-3)!$, which forces

$$|\mathcal{H}| = |\mathcal{F}| - |\mathcal{G}| = \varepsilon(n-1)! + k_2(n-2)! \pm 2kk_2(n-3)!.$$

Therefore, noting that $(n-1)!D'_{n-2} = (n-2)!D_{n-1} \pm (n-1)!(n-3)!$, we get

$$\begin{aligned}\mathrm{dp}(\mathcal{H},\mathcal{G}) &= \left(kD_{n-1} - k_2D'_{n-2} \pm 4kk_2(n-3)!\right)|\mathcal{H}| \\ &= k\varepsilon(n-1)!D_{n-1} + (k-\varepsilon)k_2(n-2)!D_{n-1} \pm 4k^2k_2(n-1)!(n-3)!. \end{aligned} \tag{15}$$

Combining (13), (14) and (15), and simplifying gives

$$\mathrm{dp}(\mathcal{G}) + \mathrm{dp}(\mathcal{H},\mathcal{G}) - \mathrm{dp}(\mathcal{T}(n,s)) \geq (1-\varepsilon)k_2(n-2)!D_{n-1} - 4k^2k_2(n-1)!(n-3)! > 0,$$

since $k_2 \geq 1$, $k \leq cn^{1/2} \leq 10^{-2}n^{1/2}$ and $D_{n-1} \geq (n-1)!/3$. Thus $\mathrm{dp}(\mathcal{F}) \geq \mathrm{dp}(\mathcal{G}) + \mathrm{dp}(\mathcal{H},\mathcal{G}) > \mathrm{dp}(\mathcal{T}(n,s))$, giving the desired estimate. This completes the proof of Theorem 1.1. □

3. Supersaturation for uniform set systems

In this section, we shall prove Theorem 1.2, but first let us examine $\mathrm{dp}(n,k,s)$. When $\binom{n}{k}-\binom{n-r+1}{k} \le s \le \binom{n}{k}-\binom{n-r}{k}$, if we write $s=\binom{n}{k}-\binom{n-r+1}{k}+\gamma\binom{n-r}{k-1}$ where $\gamma\in[0,1]$, $\mathcal{L}(n,k,s)$ consists of the full stars with centres in $[r-1]$, with a further $\gamma\binom{n-r}{k-1}$ sets from the star with centre r. Let $\mathcal{L}(i)=\{L\in\mathcal{L}(n,k,s): i\in L\}$ and $\mathcal{L}^*(i)=\{L\in\mathcal{L}(n,k,s): \min L = i\}$. One can then compute the number of disjoint pairs as
(16)

$$\mathrm{dp}(n,k,s)=\sum_{i=1}^{r-1}\mathrm{dp}(\cup_{j>i}\mathcal{L}^*(j),\mathcal{L}^*(i))=\sum_{i=1}^{r-1}\left(s-\left(\binom{n}{k}-\binom{n-i}{k}\right)\right)\binom{n-i-k}{k-1}.$$

This expression is quite unwieldy, so we shall make use of a few estimates. We first note that any set outside a star has exactly $\binom{n-k-1}{k-1}$ disjoint pairs with the star, so

$$\begin{aligned}\mathrm{dp}(n,k,s) &\le \sum_{1\le i<j\le r-1}\mathrm{dp}(\mathcal{L}(i),\mathcal{L}(j))+\sum_{1\le i\le r-1}\mathrm{dp}(\mathcal{L}(i),\mathcal{L}^*(r))\\ &\le\left(\binom{r-1}{2}+(r-1)\gamma\right)\binom{n-1}{k-1}\binom{n-k-1}{k-1}.\end{aligned}\tag{17}$$

This is only an upper bound as we overcount disjoint pairs involving sets belonging to multiple stars. For an even simpler upper bound, observe that every set belongs to at least one of the r stars, and is not disjoint from any other set in its star. In the worst case, there are an equal number of sets in each star, with each set disjoint from at most a $\left(1-\frac{1}{r}\right)$-proportion of the family. We thus have

$$\mathrm{dp}(n,k,s)\le\frac{1}{2}\left(1-\frac{1}{r}\right)s^2.\tag{18}$$

These give upper bounds on the number of disjoint pairs present in any extremal family.

3.1. **Tools.** There are two main tools we use in our proof of Theorem 1.2: a removal lemma for disjoint pairs, and the expander-mixing lemma applied to the Kneser graph. Before proving the theorem, we introduce these tools and explain how we shall use them.

3.1.1. *Removal lemma.* Using a result of Filmus [20], Das and Tran [11, Theorem 1.2] proved the following removal lemma, showing that large families with few disjoint pairs must be close to a union of stars.

Lemma 3.1 (Das and Tran)**.** *There is an absolute constant $C>1$ such that if n,k and ℓ are positive integers satisfying $n>2k\ell^2$, and $\mathcal{F}\subset\binom{[n]}{k}$ is a family of size $|\mathcal{F}|=(\ell-\alpha)\binom{n-1}{k-1}$ with at most $\left(\binom{\ell}{2}+\beta\right)\binom{n-1}{k-1}\binom{n-k-1}{k-1}$ disjoint pairs, where* $\max\{2\ell\,|\alpha|\,,|\beta|\}\le\frac{n-2k}{(20C)^2n}$*, then there is a family $\mathcal{S}$ that is the union of ℓ stars satisfying*

$$|\mathcal{F}\Delta\mathcal{S}|\le C\left((2\ell-1)\alpha+2\beta\right)\frac{n}{n-2k}\binom{n-1}{k-1}.$$

Observe that the bound on the number of disjoint pairs in the lemma is very similar to the upper bound given in (17). Thus one may interpret this result as a stability version of our previous calculation: any family with size similar to the union of $r-1$ stars without many more disjoint pairs can be made a union of $r-1$ stars by exchanging only a small number of sets. Given this stability, it is not difficult to show that the lexicographic ordering is optimal in this range.

Corollary 3.2. *There is some constant $c > 0$ such that if r, k and n are positive integers satisfying $n \geq 2c^{-1}k^2r^3$, and $s = \binom{n}{k} - \binom{n-r+1}{k} + \gamma\binom{n-r}{k-1}$, where $\gamma \in [0, \frac{c}{r}]$, any family $\mathcal{F} \subset \binom{[n]}{k}$ of size s has* $\mathrm{dp}(\mathcal{F}) \geq \mathrm{dp}(n,k,s)$.

Proof. Let C be the constant from Lemma 3.1 and choose $c = \frac{n-2k}{2(20C)^2 n}$. For the given range of s, the lexicographic initial segment has $r-1$ full stars with one small partial star, so we wish to apply Lemma 3.1 with $\ell = r-1$.

Let $\mathcal{F}$ be a subfamily of $\binom{[n]}{k}$ with s sets and the minimum number of disjoint pairs. Note that $s = (\ell+\alpha)\binom{n-1}{k-1}$, where $\gamma - \frac{kr^2}{2n} \leq \alpha \leq \gamma$. In particular, we have $|\alpha| \leq \frac{c}{r}$. By optimality of $\mathcal{F}$, and our calculation in (17), we also have $\mathrm{dp}(\mathcal{F}) \leq \mathrm{dp}(n,k,s) \leq \left(\binom{\ell}{2} + (r-1)\gamma\right)\binom{n-1}{k-1}\binom{n-k-1}{k-1}$, and hence take $\beta = (r-1)\gamma$.

We thus have $|\beta| = (r-1)\gamma \leq c < \frac{n-2k}{(20C)^2 n}$ and $2\ell|\alpha| \leq 2c = \frac{n-2k}{(20C)^2 n}$, and hence we may apply Lemma 3.1. This gives a family $\mathcal{S}$, a union of ℓ stars, such that

$$|\mathcal{F}\Delta\mathcal{S}| \leq C\left((2\ell-1)\alpha + 2\beta\right)\frac{n}{n-2k}\binom{n-1}{k-1} \leq 4cC\frac{n}{n-2k}\binom{n-1}{k-1} < \frac{1}{200}\binom{n-1}{k-1}.$$

Hence, we know an optimal family $\mathcal{F}$ must be close to a union of ℓ stars $\mathcal{S}$. We first show that $\mathcal{S} \subseteq \mathcal{F}$. If not, there is some set $F \in \mathcal{F} \setminus \mathcal{S}$ in our family, as well as a set $G \in \mathcal{S} \setminus \mathcal{F}$ missing from our family (note that $|\mathcal{F}| \geq |\mathcal{S}|$). For each star in $\mathcal{S}$, there are at most $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} \leq \frac{k^2}{n}\binom{n-1}{k-1}$ sets intersecting F, and hence F intersects at most $\frac{\ell k^2}{n}\binom{n-1}{k-1}$ sets from $\mathcal{F} \cap \mathcal{S}$. Even if F intersects every set in $\mathcal{F} \setminus \mathcal{S}$, it can intersect at most $\left(\frac{\ell k^2}{n} + \frac{1}{200}\right)\binom{n-1}{k-1} < \frac{1}{2}\binom{n-1}{k-1}$ sets in $\mathcal{F}$.

