On two problems in Ramsey-Turán theory

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Abstract

Alon, Balogh, Keevash and Sudakov proved that the $(k-1)$-partite Turán graph maximizes the number of distinct r-edge-colorings with no monochromatic K_k for all fixed k and $r = 2, 3$, among all n-vertex graphs. In this paper, we determine this function asymptotically for $r = 2$ among n-vertex graphs with sub-linear independence number. Somewhat surprisingly, unlike Alon-Balogh-Keevash-Sudakov's result, the extremal construction from Ramsey-Turán theory, as a natural candidate, does not maximize the number of distinct edge-colorings with no monochromatic cliques among all graphs with sub-linear independence number, even in the 2-colored case.

In the second problem, we determine the maximum number of triangles asymptotically in an n-vertex K_k-free graph G with $\alpha(G) = o(n)$. The extremal graphs have similar structure to the extremal graphs for the classical Ramsey-Turán problem, i.e. when the number of edges is maximized.

1 Introduction

Numerous classical problems in extremal graph theory have highly structured extremal configurations. For example, Turán [19] in 1941 proved that $\mathrm{ex}(n, K_k)$, the maximum number of edges in an n-vertex K_k-free graph, is attained only by the balanced complete $(k-1)$-partite graph, known now as the *Turán graph* $T_{n,k-1}$. Motivated by the fact that the Turán graph is particularly symmetric, admitting a $(k-1)$-partition into linear-sized independent sets, Erdős and Sós [13] introduced *Ramsey-Turán* type questions, where they investigated the maximum size of a K_k-free graph G with the additional condition that $\alpha(G) = o(|G|)$.

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Denote by $\mathrm{RT}(n, K_k, o(n))$ the Ramsey-Turán function for K_k, i.e. the maximum size of an n-vertex K_k-free graph with independence number $o(n)$. In 1970, Erdős and Sós [13] determined $\mathrm{RT}(n, K_k, o(n))$ for every odd k. The problem becomes much harder when an even clique is forbidden. For $k = 4$, Szemerédi [18], using the regularity lemma, proved that $\mathrm{RT}(n, K_4, o(n)) \leq n^2/8+o(n^2)$. It had remained an open question whether $\mathrm{RT}(n, K_4, o(n)) = \Omega(n^2)$. Bollobás and Erdős, in their seminal work [7], constructed a dense, K_4-free graph with sub-linear independence number, matching the upper bound above (see Section 2 for more details). For all even k, the order of magnitude of $\mathrm{RT}(n, K_k, o(n))$ was finally determined by Erdős, Hajnal, Sós and Szemerédi [12] in 1983. See [17] for a survey and [4, 5] for more recent developments on this topic.

In this paper, we will study Ramsey-Turán extensions of some classical results, whose extremal graphs are close to the Turán graph. See e.g. [6] for one such extension of a graph tiling problem.

1.1 Edge-colorings forbidding monochromatic cliques

Denote by $F(n, r, k)$ the maximum number of r-edge-colorings that an n-vertex graph can have without a monochromatic copy of K_k. A trivial lower bound is given by $T_{n,k-1}$ as every r-edge-coloring of a K_k-free graph is monochromatic K_k-free: $F(n, r, k) \geq r^{\mathrm{ex}(n,K_k)}$. Erdős and Rothschild [11] in 1974 conjectured that, for sufficiently large n, the above obvious lower bound is optimal for 2-edge-colorings. This was verified for $k = 3$ by Yuster [20]. In 2004, Alon, Balogh, Keevash and Sudakov [1] settled this conjecture in full, proving that, for all $k \geq 3$ and sufficiently large n, the Turán graph $T_{n,k-1}$ maximizes the number of 2-edge-colorings and 3-edge-colorings with no monochromatic K_k among all graphs:

$$F(n, 2, k) = 2^{\mathrm{ex}(n,K_k)} \quad \text{and} \quad F(n, 3, k) = 3^{\mathrm{ex}(n,K_k)}. \tag{1}$$

For 4-edge-colorings, the only two known cases are when $k = 3, 4$: an asymptotic result was given in [1] for $k = 3, 4$; the exact result was proved by Pikhurko and Yilma [15], who showed that $T_{n,4}$ and $T_{n,9}$ maximize the number of 4-edge-colorings with no monochromatic K_3 and K_4 respectively, see [16] for more recent development.

Since the Turán graph is extremal in the Erdős-Rothschild problem for $r = 2, 3$, it is natural to consider its Ramsey-Turán extension. Formally, given a function $f(n)$, we define $\mathrm{RF}(r, k, f(n))$ to be the maximum number of r-edge-colorings that an n-vertex graph with independence number at most $f(n)$ can have without a monochromatic copy of K_k. Similarly, the trivial lower bound on $\mathrm{RF}(r, k, o(n))$ is given by taking all edge-colorings on an extremal graph for Ramsey-Turán problem:

$$\mathrm{RF}(r, k, o(n)) \geq r^{\mathrm{RT}(n,K_k,o(n))+o(n^2)}. \tag{2}$$

Considering (1), it is not inconceivable that the lower bound in (2) is optimal when r is small. However, as shown in the following example, $\mathrm{RF}(r, k, o(n))$ exhibits rather different behavior than $F(n, r, k)$, even in the 2-edge-coloring case when K_4 is forbidden. Let G be a graph obtained by putting a copy of Γ in each partite set of $T_{n,2}$, where Γ is a triangle-free graph with independence number $o(n)$.[1] Since Γ is triangle-free, the neighborhood of every vertex is an independent set. Therefore, the independence number of the graph Γ is at least its maximum degree, which implies that Γ has maximum degree $o(n)$. Consider the following set of 2-edge-colorings of G. Color the edges inside one partite set red, the edges inside the other partite set blue, and color all the remaining cross-edges either red or blue. It is not hard to see that none of these colorings contain monochromatic K_4's, hence, $\mathrm{RF}(2,4,o(n)) \geq 2^{n^2/4}$, while $\mathrm{RT}(n,K_4,o(n)) = (1/8+o(1))n^2$.

The above example already suggests an obstacle in determining $\mathrm{RF}(2,k,o(n))$, that is, the subgraphs induced by each color could simultaneously have linear-sized independent set. Nonetheless, our first result reveals the asymptotic behavior of $\mathrm{RF}(2,k,o(n))$ for every integer k.

Theorem 1.1. $\mathrm{RF}(2,3,o(n)) = 2^{o(n^2)}$. *For every integer $t \geq 1$ and $i \in \{1,2,3\}$,*

$$\mathrm{RF}(2,3t+i,o(n)) = 2^{\mathrm{RT}(n,K_{4t+i},o(n))+o(n^2)}.$$

The following well-known theorem determines the asymptotic value of $\mathrm{RT}(n,K_k,o(n))$, for every $k \geq 3$. For odd k, this was proved by Erdős and Sós [13]. For $k=4$, the upper bound is due to Szemerédi [18]. In [7], Bollobás and Erdős showed that this upper bound is asymptotically sharp. These results were extended by Erdős, Hajnal, Sós, and Szemerédi [12] to every even k.

Theorem 1.2. *For every integer $k \geq 3$,*

$$\mathrm{RT}(n,K_k,o(n)) = (b_k+o(1))n^2,$$

where

$$b_k = \begin{cases} \frac{1}{2}\cdot\frac{k-3}{k-1} & \text{if } k \text{ is odd,} \\ \frac{1}{2}\cdot\frac{3k-10}{3k-4} & \text{if } k \text{ is even.} \end{cases} \tag{3}$$

The definition of b_k comes from optimizing the number of edges in a construction that we will describe in Section 2 (Construction 2.2). By Theorem 1.2, to prove Theorem 1.1, it suffices to prove the following theorem.

Theorem 1.3. $\mathrm{RF}(2,3,o(n)) = 2^{o(n^2)}$. *For every integer $t \geq 1$ and $i \in \{1,2,3\}$,*

$$\mathrm{RF}(2,3t+i,o(n)) = 2^{(b_{4t+i}+o(1))n^2}.$$

Note that since the value of the Ramsey-Turán function is only known asymptotically, we will not try to determine the exact value of $\mathrm{RF}(2,k,o(n))$. Our constructions for the lower bound in Theorem 1.3 are based on the Bollobás-Erdős graph [7].

[1] The existence of K_k-free graph Γ with $\alpha(\Gamma) = o(|\Gamma|)$ was proved by Erdős [9].

1.2 A generalized Ramsey-Turán problem

The generalized Turán-type problem, i.e. for given graphs F and H, determine $\mathrm{ex}(n, F, H)$, the maximum number of copies of F in an n-vertex H-free graph, has been studied for various choices of F and H. Erdős [10] determined $\mathrm{ex}(n, K_s, K_t)$ for all $t > s \geq 3$, showing that among all K_t-free graphs, $T_{n,t-1}$ has the maximum number of K_s's. See also Bollobás and Győri [8] for $\mathrm{ex}(n, K_3, C_5)$, and more recently, Alon and Shikhelman [2] for the cases when (F, H) are (K_3, C_5), $(K_m, K_{s,t})$, and when both F and H are trees.

Our second result studies the general function $\mathrm{RT}(F, H, f(n))$, which is the maximum number of copies of F in an H-free n-vertex graph G with $\alpha(G) \leq f(n)$. It is not hard to see that $\mathrm{RT}(K_s, K_{s+1}, o(n)) = o(n^s)$. We determine, in the following two theorems, $\mathrm{RT}(K_3, K_t, o(n))$ for every integer t.

Theorem 1.4. *Let $t \geq 6$ be an integer and $\ell = \left\lfloor \frac{t}{2} \right\rfloor$. Then as n tends to infinity,*

$$\mathrm{RT}(K_3, K_t, o(n)) = a_\ell n^3(1 + o(1)),$$

where

$$a_\ell = \begin{cases} \max\limits_{0 \leq x \leq 1} \binom{\ell-2}{3} \left(\frac{1-x}{\ell-2}\right)^3 + x\binom{\ell-2}{2} \left(\frac{1-x}{\ell-2}\right)^2 + \frac{1}{2} \cdot \left(\frac{x}{2}\right)^2 (1-x) & \text{if } t = 2\ell, \\ \left(\frac{1}{\ell}\right)^3 \binom{\ell}{3} & \text{if } t = 2\ell + 1. \end{cases} \tag{4}$$

In fact, our proof shows that all the extremal graphs should have the structure as those in Construction 2.2. The definition of a_ℓ comes from optimizing the number of K_3's in these graphs.

For the general case $t > s \geq 3$, we present a construction in the concluding remark which we believe gives the right answer. Our next result verifies the first non-trivial case.

Theorem 1.5. *For every $s \geq 3$,*

$$\mathrm{RT}(K_s, K_{s+2}, o(n)) = \left(2^{-\binom{s}{2}} + o(1)\right) \left(\frac{n}{s}\right)^s.$$

Organization. We first introduce some tools in Section 2. Then in Section 3, we prove Theorem 1.3, and in Section 4, Theorems 1.4 and 1.5.

Notation. Let $G = (V, E)$ be an n-vertex graph and $e(G) = |E(G)|$. For every $v \in V$ and $U, U' \subseteq V$, denote by $d_U(v)$ the degree of v in U. Also, let $N_U(v)$ be the set of vertices $u \in U$ such that $vu \in E(G)$. Denote by $G[U, U']$ the induced bipartite subgraph of G on partite sets U and U'. Let $k_s(G)$ be the number of K_s in G. For every $A \subseteq V(G)$ and an r-coloring of $E(G)$ with colors $\{c_1, \ldots, c_r\}$, let $G_{c_i}[A]$ be the c_i-colored subgraph of G induced by the vertex set A. We will write G_{c_i} instead of $G_{c_i}[V(G)]$. We fix throughout the paper a function $\omega(n)$ of n such that $\omega(n) \to \infty$ arbitrary slowly. If we claim that a result holds whenever $0 < a \ll b \leq 1$, then this means that there is a non-decreasing function $f : (0, 1] \to (0, 1]$ such that the result holds for all $0 < a, b \leq 1$ with $a \leq f(b)$.