On the other hand, the set G is in one of the stars of $\mathcal{S}$, which contains at least $\binom{n-1}{k-1} - |\mathcal{S} \setminus \mathcal{F}| \geq \left(1 - \frac{1}{200}\right)\binom{n-1}{k-1} > \frac{2}{3}\binom{n-1}{k-1}$ sets of $\mathcal{F}$. Hence replacing F by G in $\mathcal{F}$ strictly increases the number of intersecting pairs, thus decreasing the number of disjoint pairs, contradicting the optimality of $\mathcal{F}$.

Thus we have $\mathcal{S} \subseteq \mathcal{F}$. Let $\mathcal{H} = \mathcal{F} \setminus \mathcal{S}$. We then have

$$\mathrm{dp}(\mathcal{F}) = \mathrm{dp}(\mathcal{S}) + \mathrm{dp}(\mathcal{S},\mathcal{H}) + \mathrm{dp}(\mathcal{H}).$$

Since every set outside a union of ℓ stars is contained in exactly the same number of disjoint pairs with sets from the stars, the terms $\mathrm{dp}(\mathcal{S})$ and $\mathrm{dp}(\mathcal{S},\mathcal{H})$ are determined by ℓ and s, and independent of the structure of $\mathcal{F}$. It follows that $\mathrm{dp}(\mathcal{F})$ is minimised precisely when $\mathrm{dp}(\mathcal{H})$ is minimised. As $|\mathcal{H}| = |\mathcal{F}| - |\mathcal{S}| = \gamma\binom{n-r}{k-1} \leq \binom{n-r}{k-1}$, we may take $\mathcal{H}$ to be an intersecting family, and so $\mathrm{dp}(\mathcal{H}) = 0$ is possible. Since in $\mathcal{L}(n,k,s)$, the set $\mathcal{H}$ corresponds to the final (intersecting) partial star, it follows that $\mathcal{L}(n,k,s)$ is optimal, and so $\mathrm{dp}(\mathcal{F}) \geq \mathrm{dp}(n,k,s)$ for any family $\mathcal{F}$ of s sets. □

3.1.2. *Expander-mixing lemma.* The second tool we shall use is the expander-mixing lemma[2] of Alon and Chung [1], which relates the spectral gap of a d-regular graph to its edge distribution. In what follows, an (n,d,λ)-graph is a d-regular n-vertex graph whose largest non-trivial eigenvalue (in absolute value) is λ.

Lemma 3.3 (Alon and Chung)**.** *Let G be an (n,d,λ)-graph, and let S, T be two vertex subsets. Then*

$$\left|e(S,T) - \frac{d\,|S|\,|T|}{n}\right| \leq \lambda\sqrt{|S|\,|T|}.$$

[2]This also plays a key role in the proof of Lemma 3.1, and so in some sense is the foundation for this entire proof.

As we are interested in counting disjoint pairs, we shall apply the expander-mixing lemma to the Kneser graph, where the vertices are sets and edges represent disjoint pairs. The spectral properties of the Kneser graph were determined by Lovász [34]. In particular, the Kneser graph $KG(m, a)$ for a-uniform sets over $[m]$ is an $\left(\binom{m}{a}, \binom{m-a}{a}, \binom{m-a-1}{a-1}\right)$-graph. We shall combine this with Lemma 3.3 to obtain a useful corollary.

Corollary 3.4. *Given $1 \leq i < j \leq n$, and k-uniform families $\mathcal{F}(i)$ of subsets of $[n]$ containing i and $\mathcal{F}(j)$ of subsets of $[n]$ containing j,*

$$\mathrm{dp}(\mathcal{F}(i), \mathcal{F}(j)) \geq \left(1 - \frac{k^2}{n}\right) |\mathcal{F}(i)| \, |\mathcal{F}(j)| - \frac{3k}{2n} \left(|\mathcal{F}(i)| + |\mathcal{F}(j)|\right) \binom{n-1}{k-1}.$$

Proof. Without loss of generality we assume $i = n-1$ and $j = n$. Let $\mathcal{A} = \{F \setminus \{n-1\} : F \in \mathcal{F}(n-1),\ n \notin F\}$ and $\mathcal{B} = \{F \setminus \{n\} : F \in \mathcal{F}(n),\ n-1 \notin F\}$, and observe that $\mathrm{dp}(\mathcal{F}(n-1), \mathcal{F}(n)) = \mathrm{dp}(\mathcal{A}, \mathcal{B})$. Furthermore, we have $\mathcal{A}, \mathcal{B} \subseteq \binom{[n-2]}{k-1}$, with $|\mathcal{A}| \geq |\mathcal{F}(n-1)| - \binom{n-2}{k-2}$ and $|\mathcal{B}| \geq |\mathcal{F}(n)| - \binom{n-2}{k-2}$.

Since disjoint pairs between $\mathcal{A}$ and $\mathcal{B}$ correspond to edges between the corresponding vertex sets in the Kneser graph $KG(n-2, k-1)$, Lemma 3.3 gives

$$\mathrm{dp}(\mathcal{F}(n-1), \mathcal{F}(n)) = \mathrm{dp}(\mathcal{A}, \mathcal{B}) \geq \frac{\binom{n-k-1}{k-1}}{\binom{n-2}{k-1}} |\mathcal{A}| \, |\mathcal{B}| - \binom{n-k-2}{k-2} \sqrt{|\mathcal{A}| \, |\mathcal{B}|}.$$

We now recall that $|\mathcal{F}(n-1)| - \binom{n-2}{k-2} \leq |\mathcal{A}| \leq |\mathcal{F}(n-1)|$, with similar bounds holding for $\mathcal{B}$. We shall also remove the square root by appealing to the AM-GM inequality. Also observe that $\binom{n-k-1}{k-1} \geq \left(1 - \frac{k^2}{n}\right) \binom{n-2}{k-1}$ and $\binom{n-k-2}{k-2} \leq \frac{k}{n} \binom{n-1}{k-1}$. Hence $\mathrm{dp}(\mathcal{F}(n-1), \mathcal{F}(n))$ is at least

$$\left(1 - \frac{k^2}{n}\right) \left(|\mathcal{F}(n-1)| - \binom{n-2}{k-2}\right) \left(|\mathcal{F}(n)| - \binom{n-2}{k-2}\right) - \frac{k}{2n} \left(|\mathcal{F}(n-1)| + |\mathcal{F}(n)|\right) \binom{n-1}{k-1}.$$

Noting that $\binom{n-2}{k-2} \leq \frac{k}{n} \binom{n-1}{k-1}$, taking the main order term and collecting the negative terms then gives the desired bound. $\square$

3.2. **Proof of Theorem 1.2.** With the preliminaries in place, we now proceed with the proof of the main theorem.

Proof of Theorem 1.2. We prove the result by induction on s. For the base case, if $s \leq \binom{n}{k} - \binom{n-1}{k} = \binom{n-1}{k-1}$, then $\mathcal{L}(n, k, s)$ consists of sets that all contain the element 1. Hence $\mathrm{dp}(\mathcal{L}(n, k, s)) = 0$, which is clearly optimal.

For the induction step, we have $s = \binom{n}{k} - \binom{n-r+1}{k} + \gamma \binom{n-r}{k-1}$ for some $r \geq 2$ and $\gamma \in (0, 1]$. Letting c be the positive constant from Corollary 3.2, if $\gamma \in (0, \frac{c}{r}]$, we are done. Hence we may assume $\gamma \in (\frac{c}{r}, 1]$. Let $\mathcal{F}$ be a k-uniform set family over $[n]$ of size s with the minimum number of disjoint pairs. In particular, we must have $\mathrm{dp}(\mathcal{F}) \leq \mathrm{dp}(n, k, s)$.

For any set $F \in \mathcal{F}$, by the induction hypothesis we have $\mathrm{dp}(\mathcal{F} \setminus \{F\}) \geq \mathrm{dp}(n, k, s-1)$. Hence $\mathrm{dp}(\{F\}, \mathcal{F}) = \mathrm{dp}(\mathcal{F}) - \mathrm{dp}(\mathcal{F} \setminus \{F\}) \leq \mathrm{dp}(n, k, s) - \mathrm{dp}(n, k, s-1)$, where the right-hand side is the number of disjoint pairs involving the last set added to $\mathcal{L}(n, k, s)$. This set is in a star of size $\gamma \binom{n-r}{k-1} > \frac{\gamma}{2} \binom{n-1}{k-1}$, and hence intersects at least $\frac{\gamma}{2} \binom{n-1}{k-1}$ sets in $\mathcal{L}(n, k, s)$. Thus it follows that every set $F \in \mathcal{F}$ must also intersect at least $\frac{\gamma}{2} \binom{n-1}{k-1}$ sets in $\mathcal{F}$.