2 Preliminaries

We start with a formal definition for $\mathrm{RT}(n, H, o(n))$.

Definition 2.1. For a graph H and a function $f(n)$, let

$$\mathrm{RT}(n, H, o(f(n))) = n^2 \cdot \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{\mathrm{RT}(n, H, \varepsilon f(n))}{n^2} + o(n^2).$$

It is easy to see that the limit in the definition above exists as it is monotone in ε. Bollobás and Erdős [7] constructed a family of n-vertex K_4-free graphs with independence number $o(n)$ and $(\frac{1}{8} + o(1))n^2$ edges. We follow the description in [17] to present their construction. For a constant $\varepsilon > 0$, and sufficiently large integers d and n_0, assume $n > n_0$ is even and $\mu = \varepsilon/\sqrt{d}$. Next, partition the high-dimensional unit sphere $\mathbb{S}^d$ into $n/2$ domains, $D_1, \ldots, D_{n/2}$, of equal measure with diameter[2] less than $\mu/2$. For every $1 \le i \le n/2$, choose two points $x_i, y_i \in D_i$. Let $X = \{x_1, \ldots, x_{n/2}\}$ and $Y = \{y_1, \ldots, y_{n/2}\}$. Let $\mathrm{BE}(X, Y)$ be the graph with vertex set $X \cup Y$ and edge set as follows. For every $x, x' \in X$ and $y, y' \in Y$,

1. let $xy \in E(\mathrm{BE}(X, Y))$ if their distance is less than $\sqrt{2} - \mu$,
2. let $xx' \in E(\mathrm{BE}(X, Y))$ if their distance is more than $2 - \mu$,
3. let $yy' \in E(\mathrm{BE}(X, Y))$ if their distance is more than $2 - \mu$.

Note that the number of edges with both ends in X or Y is $o(n^2)$.

Next, for every integer $k \ge 3$, we will describe a family of n-vertex K_k-free graphs with independence number $o(n)$. As we mentioned earlier, the constant b_k defined in (3) comes from maximizing the number of edges in the construction below. In other words, some of these graphs are extremal graphs for Theorem 1.2, i.e. they have $(b_k + o(1))n^2$ edges.

Construction 2.2. Given $k \ge 3$, denote by $\mathcal{H}(n, k)$ the family of n-vertex graphs G obtained as follows. Let $\ell = \lfloor \frac{k}{2} \rfloor$, and Γ_n be an n-vertex triangle-free graph with $\alpha(\Gamma_n) = o(|\Gamma_n|)$. If k is odd, start with a complete balanced ℓ-partite graph on vertex set $V_1 \cup \ldots \cup V_\ell$. Then put a copy of $\Gamma_{|V_i|}$ in each V_i. If k is even, partition the vertex set into ℓ parts $\{V_1, \ldots, V_\ell\}$, such that $|V_1| = |V_2|$. First, let $G[V_1 \cup V_2]$ be a copy of the Bollobás-Erdős graph $\mathrm{BE}(V_1, V_2)$; then for every $i \in \{1, \ldots, \ell\}$ and $j \in \{3, \ldots, \ell\} \setminus \{i\}$, let $G[V_i, V_j]$ be a complete bipartite graph; next, for every $i \in \{3 \ldots \ell\}$, put a copy of $\Gamma_{|V_i|}$ in each V_i.

Remark 2.3. Note that in Construction 2.2, for even k, $|V_1| = |V_2|$ and $\{V_1, \ldots, V_\ell\}$ is not necessarily an equipartition.

We will also need the following definitions of regular partitions and weighted cluster graph.

[2] The diameter is the maximum distance between any two points in each domain.

Definition 2.4. Let G be a graph and $A, B \subseteq V(G)$. Denote by $d(A, B) := \frac{e(G[A,B])}{|A||B|}$ the *density* of the pair (A, B). Given $\varepsilon > 0$, a pair $X, Y \subseteq V(G)$ is *ε-regular* if for every $A \subseteq X$ and $B \subseteq Y$ with $|A| \geq \varepsilon|X|$ and $|B| \geq \varepsilon|Y|$, $|d(A, B) - d(X, Y)| \leq \varepsilon$. A vertex partition of G, $V(G) = C_1 \cup \ldots \cup C_m$ is ε-regular if all but εm^2 pairs of (C_i, C_j) are ε-regular.

Definition 2.5. For every $\varepsilon > 0$, positive integer t, and an n-vertex graph $G = (V, E)$, let $\mathcal{C} = \{C_1, \ldots, C_m\}$ be an ε-regular partition of $V(G)$ with $m \geq t$. Denote by R the *cluster graph* (with respect to ε) with vertex set $\mathcal{C}$, and C_i and C_j are adjacent if the pair (C_i, C_j) is ε-regular with density at least 10ε. We now define the *weighted cluster graph*, $R = (\mathcal{C}, w)$ (with respect to ε), on the vertex set $\mathcal{C}$ as follows. For an ε-regular pair (C_i, C_j), we will define:

$$w(C_i, C_j) := \begin{cases} 0 & \text{if } d(C_i, C_j) \leq 10\varepsilon \text{ or } (C_i, C_j) \text{ is an irregular pair,} \\ \frac{1}{2} & \text{if } 10\varepsilon < d(C_i, C_j) \leq 1/2 + 10\varepsilon, \\ 1 & \text{if } d(C_i, C_j) > 1/2 + 10\varepsilon. \end{cases}$$

Definition 2.6. A *weighted graph G* is an ordered triple (V, E, w) where $E := \binom{V}{2}$, set of all unordered pairs of vertices, and $w : E \to \{0, 1/2, 1\}$. Define $G_{1/2} = (V, E_{1/2})$ where $E_{1/2} = \{e \in E : w(e) \geq 1/2\}$ and $G_1 = (V, E_1)$ where $E_1 = \{e \in E : w(e) = 1\}$. Denote by $e(G) = \sum_{e \in E(G)} w(e)$.

For two weighted graphs $G = (V, E, w)$ and $G' = (V, E, w')$, define $G \cap G' = (V, E, w'')$ where $w''(e) = \min\{w(e), w'(e)\}$. For $X \subseteq Y \subseteq V$, we call (X, Y) a *weighted* $(|X|, |Y|)$*-clique* or *weighted complete subgraph* of size ℓ if $\binom{X}{2} \subseteq E_1$ and $\binom{Y}{2} \subseteq E_{1/2}$ and $|X| + |Y| = \ell$. Also, let the *weighted clique number* of G be the size of the largest weighted complete subgraph of G.

For a triangle $T = e_1e_2e_3$, let $w(T) = \prod_{i=1}^{3} w(e_i)$. Also, let $T(G) = \sum_{T \in \mathcal{T}} w(T)$, where $\mathcal{T}$ is the set of all triangles in G. For $X \subseteq V(G)$, denote $T(X) = \sum_{T \in G[X]} w(T)$.

We need the following two lemmas and theorem from [12], the first one has been proved in the proof of Theorem 2 in [12].

Lemma 2.7. *For every $\varepsilon > 0$, there exist $\delta > 0$ and n_0 such that for every n-vertex graph G with $n \geq n_0$, if its weighted cluster graph $R(\mathcal{C}, w)$ with respect to ε contains a weighted clique (X, Y) of size ℓ such that $\alpha(G[X]) = \delta n$, then G contains a copy of K_ℓ.*

Lemma 2.8. *For every $\varepsilon > 0$ and integer $k \geq 3$ there exists n_0 such that for every n-vertex weighted graph $G = (V, E, w)$ with $n \geq n_0$, if G does not contain a weighted complete subgraph of size k, then*

$$e(G) \leq (b_k + \varepsilon)n^2,$$

where b_k is defined in (3).

We will use the following multicolored version of the Szemerédi regularity lemma (for example, see [14]).

Theorem 2.9. *For every $\varepsilon > 0$ and integer r, there exists an M such that for every $n > M$ and every r-coloring of the edges of an n-vertex graph G with colors $\{c_1, \ldots, c_r\}$, there exists a partition of $V(G)$ into sets $V_1, \ldots, V_m$ with $||V_i| - |V_j|| \leq 1$, for some $1/\varepsilon < m < M$, which is ε-regular with respect to G_{c_i} for every $1 \leq i \leq r$.*

3 Proof of Theorem 1.3

To overcome the obstacle that all subgraphs induced by each color could have linear-sized independent sets, we need the following simple, but somewhat surprising observation.

Lemma 3.1. *For every $0 < c < 1$, $r \geq 2$, and $a \leq a_r(c) := c^{3 \cdot 2^{r-2}-1}$ the following holds. Let G be an n-vertex graph with $\alpha(G) \leq an$ and an r-edge-coloring $C : E(G) \to \{c_1, \ldots, c_r\}$. Then there exists a partition $V(G) = C_1 \cup \ldots \cup C_r$ such that $\alpha(G_{c_i}[C_i]) \leq cn$ for every $1 \leq i \leq r$.*

Proof. We fix a $c > 0$, and use induction on the number of colors r. For the base case when $r = 2$ and $a \leq c^2$, if $\alpha(G_{c_i}) \leq cn$, for some $i \in \{1, 2\}$, then we can partition $V(G)$ into $C_i = V(G)$ and $C_j = \emptyset$, where $j = \{1, 2\} \setminus \{i\}$, finishing the proof. Therefore, we may assume $\alpha(G_{c_1}) > cn$ and $\alpha(G_{c_2}) > cn$. Let $X_0 = V(G)$, $Y_0 = \emptyset$. We iterate the following operation for $i \geq 1$. At step i, if $\alpha(G_{c_1}[X_{i-1}]) \leq cn$ then we will stop. Otherwise, let I be a maximum independent set in $G_{c_1}[X_{i-1}]$. Since $\alpha(G) \leq an$, we have $\alpha(G_{c_2}[I]) \leq an$. We define $X_i := X_{i-1} \setminus I$ and $Y_i := Y_{i-1} \cup I$. Notice that $\alpha(G_{c_2}[Y_i]) \leq \alpha(G_{c_2}[Y_{i-1}]) + an$. Suppose the iteration stops after k steps, i.e. $\alpha(G_{c_1}[X_k]) \leq cn$, then $k \leq \frac{n}{cn} = 1/c$, which implies that $\alpha(G_{c_2}[Y_k]) \leq k \cdot an \leq cn$ as desired.

For the inductive step, let us assume that the lemma holds for $r-1$ colors, where $r \geq 3$. In particular, we assume that for every $a > 0$, n'-vertex graph H with $\alpha(H) \leq an'$, and $(r-1)$-edge-coloring of H with colors $c'_1, \ldots, c'_{r-1}$, there exists a partition of $V(H) = C'_1 \cup \ldots \cup C'_{r-1}$ such that $\alpha(H_{c'_i}[C'_i]) \leq a^{1/(3 \cdot 2^{r-3}-1)} n'$ for all $1 \leq i \leq r-1$.