Now suppose that $\mathcal{F}$ contains a full star; without loss of generality, assume $\mathcal{F}(1)$ consists of all $\binom{n-1}{k-1}$ sets containing the element 1. Let $\mathcal{G} = \mathcal{F} \setminus \mathcal{F}(1)$. Since $\mathcal{F}(1)$ is intersecting, and every set outside $\mathcal{F}(1)$ has exactly $\binom{n-k-1}{k-1}$ disjoint pairs with sets in $\mathcal{F}(1)$, we have

$$\mathrm{dp}(\mathcal{F}) = \mathrm{dp}(\mathcal{F}(1), \mathcal{G}) + \mathrm{dp}(\mathcal{G}) = |\mathcal{G}| \binom{n-k-1}{k-1} + \mathrm{dp}(\mathcal{G}).$$

Now $\mathcal{G}$ is a k-uniform set family over $[n] \setminus \{1\}$ of size $s' = s - \binom{n-1}{k-1}$, and so by induction $\mathrm{dp}(\mathcal{G})$ is minimised by the initial segment of the lexicographic order of size s'. However, adding back the full star $\mathcal{F}(1)$ gives the initial segment of the lexicographic order of size s, and as a result $\mathrm{dp}(\mathcal{F}) \geq \mathrm{dp}(\mathcal{L}(n,k,s)) = \mathrm{dp}(n,k,s)$.

Hence we may assume that $\mathcal{F}$ does not contain any full star. In particular, this means for any set $F \in \mathcal{F}$ and element $i \in [n]$, we have the freedom to replace F with some set containing i. We shall use such switching operations to show that $\mathcal{F}$, like $\mathcal{L}(n,k,s)$, must have a cover of size r, from which the result will easily follow.

Relabel the elements if necessary so that for every $i \in [n]$, i is the vertex of maximum degree in $\mathcal{F}|_{[n]\setminus[i-1]}$. Let $\mathcal{F}^*(i) = \{F \in \mathcal{F} : \min F = i\}$ be those sets containing i that do not contain any previous element. Define

$$X = \left\{ x \in [n] : |\mathcal{F}^*(x)| \geq \frac{\gamma}{4k} \binom{n-1}{k-1} \right\},$$

and let $\mathcal{F}_1 = \{F \in \mathcal{F} : F \cap X \neq \emptyset\}$ and $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1 = \{F \in \mathcal{F} : F \cap X = \emptyset\}$. We shall show that X is a cover for $\mathcal{F}$ (that is, $\mathcal{F}_1 = \mathcal{F}$ and $\mathcal{F}_2 = \emptyset$), but to do so we shall first have to establish a few claims. The first shows that X cannot be too big.

Claim 3.5. $|X| \leq \frac{4kr}{\gamma}$.

Proof. Observe that the families $\{\mathcal{F}^*(x) : x \in X\}$ partition $\mathcal{F}_1$. Hence we have

$$r\binom{n-1}{k-1} \geq s = |\mathcal{F}| \geq |\mathcal{F}_1| = \sum_{x \in X} |\mathcal{F}^*(x)| \geq \frac{\gamma}{4k}\binom{n-1}{k-1} |X|,$$

from which the claim immediately follows. □

The next claim asserts that every set in $\mathcal{F}$ must intersect many sets in $\mathcal{F}_1$.

Claim 3.6. *Every set $F \in \mathcal{F}$ intersects at least $\frac{\gamma}{4}\binom{n-1}{k-1}$ sets in $\mathcal{F}_1$.*

Proof. First observe that any element $i \in [n]$ is contained in fewer than $\frac{\gamma}{4k}\binom{n-1}{k-1}$ sets in $\mathcal{F}_2$. Indeed, the elements $x \in X$ have all of their sets in $\mathcal{F}_1$, and hence have $\mathcal{F}_2$-degree zero. Thus the $\mathcal{F}_2$-degree of any element is its degree in $\mathcal{F}|_{[n]\setminus X}$. If the element of largest $\mathcal{F}_2$-degree was contained in at least $\frac{\gamma}{4k}\binom{n-1}{k-1}$ sets from $\mathcal{F}_2$, then it would have been in X, giving a contradiction.

Now recall that every set $F \in \mathcal{F}$ must intersect at least $\frac{\gamma}{2}\binom{n-1}{k-1}$ sets in $\mathcal{F}$. The number of sets in $\mathcal{F}_2$ it can intersect is at most

$$\sum_{i \in F} |\mathcal{F}_2(i)| \leq k \cdot \frac{\gamma}{4k}\binom{n-1}{k-1} = \frac{\gamma}{4}\binom{n-1}{k-1}.$$

Hence the remaining $\frac{\gamma}{4}\binom{n-1}{k-1}$ intersections must come from sets in $\mathcal{F}_1$. □

The following claim combines our previous results with the expander-mixing corollary to provide much sharper bounds on the size of X.

Claim 3.7. $|X| \leq \frac{8r}{\gamma}$.

Proof. For every $i \in X$, we shall estimate $\mathrm{dp}(\mathcal{F}^*(i), \mathcal{F}_1)$. Since $\{\mathcal{F}^*(x) : x \in X\}$ is a partition of $\mathcal{F}_1$, we have $\mathrm{dp}(\mathcal{F}^*(i), \mathcal{F}_1) = \sum_{j \in X \setminus \{i\}} \mathrm{dp}(\mathcal{F}^*(i), \mathcal{F}^*(j))$. Applying Corollary 3.4, we get

$$\begin{aligned}
\mathrm{dp}(\mathcal{F}^*(i), \mathcal{F}_1) &= \sum_{j \in X \setminus \{i\}} \mathrm{dp}(\mathcal{F}^*(i), \mathcal{F}^*(j)) \\
&\geq \sum_{j \in X \setminus \{i\}} \left[\left(1 - \frac{k^2}{n}\right) |\mathcal{F}^*(i)|\, |\mathcal{F}^*(j)| - \frac{3k}{2n} \left(|\mathcal{F}^*(i)| + |\mathcal{F}^*(j)|\right) \binom{n-1}{k-1} \right] \\
&\geq \left(1 - \frac{k^2}{n}\right) \left(|\mathcal{F}_1| - |\mathcal{F}^*(i)|\right) |\mathcal{F}^*(i)| - \frac{3k}{2n} \left(|X|\, |\mathcal{F}^*(i)| + |\mathcal{F}_1|\right) \binom{n-1}{k-1}.
\end{aligned}$$

By averaging, some $F \in \mathcal{F}^*(i)$ is disjoint from at least

$$\left(1 - \frac{k^2}{n}\right) \left(|\mathcal{F}_1| - |\mathcal{F}^*(i)|\right) - \frac{3k}{2n} \left(|X| + \frac{|\mathcal{F}_1|}{|\mathcal{F}^*(i)|}\right) \binom{n-1}{k-1}$$

sets in $\mathcal{F}_1$. By Claim 3.5, $|X| \leq \frac{4kr}{\gamma}$. Since $|\mathcal{F}_1| \leq s \leq r\binom{n-1}{k-1}$, and $|\mathcal{F}^*(i)| \geq \frac{\gamma}{4k}\binom{n-1}{k-1}$, we can lower bound this expression by

$$\mathrm{dp}(\{F\}, \mathcal{F}_1) \geq \left(1 - \frac{k^2}{n}\right) \left(|\mathcal{F}_1| - |\mathcal{F}^*(i)|\right) - \frac{12k^2 r}{\gamma n} \binom{n-1}{k-1}.$$

Recalling that $\gamma \geq \frac{c}{r}$, we find that F intersects at most

$$|\mathcal{F}_1| - \mathrm{dp}(\{F\}, \mathcal{F}_1) \leq |\mathcal{F}^*(i)| + \frac{k^2}{n} |\mathcal{F}_1| + \frac{12k^2 r}{\gamma n} \binom{n-1}{k-1} \leq |\mathcal{F}^*(i)| + \frac{13k^2 r^2}{cn} \binom{n-1}{k-1}$$

sets from $\mathcal{F}_1$. By Claim 3.6, this quantity must be at least $\frac{\gamma}{4}\binom{n-1}{k-1}$, which gives

$$|\mathcal{F}^*(i)| \geq \left(\frac{\gamma}{4} - \frac{13k^2 r^2}{cn}\right) \binom{n-1}{k-1} \geq \frac{\gamma}{8} \binom{n-1}{k-1},$$

since $n > Ck^2 r^3$ for some large enough constant C.

Hence for every $i \in X$, we in fact have the much stronger bound $|\mathcal{F}^*(i)| \geq \frac{\gamma}{8}\binom{n-1}{k-1}$. Repeating the calculation of Claim 3.5 with this new bound gives $|X| \leq \frac{8r}{\gamma}$, as required. $\square$

Our next claim shows that X is indeed a cover for $\mathcal{F}$.

Claim 3.8. *X is a cover for $\mathcal{F}$; that is, $\mathcal{F}_1 = \mathcal{F}$ and $\mathcal{F}_2 = \emptyset$.*

Proof. Suppose for contradiction we had some set $F \in \mathcal{F}_2$. By Claim 3.6, at least $\frac{\gamma}{4}\binom{n-1}{k-1}$ sets in $\mathcal{F}_1$ must intersect F. However, each such set must contain at least one element of X, which by Claim 3.7 has size at most $\frac{8r}{\gamma}$, together with one element from F. Hence there are at most $k\,|X|\binom{n-2}{k-2} \leq \frac{8k^2 r}{\gamma n}\binom{n-1}{k-1}$ sets in $\mathcal{F}_1$ intersecting F. Since $\gamma \geq \frac{c}{r}$ and $n > Ck^2 r^3$ for some large enough constant C, this is less than $\frac{\gamma}{4}\binom{n-1}{k-1}$, giving the desired contradiction. $\square$

Now observe that every set in $\mathcal{F}^*(i)$ meets X in the element i. If it intersects X in further elements, there are at most $|X| \leq \frac{8r}{\gamma}$ choices from the other element, and at most $\binom{n-2}{k-2} \leq \frac{k}{n}\binom{n-1}{k-1}$ choices for the rest of the set. Hence at most $\frac{8kr}{\gamma n}\binom{n-1}{k-1} < \frac{\gamma}{8}\binom{n-1}{k-1} \leq |\mathcal{F}^*(i)|$ sets in $\mathcal{F}^*(i)$ meet X in at least two elements, and thus there must be some set $F_i \in \mathcal{F}^*(i)$ such that $F_i \cap X = \{i\}$. We shall use this fact to establish the following claim.

Claim 3.9. *For all $i, j \in X$, $|\mathcal{F}^*(j)| - |\mathcal{F}^*(i)| \leq \frac{8k^2 r}{\gamma n}\binom{n-1}{k-1}$.*

Proof. Suppose for contradiction $|\mathcal{F}^*(j)| > |\mathcal{F}^*(i)| + \frac{8k^2r}{\gamma n}\binom{n-1}{k-1}$. Let $F_i \in \mathcal{F}^*(i)$ be such that $F_i \cap X = \{i\}$. Then F_i intersects only those sets containing i together with sets containing some other element in X and some element in F_i. This gives a total of at most

$$|\mathcal{F}^*(i)| + k\,|X|\binom{n-2}{k-2} \leq |\mathcal{F}^*(i)| + \frac{8k^2r}{\gamma n}\binom{n-1}{k-1} < |\mathcal{F}^*(j)|$$

sets. On the other hand, if we replace F_i by some set G containing j (which we may do, since we assume the family $\mathcal{F}(j)$ is not a full star), we would gain at least $|\mathcal{F}^*(j)|$ intersecting pairs. Hence $\mathcal{F} \cup \{G\} \setminus \{F_i\}$ is a family of s sets with strictly fewer disjoint pairs, contradicting the optimality of $\mathcal{F}$. □

This claim shows that the sets in $\mathcal{F}$ are roughly equally distributed over the families $\mathcal{F}^*(i)$, $i \in X$. To simplify the notation, we let $m = |X|$, and so we have $X = [m]$. By Claim 3.7, $m \leq \frac{8r}{\gamma}$. We shall now proceed to lower-bound the number of disjoint pairs in $\mathcal{F}$. Note that $\mathrm{dp}(\mathcal{F}) = \sum_{1 \leq i < j \leq m} \mathrm{dp}(\mathcal{F}^*(i), \mathcal{F}^*(j))$. We shall use Corollary 3.4 to bound these summands. We let $s_i = |\mathcal{F}^*(i)|\binom{n-1}{k-1}^{-1}$ and set $\overline{s} = s\binom{n-1}{k-1}^{-1} = \sum_i s_i$. Note that $s = \binom{n}{k} - \binom{n-r+1}{k} + \gamma\binom{n-r}{k-1}$ and $\gamma \in \left[\frac{c}{r}, 1\right]$ implies $r - 1 \leq r - 1 + \gamma - \frac{kr^2}{2n} \leq \overline{s} \leq r$.

We then have

$$\begin{aligned}
\mathrm{dp}(\mathcal{F}) &= \sum_{1 \leq i < j \leq m} \mathrm{dp}(\mathcal{F}^*(i), \mathcal{F}^*(j)) \\
&\geq \sum_{1 \leq i < j \leq m} \left[\left(1 - \frac{k^2}{n}\right)|\mathcal{F}^*(i)|\,|\mathcal{F}^*(j)| - \frac{3k}{2n}\left(|\mathcal{F}^*(i)| + |\mathcal{F}^*(j)|\right)\binom{n-1}{k-1}\right] \\
&\geq \left[\left(1 - \frac{k^2}{n}\right)\sum_{i<j} s_i s_j - \frac{3k}{2n}\sum_{i<j}(s_i + s_j)\right]\binom{n-1}{k-1}^2 \\
&\geq \left[\frac{1}{2}\left(1 - \frac{k^2}{n}\right)\left(\overline{s}^2 - \sum_i s_i^2\right) - \frac{3km\overline{s}}{2n}\right]\binom{n-1}{k-1}^2 \\
&\geq \frac{1}{2}\left[\overline{s}^2 - \frac{k^2\overline{s}^2}{n} - \sum_i s_i^2 - \frac{3km\overline{s}}{n}\right]\binom{n-1}{k-1}^2.
\end{aligned}$$

Since $\sum_i s_i = \overline{s}$, there must be some ℓ with $s_\ell \leq \frac{\overline{s}}{m}$, and Claim 3.9 then implies that for every i, $s_i \leq \frac{\overline{s}}{m} + \frac{8k^2r}{\gamma n}$. Hence $\sum_i s_i^2 \leq (\max_i s_i)\sum_i s_i \leq \left(\frac{\overline{s}}{m} + \frac{8k^2r}{\gamma n}\right)\overline{s}$, giving

$$\mathrm{dp}(\mathcal{F}) \geq \frac{1}{2}\left(1 - \frac{1}{m} - \frac{k^2}{n} - \frac{8k^2r}{\gamma\overline{s}n} - \frac{3km}{\overline{s}n}\right)\left(\overline{s}\binom{n-1}{k-1}\right)^2. \tag{19}$$

Claim 3.10. *$|X| = r$; that is, $\mathcal{F}$ has a cover of size r.*

Proof. Since $s > \binom{n}{k} - \binom{n-r+1}{k}$, $\mathcal{F}$ cannot be covered by $r - 1$ elements. Hence we must have $m = |X| \geq r$.

Now recall we have $s = \overline{s}\binom{n-1}{k-1}$, $\overline{s} \geq r - 1$, $\gamma \geq \frac{c}{r}$, $m \leq \frac{8r}{\gamma} \leq 8c^{-1}r^2$ and $n \geq Ck^2r^3$ for some sufficiently large constant C. Substituting these bounds into (19), we find

$$\mathrm{dp}(\mathcal{F}) > \frac{1}{2}\left(1 - \frac{1}{m} - \frac{1}{r(r+1)}\right)s^2.$$

However, by (18), we must have

$$\mathrm{dp}(\mathcal{F}) \leq \frac{1}{2}\left(1 - \frac{1}{r}\right) s^2.$$

These two bounds together imply $\frac{1}{m} + \frac{1}{r(r+1)} > \frac{1}{r}$, which in turn gives $m < r + 1$. This shows $m = r$, and X is thus a cover of size r. $\square$

Hence it follows that $\mathcal{F}$ is covered by some r elements, which we may without loss of generality assume to be $[r]$. We now finish with a similar argument as in the proof of Corollary 3.2: let $\mathcal{S}$ be the union of the r stars with centres in $[r]$, and let $\mathcal{G} = \mathcal{S} \setminus \mathcal{F}$ be the missing sets. Then $\mathrm{dp}(\mathcal{F}) = \mathrm{dp}(\mathcal{S}) - \mathrm{dp}(\mathcal{G}, \mathcal{S}) + \mathrm{dp}(\mathcal{G})$ is minimised when $\mathcal{G}$ is an intersecting family of sets that each meet $[r]$ in precisely one element, which is the case for $\mathcal{F} = \mathcal{L}(n, k, s)$. Hence $\mathrm{dp}(\mathcal{F}) \geq \mathrm{dp}(n, k, s)$, completing the proof of the theorem. $\square$

The problem of minimising the number of disjoint pairs can be viewed as an isoperimetric inequality in the Kneser graph. The following lemma links isoperimetric problems for small and large families (see, for instance, [10, Lemma 2.3]).

Lemma 3.11. *Let $G = (V, E)$ be a regular graph on n vertices. Then $S \subset V$ minimises the number of edges $e(S)$ over all sets of $|S|$ vertices if and only if $V \setminus S$ minimises the number of edges over all sets of $n - |S|$ vertices.*

The following corollary, which is a direct consequence of Theorem 1.2 and Lemma 3.11, shows that the complements of the lexicographical initial segments, which are isomorphic to initial segments of the colexicographical order, are optimal when s is close to $\binom{n}{k}$.