Now, we will prove the lemma for r colors. Fix an arbitrary r-edge-coloring of G, we can assume that $\alpha(G_{c_1}) > cn$, otherwise $C_1 = V(G)$ and $C_2 = \ldots = C_r = \emptyset$. Let $C_{1,0} = V(G)$ and $C_{i,0} = \emptyset$ for all $2 \leq i \leq r$. We iterate the following operation. At step k, if $\alpha(G_{c_1}[C_{1,k-1}]) \leq cn$ then we will stop. Otherwise, let I be a maximum independent set of $G_{c_1}[C_{1,k-1}]$ with $n' > cn$ vertices. Since $\alpha(G) \leq a \cdot n$, for some constant $a \leq a_r(c)$, we have $\alpha(G_{c_2}[I] \cup \ldots \cup G_{c_r}[I]) \leq an < an'/c$. We can apply the induction hypothesis to the graph $G[I]$. Therefore, there exists a partition of $I = I_2 \cup \ldots \cup I_r$ such that for every $2 \leq i \leq r$, we have

$$\alpha(G_{c_i}[I_i]) \leq \left(\frac{a}{c}\right)^{\frac{1}{3 \cdot 2^{r-3}-1}} n' \leq c^{\frac{3 \cdot 2^{r-2}-2}{3 \cdot 2^{r-3}-1}} n' \leq c^{\frac{3 \cdot 2^{r-2}-2}{3 \cdot 2^{r-3}-1}} n. \tag{5}$$

Then, we define $C_{1,k} := C_{1,k-1} \setminus I$ and $C_{i,k} := C_{i,k-1} \cup I_i$, for $2 \leq i \leq r$. By (5),

$$\alpha(G_{c_i}[C_{i,k}]) \leq \alpha(G_{c_i}[C_{i,k-1}]) + c^{\frac{3 \cdot 2^{r-2}-2}{3 \cdot 2^{r-3}-1}} n.$$

Let us assume that the iteration stops after l steps, i.e. $\alpha(G_{c_1}[C_{1,l}]) \le cn$. Note that $l \le \frac{n}{cn} = 1/c$, which implies that for every $2 \le i \le r$,

$$\alpha(G_{c_i}[C_{i,l}]) \le l \cdot c^{\frac{3\cdot 2^{r-2}-2}{3\cdot 2^{r-3}-1}} n \le \frac{1}{c} \cdot c^{\frac{3\cdot 2^{r-2}-2}{3\cdot 2^{r-3}-1}} n = c^{\frac{3\cdot 2^{r-2}-2}{3\cdot 2^{r-3}-1}-1} n = cn.$$

□

Proof of Theorem 1.3. (Lower bound) Fix an arbitrary $t \ge 1$. For each $i = 1, 2, 3$, by Theorem 1.2, there is a $\lfloor \frac{4t+i}{2} \rfloor$-partite graph G_i, with partite sets $\{V_1, \ldots, V_{\lfloor (4t+i)/2 \rfloor}\}$, in $\mathcal{H}(n, 4t+i)$ from Construction 2.2 such that $e(G_i) = \mathrm{RT}(n, 4t+i, o(n)) = (b_{4t+i} + o(1))n^2$. We take the following set of colorings.

- $i = 1$: Set $V(G_1) = V_1 \cup \ldots \cup V_{2t}$. Color $G_1[V_p]$, for $1 \le p \le t$, red, color $G_1[V_q]$, for $t+1 \le q \le 2t$, blue, and all cross-edges in $G_1[V_p, V_q]$, $1 \le p < q \le 2t$, in either red or blue.
- $i = 2$: Set $V(G_2) = V_1 \cup \ldots \cup V_{2t+1}$. Color $G_2[V_p]$, for $1 \le p \le t+1$, red, color $G_2[V_q]$, for $t+2 \le q \le 2t+1$, blue, and all cross-edges in $G_2[V_p, V_q]$, $1 \le p < q \le 2t+1$, in either red or blue.
- $i = 3$: Set $V(G_3) = V_1 \cup \ldots \cup V_{2t+1}$. Color $G_3[V_p]$, for $1 \le p \le t$, red, color $G_3[V_q]$, for $t+1 \le q \le 2t+1$, blue, and all cross-edges in $G_3[V_p, V_q]$, $1 \le p < q \le 2t+1$, in either red or blue.

In all three cases, the total number of edges inside all V_i's is $o(n^2)$. Therefore, the total number of cross-edges is $RT(n, 4t+i, o(n)) - o(n^2)$, which implies that we obtain $2^{\mathrm{RT}(n,4t+i,o(n))-o(n^2)}$ 2-edge-colorings. Hence, we are left to show that all these colorings are monochromatic K_{3t+i}-free. For $i = 1, 3$, note that every blue (red resp.) clique have at most one vertex from each V_p (V_q resp.), and at most two vertices from each V_q (V_p resp.). Hence, the size of the largest blue (red resp.) clique is at most $t + 2 \cdot (t + \lfloor \frac{i}{2} \rfloor) < 3t + i$ ($2t + t + \lfloor \frac{i}{2} \rfloor < 3t + i$ resp.). For the case when $i = 2$, fix arbitrary p, q such that $1 \le p \le t+1$ and $t+2 \le q \le 2t+1$. Note that to get a blue clique, we can have at most 1 vertex from each V_p and 2 vertices from each V_q, hence, the largest blue clique has size at most $1 \cdot (t+1) + 2 \cdot t = 3t+1$. For a red clique, we can have at most 1 vertex from each V_q, at most a K_3 from $V_1 \cup V_2$, and at most 2 vertices from each V_p, $p \ne 1, 2$. Thus, the largest red clique is of size at most $3 + 2 \cdot (t-1) + 1 \cdot t = 3t+1$.

(Upper bound) We will prove that for a given constant $\varepsilon > 0$, there exists $\gamma > 0$ and $n_0 > 0$ such that for any $n \ge n_0$ the following holds. Let G be an n-vertex graph with $\alpha(G) \le \gamma^2 n$, the number of 2-edge-colorings of G without a monochromatic K_{3t+i} is at most $2^{(b_{4t+i}+\varepsilon)n^2}$, for $t \ge 1$ and $i = 1, 2, 3$. Also, the number of 2-edge-colorings with no monochromatic K_3 is at most $2^{\varepsilon n^2}$. Throughout the proof, constants are chosen from right to left according to the following hierarchy:

$$0 < 1/n_0 \ll \gamma \ll \delta \ll 1/M \ll \varepsilon_2 \ll \varepsilon_1 \ll \varepsilon < 1, \tag{6}$$

where δ and M resp. are the constants returned from Lemma 2.7 and Theorem 2.9 resp. with ε_2 playing the role of ε.