Corollary 3.12. *There exists a positive constant C such that the following statement holds. Provided $n \geq Ck^2r^3$ and $\binom{n-r}{k} \leq s \leq \binom{n}{k}$, $\binom{[n]}{k} \setminus \mathcal{L}(n, k, \binom{n}{k} - s)$ minimises the number of disjoint pairs among all systems of s sets in $\binom{[n]}{k}$.*

4. Typical structure of set systems with given matching number

4.1. **Families with no matching of size s.** In this section we describe the structure of k-uniform set families without matchings of size s. The following lemma, which follows readily from [3, Lemmas 2.2 and 2.3], gives a sufficient condition for the trivial extremal families to be typical.

Lemma 4.1. *Let $\mathbf{P}$ be a decreasing property. Let N_0 denote the size of the extremal (that is, largest) family with property $\mathbf{P}$, N_1 the size of the largest non-extremal maximal family, and suppose two distinct extremal families have at most N_2 members in common. Suppose further that the number of extremal families is T, and there are at most M maximal families. Provided*

$$2 \log M + \max(N_1, N_2) - N_0 \to -\infty, \tag{20}$$

the number of families with property $\mathbf{P}$ is $(T + o(1))2^{N_0}$.

We will apply Lemma 4.1 with $\mathbf{P}$ being the property of avoiding a matching of size s or, equivalently, of not containing s pairwise disjoint sets. To do so, we first bound the number of maximal families with no matching of size s.

Proposition 4.2. *The number of maximal k-uniform families over $[n]$ with no matching of size s is at most $\binom{n}{k}^{\binom{sk}{k}}$.*

Proof. Given $\mathcal{F} \subset \binom{[n]}{k}$, let $\mathcal{I}(\mathcal{F}) = \left\{G \in \binom{[n]}{k} : \mathcal{F} \cup \{G\} \text{ does not have } s \text{ pairwise disjoint sets}\right\}$. Note that $\mathcal{F}$ does not contain a matching of size s if and only if $\mathcal{F} \subset \mathcal{I}(\mathcal{F})$, while $\mathcal{F}$ is maximal if and only if $\mathcal{I}(\mathcal{F}) = \mathcal{F}$. Given a maximal family $\mathcal{F}$, we say that $\mathcal{G} \subset \mathcal{F}$ is a *generating family* of $\mathcal{F}$ if $\mathcal{I}(\mathcal{G}) = \mathcal{F}$.

Let $\mathcal{F}_0 = \{F_1, \ldots, F_m\} \subset \mathcal{F}$ be a minimal generating family of $\mathcal{F}$. By the minimality of $\mathcal{F}_0$, we must have $\mathcal{I}(\mathcal{F}_0 \setminus \{F_i\}) \supsetneq \mathcal{F} = \mathcal{I}(\mathcal{F}_0)$, for each $1 \le i \le m$. Hence we can find some set $G_{i,s-1} \in \mathcal{I}(\mathcal{F}_0 \setminus \{F_i\}) \setminus \mathcal{I}(\mathcal{F}_0)$. It follows that there exist $s-2$ sets $G_{i,1}, \ldots, G_{i,s-2}$ in $\mathcal{F}_0$ such that $F_i, G_{i,1}, \ldots, G_{i,s-2}$ and $G_{i,s-1}$ are pairwise disjoint, while for every $j \neq i$, $F_j, G_{i,1}, \ldots, G_{i,s-2}$ and $G_{i,s-1}$ are not pairwise disjoint. In other words, if we let $G_i = G_{i,1} \cup \ldots \cup G_{i,s-1}$, then $F_i \cap G_i = \emptyset$ and $F_j \cap G_i \neq \emptyset$ for $j \neq i$. Given these conditions, we may apply the Bollobás set-pairs inequality [6] to bound the size of $\mathcal{F}_0$.

Theorem 4.3 (Bollobás)**.** *Let $A_1, \ldots, A_m$ be sets of size a and $B_1, \ldots, B_m$ sets of size b such that $A_i \cap B_i = \emptyset$ and $A_i \cap B_j \neq \emptyset$ for every $i \neq j$. Then $m \le \binom{a+b}{a}$.*

We apply this to the pairs $\{(A_i, B_i)\}_{i=1}^m$, where for $1 \le i \le m$ we take $A_i = F_i$ and $B_i = G_i$. The conditions of Theorem 4.3 are satisfied, and hence we deduce $m \le \binom{sk}{k}$.

We map each maximal family $\mathcal{F}$ to a minimal generating family $\mathcal{F}_0 \subset \mathcal{F}$. This map is injective because $\mathcal{I}(\mathcal{F}_0) = \mathcal{F}$. We have shown that $|\mathcal{F}_0| \le \binom{sk}{k}$, and thus the number of maximal families is bounded from above by $\sum_{i=0}^{\binom{sk}{k}} \binom{\binom{n}{k}}{i} \le \binom{n}{k}^{\binom{sk}{k}}$, as desired. $\square$

Proof of Theorem 1.3. We shall verify that the condition (20) from Lemma 4.1 holds. A result of Frankl [21, Theorem 1.1] states that when $n \ge (2s-1)k - s + 1$, the extremal families with no s pairwise disjoint sets are isomorphic to $\left\{F \in \binom{[n]}{k} : F \cap [s-1] \neq \emptyset\right\}$, and consequently we may take $N_0 = \binom{n}{k} - \binom{n-s+1}{k}$ and $T = \binom{n}{s-1}$. Moreover, it is not difficult to see that the intersection of any two extremal families has size at most $N_2 = \binom{n}{k} - \binom{n-s+1}{k} - \binom{n-s}{k-1}$. Furthermore, a result due to Frankl and Kupavskii [23, Theorem 5] implies that $N_1 \le \binom{n}{k} - \binom{n-s+1}{k} - \frac{1}{s+1}\binom{n-k-s+1}{k-1}$ for $n \ge 2sk - s$. Hence $\max(N_1, N_2) \le \binom{n}{k} - \binom{n-s+1}{k} - \frac{1}{s+1}\binom{n-k-s+1}{k-1}$. In addition, Proposition 4.2 shows that we may use the estimate $\log M \le n\binom{sk}{k}$. Altogether we have

$$\begin{aligned}
2\log M + \max(N_1, N_2) - N_0 &\le 2n\binom{sk}{k} - \frac{1}{s+1}\binom{n-k-s+1}{k-1} \\
&= 2n\binom{sk}{k} - \frac{k}{(s+1)(n-k-s+2)}\binom{n-k-s+2}{k} \\
&\le 2n\binom{sk}{k}\left[1 - \frac{k}{2(s+1)(n-k-s+2)n}\left(\frac{n-k-s+2}{sk}\right)^k\right]. \qquad (21)
\end{aligned}$$

As $n \ge 2sk + 38s^4$ and $s \ge 2$, we find $\frac{n-k-s+2}{sk} \ge \frac{3}{2}\left(1 + \frac{74s^3}{3k}\right)$, and hence

$$\left(\frac{n-k-s+2}{sk}\right)^{k-2} \ge \left(\frac{3}{2}\right)^{k-2}\left(1 + \frac{74s^3}{3k}\right)^{k-2} \ge \frac{k}{2} \cdot \frac{74(k-2)s^3}{3k} = \tfrac{37}{3}(k-2)s^3.$$

This implies

$$\begin{aligned}
\frac{k}{2(s+1)(n-k-s+2)n}\left(\frac{n-k-s+2}{sk}\right)^k &= \frac{n-k-s+2}{2(s+1)s^2kn}\left(\frac{n-k-s+2}{sk}\right)^{k-2} \\
&\ge \frac{37s(k-2)(n-k-s+2)}{6(s+1)kn} \ge \frac{37}{36}
\end{aligned}$$

as $s/(s+1) \geq 2/3, (k-2)/k \geq 1/3$ and $(n-k-s+2)/n \geq 3/4$. Substituting this inequality into (21), we obtain

$$2 \log M + \max(N_1, N_2) - N_0 \leq -\frac{1}{18} n \binom{sk}{k} \to -\infty. \qquad \square$$

4.2. **Intersecting set systems.** In this section we shall use the removal lemma for disjoint sets (Lemma 3.1) to show that intersecting set systems in $\binom{[n]}{k}$ are typically trivial when $n \geq 2k + C\sqrt{k \ln k}$ for some positive constant C. Since the number of trivial intersecting families is

$$n \cdot 2^{\binom{n-1}{k-1}} \pm \binom{n}{2} \cdot 2^{\binom{n-2}{k-2}} = (n + o(1)) 2^{\binom{n-1}{k-1}},$$

it suffices to prove that there are $o(2^{\binom{n-1}{k-1}})$ non-trivial intersecting families.

We need a few classic theorems from extremal set theory. The first is a theorem of Hilton and Milner [25], bounding the cardinality of a non-trivial uniform intersecting family.

Theorem 4.4 (Hilton and Milner)**.** *Let $\mathcal{F} \subset \binom{[n]}{k}$ be a non-trivial intersecting family with $k \geq 2$ and $n \geq 2k+1$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$.*

The next result we require is a theorem of Kruskal [31] and Katona [27]. For a family $\mathcal{F} \subset \binom{[n]}{r}$, its *$s$-shadow* in $\binom{[n]}{s}$, denoted $\partial^{(s)}\mathcal{F}$, is the family of those s-sets contained in some member of $\mathcal{F}$. For $x \in \mathbb{R}$ and $r \in \mathbb{N}$, we define the *generalised binomial coefficient* $\binom{x}{r}$ by setting

$$\binom{x}{r} = \frac{x(x-1)\dots(x-r+1)}{r!}.$$

The following convenient formulation of the Kruskal-Katona theorem is due to Lovász [33].

Theorem 4.5 (Lovász)**.** *Let n, r and s be positive integers with $s \leq r \leq n$. If $\mathcal{F}$ is a subfamily of $\binom{[n]}{r}$ with $|\mathcal{F}| = \binom{x}{r}$ for some real number $x \geq r$, then $\left|\partial^{(s)}\mathcal{F}\right| \geq \binom{x}{s}$.*

With these results in hand, we now prove Theorem 1.4.

Proof of Theorem 1.4. The statement has been established for $n \geq 3k + 8 \ln k$ in [3, Theorem 1.4], and so we may assume $n = 2k + s$ for some integer s with $C\sqrt{k \ln k} \leq s \leq k + 8 \ln k$.

For each $\ell \in \mathbb{N}$, let N_ℓ denote the number of maximal non-trivial intersecting families of size $\binom{n-1}{k-1} - \ell$. By Theorem 4.4, we know $N_\ell = 0$ for $\ell < \binom{n-k-1}{k-1} - 1$. By taking a simple union bound over the subfamilies of these families, we can bound the number of non-trivial intersecting families by

$$\sum_{\ell=\binom{n-k-1}{k-1}-1}^{\binom{n-1}{k-1}} N_\ell 2^{\binom{n-1}{k-1}-\ell} = \left(\sum_\ell N_\ell 2^{-\ell} \right) 2^{\binom{n-1}{k-1}},$$

so it suffices to show $\sum_\ell N_\ell 2^{-\ell} = o(1)$.

By a result of Balogh et al. [3, Proposition 2.2], we know the total number of maximal intersecting families can be bounded by $\sum_\ell N_\ell \leq 2^{\frac{1}{2}n\binom{2k}{k}}$, and so we have

$$\sum_{\ell \geq n\binom{2k}{k}} N_\ell 2^{-\ell} \leq 2^{-n\binom{2k}{k}} \cdot \sum_{\ell \geq n\binom{2k}{k}} N_\ell \leq 2^{-\frac{1}{2}n\binom{2k}{k}} = o(1).$$

Hence it suffices to show

$$\sum_{\ell=\binom{n-k-1}{k-1}-1}^{n\binom{2k}{k}} N_\ell 2^{-\ell} = o(1). \tag{22}$$

We fix some integer ℓ with $\binom{n-k-1}{k-1} - 1 \le \ell \le n\binom{2k}{k}$, and fix some maximal intersecting family $\mathcal{F}$ of size $\binom{n-1}{k-1} - \ell$. Let $\mathcal{S}$ be the star that minimises $|\mathcal{F}\Delta\mathcal{S}|$, and without loss of generality assume that n is the center of $\mathcal{S}$. Let $\mathcal{A} = \mathcal{F} \setminus \mathcal{S}$, and $t = |\mathcal{A}|$. Let $\mathcal{B} = \mathcal{S} \setminus \mathcal{F}$, and note that $|\mathcal{B}| = t + \ell$.

Let $\mathcal{P} = \{[n-1] \setminus A : A \in \mathcal{A}\}$, and observe that $\mathcal{P} \subseteq \binom{[n-1]}{n-k-1}$, since $n \notin A$ for all $A \in \mathcal{A}$. Let $\mathcal{Q} = \{B \setminus \{n\} : B \in \mathcal{B}\} \subseteq \binom{[n-1]}{k-1}$. We claim that $\partial^{(k-1)}\mathcal{P} = \mathcal{Q}$.

Indeed, suppose $H \in \partial^{(k-1)}\mathcal{P}$. Then there is some $A \in \mathcal{A}$ such that $H \subset [n-1] \setminus A$, and so $H \cap A = \emptyset$. As $n \notin A$, this forces $(\{n\} \cup H) \cap A = \emptyset$, and so $\{n\} \cup H \notin \mathcal{F}$. Hence $\{n\} \cup H \in \mathcal{B}$, giving $H \in \mathcal{Q}$.

For the opposite direction, suppose $H \notin \partial^{(k-1)}\mathcal{P}$. Then, following the same argument as above, $(\{n\} \cup H) \cap A \neq \emptyset$ for all $A \in \mathcal{A}$. By maximality of $\mathcal{F}$, we must have $\{n\} \cup H \in \mathcal{F}$, and thus $\{n\} \cup H \notin \mathcal{B}$, resulting in $H \notin \mathcal{Q}$.

We shall show that $\ell \ge 2nt$. First let us see why this implies (22). For each family $\mathcal{F}$ counted by N_ℓ, it suffices to provide the star $\mathcal{S}$ and the family $\mathcal{A}$ outside the star[3]. Indeed, since $\mathcal{Q} = \partial^{(k-1)}\mathcal{P}$, we can compute $\mathcal{F} \cap \mathcal{S}$, and hence completely determine $\mathcal{F}$. Moreover, $|\mathcal{A}| = t \le \ell/(2n)$. Thus

$$N_\ell 2^{-\ell} \le n \cdot \binom{n-1}{k}^{\ell/(2n)} 2^{-\ell} < n \cdot 2^{-\ell/2}, \text{ and so } \sum_{\ell=\binom{n-k-1}{k-1}-1}^{n\binom{2k}{k}} N_\ell 2^{-\ell} \le \frac{2n}{\sqrt{2}-1} \cdot 2^{-\frac{1}{2}\binom{n-k-1}{k-1}} = o(1).$$

It remains to show $\ell \ge 2nt$. Letting $\mathcal{P}$ and $\mathcal{Q}$ be as above, recall that $\mathcal{Q} = \partial^{(k-1)}\mathcal{P}$. According to Theorem 4.5, if x is a real number so that $t = |\mathcal{P}| = \binom{x}{n-k-1}$, then $\ell + t = |\mathcal{Q}| \ge \binom{x}{k-1}$.

Now observe that by Lemma 3.1, we have $t \le C'n\ell$ for some absolute constant C'. Since $\ell \le n\binom{2k}{k}$, this implies $t \le C'n^2\binom{2k}{k}$. Since $n = 2k + s$, we have $t = \binom{x}{n-k-1} = \binom{x}{k+s-1}$. We next show that $x < 2k + \lfloor \frac{3}{4}s \rfloor$. If not, then

$$\begin{aligned} t = \binom{x}{k+s-1} \ge \binom{2k + \lfloor \frac{3}{4}s \rfloor}{k+s-1} &= \binom{2k}{k} \cdot \frac{\binom{2k+s-1}{k+s-1}}{\binom{2k}{k}} \cdot \frac{\binom{2k+\lfloor \frac{3}{4}s \rfloor}{k+s-1}}{\binom{2k+s-1}{k+s-1}} \\ &= \binom{2k}{k} \cdot \prod_{j=1}^{s-1} \frac{2k+j}{k+j} \cdot \prod_{j=\lfloor \frac{3}{4}s \rfloor + 1}^{s-1} \frac{k-s+1+j}{2k+j} \\ &\ge \binom{2k}{k} \left(\frac{2k+s}{k+s}\right)^{s-1} \left(\frac{k-\frac{1}{4}s}{2k+\frac{3}{4}s}\right)^{\frac{1}{4}s}. \end{aligned}$$

The bases of the exponential factors are minimised when s is as large as possible; substituting $s \le k + 8\ln k < 1.1k$, we can lower bound the coefficient of $\binom{2k}{k}$ by

$$\frac{2.1}{3.1}\left(\frac{3.1}{2.1}\right)^s \left(\frac{0.725}{2.825}\right)^{\frac{1}{4}s} > \frac{2}{3}1.05^s > n^3$$

[3]For every choice of $\mathcal{F}$ there is a unique $\mathcal{A}$, but not every $\mathcal{A}$ corresponds to a maximal family $\mathcal{F}$

as $s > C\sqrt{k \ln k} \geq 100 \ln n$, contradicting our upper bound $t \leq C' n^2 \binom{2k}{k}$.