For any fixed 2-edge-coloring of G, $\phi : E(G) \to \{\phi_1, \phi_2\}$, apply Lemma 3.1 with $r = 2$, $c = \gamma$, and $a_2(c) = \gamma^2$. Let $\{A, B\}$ be the resulting partition such that

$$\alpha(G_{\phi_1}[A]) \le \gamma n, \quad \text{and} \quad \alpha(G_{\phi_2}[B]) \le \gamma n. \tag{7}$$

We then apply Theorem 2.9, with ε_2 playing the role of ε, to G with coloring ϕ, and let $\mathcal{P} = \{P_1, \dots, P_m\}$ be the resulting partition of $V(G)$, where $M > m \ge 1/\varepsilon_2$. Note that we may assume the regularity partition $\mathcal{P}$ refines the $\{A, B\}$-partition. Let R_{ϕ_1} and R_{ϕ_2} be the ϕ_1-colored and ϕ_2-colored weighted cluster graphs respectively, both on vertex set $\{p_1, \dots, p_m\}$, where the vertex p_i represents the vertex set P_i, for all $i \in [m]$. We have

$$\begin{aligned} \text{number of ways to fix an } \{A, B\}\text{-partition of } V(G) &\le 2^n, \\ \text{number of ways to fix a } \mathcal{P}\text{-partition of } V(G) &\le m^n, \\ \text{number of ways to fix } R_{\phi_1} \text{ and } R_{\phi_2} &\le \left(2^{\binom{m}{2}}\right)^4. \end{aligned} \tag{8}$$

Now, we will count the number of colorings with a fixed $\{A, B\}$-partition, $\mathcal{P}$-partition and weighted cluster graphs R_{ϕ_1} and R_{ϕ_2}. First, note that the number of edges of the graph G with both ends in one of the P_i's, between irregular or sparse pairs is at most

$$m \cdot \left(\frac{n}{m}\right)^2 + \varepsilon_2 \cdot m^2 \cdot \left(\frac{n}{m}\right)^2 + 10 \cdot \varepsilon_2 \cdot m^2 \cdot \left(\frac{n}{m}\right)^2 \le \varepsilon_2 n^2 + \varepsilon_2 n^2 + 10 \cdot \varepsilon_2 n^2. \tag{9}$$

Hence, the number of ways to color these edges is at most $2^{12\varepsilon_2 n^2}$. From now on, we will only consider the rest of the edges of G, i.e. the edges between pairs of clusters that are adjacent in $R_{\phi_1} \cup R_{\phi_2}$. Note that there is a unique way to color edges in $R_{\phi_1} \Delta R_{\phi_2}$. Thus the number of 2-edge-colorings corresponding to the fixed $\{A, B\}$-partition, $\mathcal{P}$-partition and weighted cluster graphs R_{ϕ_1} and R_{ϕ_2} is at most

$$2^{\left(\frac{n}{m}\right)^2 e(R_{\phi_1} \cap R_{\phi_2}) + 12\varepsilon_2 n^2}. \tag{10}$$

To complete the proof, it remains to show that

(i) when ϕ is monochromatic K_3-free, $e(R_{\phi_1} \cap R_{\phi_2}) = 0$, and

(ii) when ϕ is monochromatic K_{3t+i}-free for $t \ge 1$ and $i = 1, 2, 3$,

$$e(R_{\phi_1} \cap R_{\phi_2}) \le (b_{4t+i} + \varepsilon_1) \cdot m^2, \tag{11}$$

where b_{4t+i} is defined in (3). Indeed, since the choice of G is arbitrary, (i) together with (8) and (10), implies

$$\mathrm{RF}(2, 3, \gamma^2 n) \;\le\; 2^n \cdot m^n \cdot 2^{4 \cdot \binom{m}{2}} \cdot 2^{12\varepsilon_2 n^2} \le 2^{\varepsilon n^2},$$

and (ii), together with (8) and (10), implies

$$\mathrm{RF}(2, 3t + i, \gamma^2 n) \;\le\; 2^n \cdot m^n \cdot 2^{4 \cdot \binom{m}{2}} \cdot 2^{(b_{4t+i} + \varepsilon_1) n^2 + 12\varepsilon_2 n^2} \le 2^{(b_{4t+i} + \varepsilon) n^2}.$$

To see (i), notice that if there is an edge, say $uv \in E(R_{\phi_1} \cap R_{\phi_2})$, then, without loss of generality, we may assume that $u \in A$. Therefore, by setting $X = \{u\}$ and $Y = \{u, v\}$, it follows from (7) that we have a ϕ_1-colored weighted $(1, 2)$-clique (X, Y) with $\alpha(G_{\phi_1}[X]) \le \gamma n$, which by Lemma 2.7 yields a monochromatic K_3 in ϕ, a contradiction.

For (ii), suppose that (11) is not satisfied. Since $m > 1/\varepsilon_2 > n_1$, we can apply Lemma 2.8 to the graph $R_{\phi_1} \cap R_{\phi_2}$, with ε_1 playing the role of ε. Hence, the graph $R_{\phi_1} \cap R_{\phi_2}$ has a weighted complete subgraph (X, Y) of size $4t + i$, and we shall find a monochromatic K_{3t+i} in G using (X, Y), which is a contradiction. Let $x = |X|$, $y = |Y|$ and $X = \{p_1 \cup \ldots \cup p_x\}$. Without loss of generality, we may assume that $p_1 \cup \ldots \cup p_{\lceil x/2 \rceil} := X' \subseteq A$, i.e. $\alpha(G_{\phi_1}[P_1 \cup \ldots \cup P_{\lceil x/2 \rceil}]) \le \gamma n$. We have thus found a weighted clique (X', Y) in G_{ϕ_1} such that $\alpha(G_{\phi_1}[X']) \le \gamma n$. Hence, Lemma 2.7 shows that G_{ϕ_1} contains a copy of $K_{\lceil x/2 \rceil + y}$.

Claim 3.2. $\lfloor \frac{x}{2} \rfloor \le t$.