Suppose, then, that $x \leq 2k + \lfloor \frac{3}{4} s \rfloor - 1$. Since $t = \binom{x}{k+s-1}$ and $\ell + t \geq \binom{x}{k-1}$, we have

$$\frac{\ell}{t} \geq \frac{\binom{x}{k-1}}{\binom{x}{k+s-1}} - 1 = \prod_{j=k}^{k+s-1} \frac{j}{x+1-j} - 1.$$

This product is decreasing in x, so we can substitute our upper bound $x \leq 2k + \lfloor \frac{3}{4} s \rfloor - 1$ to find

$$\frac{\ell}{t} \geq \prod_{j=k}^{k+s-1} \frac{j}{2k + \lfloor \frac{3}{4} s \rfloor - j} - 1 = \prod_{j=\lfloor \frac{3}{4} s \rfloor + 1}^{s-1} \frac{k+j}{k + \lfloor \frac{3}{4} s \rfloor - j} - 1 \geq \left(1 + \frac{3s}{4k}\right)^{\frac{1}{4}s - 1} - 1.$$

This is increasing in s, so plugging in the lower bound $s \geq C\sqrt{k \ln k}$, we have

$$\frac{\ell}{t} \geq \left(1 + \frac{3}{4} C \sqrt{\frac{\ln k}{k}}\right)^{\frac{C}{4}\sqrt{k \ln k} - 1} - 1 > e^{\frac{3C}{64} \ln k} - 1 \geq 8k > 2n,$$

as required. This completes the proof. $\square$

5. Concluding remarks

We close by offering some final remarks and open problems related to the supersaturation problems discussed in this paper.

5.1. **Supersaturation for permutations.** Theorem 1.1 shows, for $k \leq cn^{1/2}$ and s (very) close to $k(n-1)!$, one minimises the number of disjoint pairs in a family of s permutations by selecting them from pairwise-disjoint cosets. This leaves large gaps between the ranges where we know the answer to the supersaturation problem, and it would be very interesting to determine the correct behaviour throughout. For instance, which family of $1.5(n-1)!$ permutations minimises the number of disjoint pairs?

Note that the derangement graph is d_n-regular, and so we can apply Lemma 3.11 to determine the optimal families for sizes close to $k(n-1)!$ when $k \geq n - cn^{1/2}$ by taking complements. However, the complement of a union of pairwise disjoint cosets is again a union of pairwise disjoint cosets, and hence there may well be a nested sequence of optimal families for this problem. One candidate would be the initial segments of the lexicographic order on S_n, where $\pi < \sigma$ if and only if $\pi_j < \sigma_j$ for $j = \min\{i \in [n] : \pi_i \neq \sigma_i\}$.

5.2. **Set systems of very large uniformity.** For set families, we improved the range of uniformities for which the small initial segments of the lexicographic order are known to be optimal. In Corollary 3.2, which applies when $n = \Omega(r^3 k^2)$, we handled the case where the family is a little larger than the union of r stars. However, if one restricts the size of the set families even further, one can obtain optimal bounds on n. For instance, Katona, Katona and Katona [28] showed that adding one set to a full star is always optimal.

Proposition 5.1. *Suppose $n \geq 2k+1$. Any system $\mathcal{F} \subseteq \binom{[n]}{k}$ with $|\mathcal{F}| = \binom{n-1}{k-1} + 1$ contains at least $\binom{n-k-1}{k-1}$ disjoint pairs.*

By applying the removal lemma (Lemma 3.1), we can extend this exact result to a larger range of family sizes.

Proposition 5.2. *For some positive constant c, the following holds. Provided $n \geq 2k+2$ and $0 \leq s \leq \binom{n-1}{k-1} + c \cdot \frac{n-2k}{n}\binom{n-k-1}{k-1}$, $\mathcal{L}(n,k,s)$ minimises the number of disjoint pairs among all systems of s sets in $\binom{[n]}{k}$.*

Proof. Let C be the positive constant from Lemma 3.1 and set $c = (20C)^{-2}$. Suppose $\mathcal{F} \subseteq \binom{[n]}{k}$ is a family with $|\mathcal{F}| = \binom{n-1}{k-1} + t$ for some $1 \leq t \leq c \cdot \frac{n-2k}{n}\binom{n-k-1}{k-1}$. Letting $s = \binom{n-1}{k-1} + t$, we shall show that $\mathrm{dp}(\mathcal{F}) \geq \mathrm{dp}(\mathcal{L}_{n,k}(s)) = t\binom{n-k-1}{k-1}$. Suppose otherwise that $\mathrm{dp}(\mathcal{F}) < t\binom{n-k-1}{k-1}$. By Lemma 3.1, there exists a star $\mathcal{S}$ such that $|\mathcal{F}\Delta\mathcal{S}| \leq \frac{1}{2}\binom{n-k-1}{k-1}$. It follows that $|\mathcal{F}\cap\mathcal{S}| = \binom{n-1}{k-1} - p$ for some integer p with $0 \leq p \leq \frac{1}{2}\binom{n-k-1}{k-1}$. As $|\mathcal{F}| = \binom{n-1}{k-1} + t$ and $|\mathcal{F}\cap\mathcal{S}| = \binom{n-1}{k-1} - p$, we must have $|\mathcal{F}\setminus\mathcal{S}| = p + t$. Since each set in $\mathcal{F}\setminus\mathcal{S}$ is disjoint from exactly $\binom{n-k-1}{k-1}$ sets in the star $\mathcal{S}$ and $|\mathcal{F}\cap\mathcal{S}| = \binom{n-1}{k-1} - p$, we conclude $\mathrm{dp}(F, \mathcal{F}\cap\mathcal{S}) \geq \binom{n-k-1}{k-1} - p > 0$ for all $F \in \mathcal{F}\setminus\mathcal{S}$. Thus

$$\begin{aligned}
\mathrm{dp}(\mathcal{F}) &\geq \sum_{F\in\mathcal{F}\setminus\mathcal{S}} \mathrm{dp}(F, \mathcal{F}\cap\mathcal{S}) \geq |\mathcal{F}\setminus\mathcal{S}|\left(\binom{n-k-1}{k-1} - p\right) \\
&= (p+t)\left(\binom{n-k-1}{k-1} - p\right) = t\binom{n-k-1}{k-1} + p\left(\binom{n-k-1}{k-1} - p - t\right) \\
&\geq t\binom{n-k-1}{k-1},
\end{aligned}$$

where the last inequality holds since $p \leq \frac{1}{2}\binom{n-k-1}{k-1}$ and $t \leq c \cdot \frac{n-2k}{n}\binom{n-k-1}{k-1}$. □

5.3. **A counterexample to the Bollobás–Leader conjecture.** Finally, it remains to extend the set supersaturation results to larger values of k. Are small initial segments of the lexicographic order still optimal when $k > \sqrt{n}$?

This is not the case when $n = 3k-1$, as the following construction shows. Let $s = \binom{n-1}{k-1} + \binom{2k-1}{k} - 1$. Then $\mathcal{L}(n,k,s)$ consists of one full star, and $\binom{2k-1}{k} - 1$ sets from another star, each of which is disjoint from $\binom{n-k-1}{k-1} = \binom{2k-2}{k-1}$ sets from the full star. Hence $\mathrm{dp}(\mathcal{L}(n,k,s)) = \left(\binom{2k-1}{k} - 1\right)\binom{2k-2}{k-1}$.

Now instead let $\mathcal{F}'$ be the family consisting of the $\mathcal{S}_1$, the full star with centre 1, and all but one k-element subset of $\{2,3,\dots,2k\}$. Since $\mathcal{F}'$ again consists of a full star and an intersecting family of size $\binom{2k-1}{k} - 1$, we have $\mathrm{dp}(\mathcal{F}') = \mathrm{dp}(\mathcal{L}(n,k,s))$. Now form the family $\mathcal{F}$ from $\mathcal{F}'$ by replacing the set $A = \{1, 2k+1, \dots, 3k-1\}$ with the missing k-set B from $\{2,3,\dots,2k\}$. We lose $\binom{2k-1}{k} - 1$ disjoint pairs when we remove A, and gain only $\binom{n-k-1}{k-1} - 1 = \binom{2k-2}{k-1} - 1$ disjoint pairs when we add B. As $\binom{2k-2}{k-1} < \binom{2k-1}{k}$, it follows that $\mathrm{dp}(\mathcal{F}) < \mathrm{dp}(\mathcal{L}(n,k,s))$, showing the initial segment of the lexicographic order is not optimal.

Bollobás and Leader [7] conjectured that the solution to the supersaturation problem is always given by an ℓ-ball. Given n, k and s, an ℓ-ball of size s is a family $\mathcal{B}_\ell(n,k,s)$ of s sets such that there is some r with $\left\{F \in \binom{[n]}{k} : |F\cap[r]| \geq \ell\right\} \subseteq \mathcal{B}_\ell(n,k,s) \subseteq \left\{F \in \binom{[n]}{k} : |F\cap[r+1]| \geq \ell\right\}$. In particular, the initial segments of the lexicographic order are 1-balls, while their complements are isomorphic to k-balls.

We have shown that the construction $\mathcal{F}$ given above has fewer disjoint pairs than the 1-balls of size $s = |\mathcal{F}|$. Computer-aided calculations show that for $n = 3k-1$, $s = \binom{n-1}{k-1} + \binom{2k-1}{k} - 1$ and $5 \leq k \leq 15$, the 1-balls have far fewer disjoint pairs than the ℓ-balls for $\ell \geq 2$, showing that $\mathcal{F}$ gives a counterexample to the Bollobás–Leader conjecture for these parameters. The numerical evidence suggests that $\mathcal{F}$ should be a counterexample for all $k \geq 5$, but it is difficult to estimate the number of disjoint pairs in $\mathcal{B}_\ell(3k-1, k, s)$ for $\ell \geq 2$, and so we have been unable to prove this.

References

[1] N. Alon and F. R. K. Chung, Explicit constructions of linear sized tolerant networks, *Discrete Math.* **2** (1988), 15–19.

[2] J. Balogh and H. Liu, On the number of K_4-saturating edges, *J. Combin. Theory, Ser. B* **109** (2014), 250–257.

[3] J. Balogh, S. Das, M. Delcourt, H. Liu and M. Sharifzadeh, Intersecting families of discrete structures are typically trivial, *J. Combin. Theory, Ser. A* **132** (2015), 224–245.

[4] J. Balogh, S. Petrickova and A. Zs. Wagner, Families in posets minimizing the number of comparable pairs, *arXiv:1703.05427* (2017).

[5] J. Balogh and A. Zs. Wagner, Kleitman's conjecture about families of given size minimizing the number of k-chains *arXiv:1609.02262* (2016).

[6] B. Bollobás, On generalized graphs, *Acta Math. Acad. Sci. Hungar.* **16** (1965), 447–452.

[7] B. Bollobás and I. Leader, Set systems with few disjoint pairs, *Combinatorica* **23** (2003), 559–570.

[8] O. Chervak, O. Pikhurko and K. Staden, Minimum number of additive tuples in groups of prime order, *arXiv:1710.01936* (2017).

[9] S. Das, W. Gan and B. Sudakov, Sperner's Theorem and a problem of Erdős-Katona-Kleitman, *Combin. Probab. Comput.* **24** (2015), 585-608.

[10] S. Das, W. Gan and B. Sudakov, The minimum number of disjoint pairs in set systems and related problems, *Combinatorica* **36** (2016), 623–660.

[11] S. Das and T. Tran, Removal and stability for Erdős-Ko-Rado, *SIAM J. Discrete Math.* **30** (2016), 1102–1114.

[12] M. Deza and P. Frankl, On the maximum number of permutations with given maximal or minimal distance, *J. Combin. Theory, Ser. A* **22** (1977), 352–360.

[13] A. P. Dove, J. R. Griggs, R. J. Kang and J.-S. Sereni, Supersaturation in the Boolean lattice, *Integers* **14A** (2014), #A4.

[14] D. Ellis, A proof of the Cameron-Ku conjecture, *J. London Math. Soc.* **85** (2012), 165–190.

[15] D. Ellis, Y. Filmus and E. Friedgut, A quasi-stability result for dictatorships in S_n, *Combinatorica* **35** (2015), 573–618.

[16] D. Ellis, E. Friedgut and H. Pilpel, Intersecting families of permutations, *J. Amer. Math. Soc.* **24** (2011), 649–682.

[17] P. Erdős, On a theorem of Rademacher-Turán, *Illinois J. Math* **6** (1962), 122–127.

[18] P. Erdős, On the number of complete subgraphs contained in certain graphs, *Magy. Tud. Acad. Mat. Kut. Int. Közl.* **7** (1962), 459–474.

[19] P. Erdős, C. Ko and R. Rado, Intersection theorems for systems of finite sets, *Q. J. Math.*, **12** (1961), 313–320.

[20] Y. Filmus, Friedgut–Kalai–Naor theorem for slices of the Boolean cube, *Chicago J. Theoret. Comput. Sci.* (2016), article 14, pages 1–17.

[21] P. Frankl, Improved bounds for Erdős' matching conjecture, *J. Combin. Theory, Ser. A* **120** (2013), 1068–1072.

[22] P. Frankl, Y. Kohayakawa and V. Rödl, A note on supersaturated set systems, *European J. Combin.* **51** (2016), 190–199.

[23] P. Frankl and A. Kupavskii, Families with no s pairwise disjoint sets, *J. London Math. Soc.* **95** (2017), 875–894.

[24] P. Frankl and A. Kupavskii, Counting intersecting and pairs of cross-intersecting families, *Comb. Prob. Comput.* **27** (2018), 60–68.

[25] A. J. Hilton and E. C. Milner, Some intersection theorems for systems of finite sets, *Q. J. Math.* **18** (1967), 369–384.

[26] S. Huczynska, Beyond Sum-Free Sets in the Natural Numbers, *Electron. J. Comb.* **21(1)** (2014), #P1.21.

[27] G. Katona, A theorem of finite sets, *Theory of graphs*, (Proc. Colloq., Tihany, 1966), Academic Press, New York, 1968, pages 187–207.

[28] G. O. H. Katona, G. Y. Katona and Z. Katona, Most probably intersecting families of subsets, *Comb. Prob. Comput.* **21** (2012), 219–227.

[29] N. Khadžiivanov and V. Nikiforov, Solution of a problem of P. Erdős about the maximum number of triangles with a common edge in a graph (Russian), *C. R. Acad. Bulgare Sci.* **32** (1979), 1315–1318.

[30] D. Kleitman, A conjecture of Erdős-Katona on commensurable pairs among subsets of an n-set, *Theory of Graphs, Proc. Colloq., Tihany*, (1966), 215–218.

[31] J. B. Kruskal, The number of simplices in a complex, *Mathematical optimization techniques*, Univ. of California Press, Berkeley, Calif., 1963, pages 251–278.

[32] H. Liu, O. Pikhurko and K. Staden, The exact minimum number of triangles in graphs of given order and size, *arXiv:1712.00633* (2017).

[33] L. Lovász, Combinatorial problems and exercises, 2nd ed., AMS Chelsea Publishing, Providence, RI, 2007.

[34] L. Lovász, On the Shannon capacity of a graph, *IEEE T. Inform. Theory* **25** (1979), 1–7.

[35] L. Lovász and M. Simonovits, On the number of complete subgraphs of a graph, II. *Studies in pure mathematics*, 459–495, Birkhuser, Basel, 1983.

[36] W. Mantel, Problem 28, *Winkundige Opgaven* **10** (1907), 60–61.

[37] D. Mubayi, Counting substructures I: color critical graphs, *Adv. Math.* **225** (2010), 2731–2740.

[38] V. Nikiforov, The number of cliques in graphs of given order and size, *Trans. Amer. Math. Soc.* **363** (2011), 1599–1618.

[39] J. Noel, A. Scott and B. Sudakov, Supersaturation in posets and applications involving the container method, *J. Combin. Theory, Ser. A* **154** (2018), 247–284.

[40] O. Pikhurko and Z. Yilma, Supersaturation problem for color-critical graphs, *J. Combin. Theory, Ser. B* **123** (2017), 148–185.

[41] A. Razborov, On the minimal density of triangles in graphs, *Combin. Probab. Comput.* **17** (2008), 603–618.

[42] C. Reiher, The clique density theorem, *Ann. Math.* **184** (2016), 683–707.

[43] P. Renteln, On the Spectrum of the Derangement Graph, *Electron. J. Combin.* **14** (2007), R82.

[44] W. Samotij, Subsets of posets minimising the number of chains, *arXiv:1708.02436* (2017).

[45] W. Samotij and B. Sudakov, On the number of monotone sequences, *J. Combin. Theory, Ser. B* **115** (2015), 132–163.

[46] W. Samotij and B. Sudakov, The number of additive triples in subsets of abelian groups, *Math. Proc. Cambridge Philos. Soc.* **160** (2016), 495–512.

József Balogh, Department of Mathematical Sciences, University of Illinois at Urbana-Champaign, IL, USA, and Moscow Institute of Physics and Technology, 9 Institutskiy per., Dolgoprodny, Moscow Region, 141701, Russian Federation.
Shagnik Das, Institut für Mathematik, Freie Universität Berlin, Germany.
Hong Liu and Maryam Sharifzadeh, Mathematics Institute, University of Warwick, UK.
Tuan Tran, Department of Mathematics, ETH, Switzerland.

E-mail addresses: `jobal@illinois.edu`, `shagnik@mi.fu-berlin.de`,
{`h.liu.9`, `m.sharifzadeh`}`@warwick.ac.uk`, `manh.tran@math.ethz.ch`