Proof. Suppose that $\lfloor \frac{x}{2} \rfloor \ge t + 1$. Recall from the definition of weighted clique that $X \subseteq Y$, i.e. $x \le y$, and $x + y = 4t + i$. Thus,

$$4t + 3 \ge 4t + i = x + y \ge 2x \ge 4 \left\lfloor \frac{x}{2} \right\rfloor \ge 4(t + 1),$$

a contradiction. □

Claim 3.2 then implies that the monochromatic clique corresponding to (X', Y) we found in G_{ϕ_1} is of order

$$\left\lceil \frac{x}{2} \right\rceil + y = x + y - \left\lfloor \frac{x}{2} \right\rfloor \ge 4t + i - t = 3t + i, \tag{12}$$

a contradiction. □

4 Proof of Theorems 1.4 and 1.5

Proof of Theorem 1.5. (Lower bound) Let $D_1, \ldots, D_{n/s}$ be a partition of the high-dimensional unit sphere of equal measure with small diameter as in the Bollobás-Erdős graph construction. Let G be an n-vertex graph with a balanced vertex partition $V_1, \ldots, V_s$, where each V_i consists of one point from each of the n/s domains $D_1, \ldots, D_{n/s}$. For every pair of distinct integers $i, j \in [s]$, let $G[V_i \cup V_j]$ be a copy of $\mathrm{BE}(V_i, V_j)$. Note that each $G[V_i]$ is triangle-free and each $G[V_i \cup V_j]$ is K_4-free. We claim that G is K_{s+2}-free. Indeed, let F be a largest clique in G and let $g_i = |V(F) \cap V_i|$. Since $G[V_i]$ is triangle-free, each $g_i \le 2$. If $|V(F)| \ge s+2$, then there exists at least two indices p, q such that $g_p = g_q = 2$, which contradicts to $G[V_p, V_q]$ being K_4-free. We will count the number of K_s with exactly one vertex from each V_i. Fix a vertex $v_1 \in V_1$, a uniformly at random chosen $v_2 \in V_2$ is adjacent to v_1 if v_2 is in the cap (almost a hemisphere) centered at v_1 with measure $1/2 - o(1)$, which happens with probability $1/2 - o(1)$. Now we fix a clique on vertex set $\{v_1, \ldots, v_{\ell-1}\}$ with $\ell \ge 2$ and $v_i \in V_i$.

The number of vertices in V_ℓ that are in $\bigcap_{i=1}^{\ell-1} N(v_i)$ is at least $2^{-(\ell-1)}n/s - o(n)$. Therefore, we have

$$k_s(G) \geq \prod_{i=1}^{s} \left[\left(2^{-(i-1)} - o(1)\right) \frac{n}{s} \right] = \left(2^{-\binom{s}{2}} - o(1)\right) \left(\frac{n}{s}\right)^s.$$

(Upper bound) We will prove that for a given $\varepsilon > 0$ and integer $s \geq 3$, there exists $\gamma > 0$ and n_0 such that for any $n \geq n_0$ the following holds. Let G be an n-vertex K_{s+2}-free graph with $\alpha(G) \leq \gamma n$, the number of edges of copies of K_s in G is at most $(2^{-\binom{s}{2}} + \varepsilon)(n/s)^s$.

Fix constants from right to left according to the following hierarchy,

$$0 < 1/n_0 \ll \gamma \ll \delta \ll 1/M \ll \varepsilon_1 \ll \varepsilon < 1,$$

where δ and M resp. are the constants returned from Lemma 2.7 and Theorem 2.9 resp. with ε_1 playing the role of ε.

First, we apply Theorem 2.9, with ε_1 playing the role of ε, to the graph G and let $\mathcal{P} = \{P_1, \dots, P_m\}$ be the resulting partition of $V(G)$, where $M > m \geq 1/\varepsilon_1$, and let R be the weighted cluster graph with respect to ε_1. We call an edge in R *heavy* if it has weight 1. We claim that the graph R does not contain any weighted $(1, s+1)$- or $(2, s)$-clique. Otherwise, we apply Lemma 2.7 to the graph R with ε_1 playing the role of ε. Since $\alpha(G[X]) \leq \alpha(G) \leq \gamma n \leq \delta n$, G contains a copy of K_{s+2}, a contradiction. In other words, we have that R is K_{s+1}-free and does not have a copy of K_s with at least one heavy edge.

Now, we can count the total number of copies of K_s in G. Note that similarly to (9), the total number of edges inside all clusters, between irregular pairs, or sparse pairs is at most $12 \cdot \varepsilon_1 n^2$. Therefore, the total number of copies of K_s with at least one such edge is at most $12 \cdot \varepsilon_1 n^2 \cdot n^{s-2}$. Since R is K_{s+1}-free, by the result of Erdős [10], it has at most $(m/s)^s$ copies of K_s. Also, since R does not have a copy of K_s with a heavy edge, it implies that each K_s in R has weight at most $(1/2 + 10 \cdot \varepsilon_1)^{\binom{s}{2}} \leq 2^{-\binom{s}{2}} + \varepsilon/2$, where the last inequality holds as $\varepsilon_1 \ll \varepsilon$. Hence, we have that the number of K_s in G is at most

$$\left(2^{-\binom{s}{2}} + \frac{\varepsilon}{2}\right) \cdot \left(\frac{n}{m}\right)^s \cdot \left(\frac{m}{s}\right)^s + 12 \cdot \varepsilon_1 n^s \leq \left(2^{-\binom{s}{2}} + \varepsilon\right) \left(\frac{n}{s}\right)^s,$$

as desired. □

The following lemma is the main step for proving Theorem 1.4.

Lemma 4.1. *For every integer $t \geq 4$ and n-vertex weighted graph $G = (V, E, w)$ (as in Definition 2.6) with no weighted complete subgraph of size t, we have*

$$T(G) \leq a_t n^3,$$

where a_t is as in (4).

Proof. Let $G = (V, E, w)$ be an n-vertex weighted graph that satisfies the hypothesis and is extremal, i.e. has the maximum number of triangles. First, we will apply two rounds of the so-called symmetrization method to the graph G. For $v, v' \in V(G)$, denote $T_v(G) = \sum_{v \in T \in G} w(T)$, the number of weighted triangles containing v. Similarly, define $T_{vv'}(G) = \sum_{v,v' \in T \in G} w(T)$. Let $V(G) = \{v_1, \ldots, v_n\}$ such that $T_{v_1}(G) \geq \ldots \geq T_{v_n}(G)$. Define $S_1(i,j)$, for $i \neq j \in [n]$ to be the following operation: if $v_iv_j \notin G_{1/2}$ then we replace v_j with a copy of v_i, i.e. change $w(v_jv_k)$ to $w(v_iv_k)$ for all $k \neq i, j$. If $i < j$, then the number of triangles changes by $T_{v_i}(G) - T_{v_j}(G) \geq 0$. Since G is extremal, we have that $T_{v_i}(G) = T_{v_j}(G)$ for any $v_iv_j \notin G_{1/2}$. Consequently, the following process, denoted by S_1, is finite: apply $S_1(i,j)$ for every $1 \leq i < j \leq n$ with $v_iv_j \notin G_{1/2}$. Note that S_1 will not increase the weighted clique number and keep the same number of triangles. After S_1, in the resulting graph $v_iv_j \notin G_{1/2}$ is an equivalence relation. Denote by $\mathcal{A} = \{A_1, \ldots, A_m\}$ the equivalence classes of this relation, i.e. two vertices u and v are in the same class if and only if $uv \notin G_{1/2}$. Therefore, for fixed $1 \leq i, j \leq m$, all the edges between A_i and A_j have equal weights, which we denote by $w(A_iA_j)$, and for all vertices $x, x' \in A_i$ and $y, y' \in A_j$, we have

$$T_x(G) = T_{x'}(G) \quad \text{and} \quad T_{xy}(G) = T_{x'y'}(G).$$

Therefore, we can define $T_{A_i}(G) = T_x(G)$ and $T_{A_iA_j}(G) = T_{xy}(G)$. Note that if (X, Y) is one of the largest weighted complete subgraphs of G, then $|Y| = m$.

We summarize the structure of G as follows: Let H be a weighted complete graph on vertex set $\{a_1, \ldots, a_m\}$ with all its edges having weight either 1 or 1/2, and $w(a_ia_j) = w(A_iA_j)$. The graph G is a blow-up of H where we replace each a_i with a set of $|A_i|$ vertices, and inside each A_i the weight of all edges is zero.

Our next goal is to show that a second round of symmetrization can be carried out in G, in other words, in H, $w(a_ia_j) = 1/2$ is an equivalence relation. Without loss of generality we may assume $T_{A_1}(G) \geq \ldots \geq T_{A_m}(G)$. For every $1 \leq i, j \leq m$, define $S_2(i,j)$ to be the following operation: Change $w(A_jA_k)$ to $w(A_iA_k)$ for all $k \neq i, j$, and denote G_{A_i} the resulting graph. Define G_{A_j} analogously as the graph obtained from applying $S_2(j,i)$ to G. The following claim states that when $w(A_iA_j) = 1/2$, we can replace vertices in A_i with copies of vertices in A_j, or the other way around, without decreasing the number of triangles.

Claim 4.2. For every pair of integers $1 \leq i < j \leq m$ with $w(A_iA_j) = 1/2$,
(i) $T_{A_i}(G) = T_{A_j}(G)$;
(ii) $k_3(G_{A_i}) = k_3(G_{A_j}) = k_3(G)$.

Proof. Define

$$T^o_{A_i}(G) = T_{A_i}(G) - T_{A_iA_j}(G) \quad \text{and} \quad G' = G \setminus \{A_i \cup A_j\}.$$

Since $T_{A_i}(G) \geq T_{A_j}(G)$, we have

$$T^o_{A_i}(G) + T_{A_iA_j}(G) \geq T^o_{A_j}(G) + T_{A_iA_j}(G) \quad \Leftrightarrow \quad T^o_{A_i}(G) \geq T^o_{A_j}(G). \tag{13}$$

For (i), it suffices to show

$$T^o_{A_i}(G) = T^o_{A_j}(G). \tag{14}$$

Note that

$$\begin{aligned} k_3(G) &= k_3(G') + |A_i| \cdot T^o_{A_i}(G) + |A_j| \cdot T^o_{A_j}(G) + |A_i| \cdot |A_j| \cdot T_{A_iA_j}(G), \qquad (15) \\ k_3(G_{A_i}) &= k_3(G') + (|A_i| + |A_j|) \cdot T^o_{A_i}(G) + |A_i| \cdot |A_j| \cdot T_{A_iA_j}(G_{A_i}). \qquad (16) \end{aligned}$$

Then, since G is extremal,

$$0 \geq k_3(G_{A_i}) - k_3(G) = |A_j| \cdot (T^o_{A_i}(G) - T^o_{A_j}(G)) + |A_i| \cdot |A_j| \cdot (T_{A_iA_j}(G_{A_i}) - T_{A_iA_j}(G)).$$

Therefore, by (13), we only need to show $T_{A_iA_j}(G_{A_i}) \geq T_{A_iA_j}(G)$. Let

$$\begin{aligned} V_{1,1/2} &= \left\{ A_\ell : w(A_iA_\ell) = 1 \text{ and } w(A_jA_\ell) = \frac{1}{2} \right\}, \\ V_{1/2,1} &= \left\{ A_\ell : w(A_iA_\ell) = \frac{1}{2} \text{ and } w(A_jA_\ell) = 1 \right\}, \\ V_{1/2,1/2} &= \left\{ A_\ell : w(A_iA_\ell) = \frac{1}{2} \text{ and } w(A_jA_\ell) = \frac{1}{2} \right\}, \\ V_{1,1} &= \left\{ A_\ell : w(A_iA_\ell) = 1 \text{ and } w(A_jA_\ell) = 1 \right\}. \end{aligned}$$

Denote by $|V_{p,q}| = \sum_{A_\ell \in V_{p,q}} |A_\ell|$ for $p, q \in \{1/2, 1\}$. We have

$$T_{A_iA_j}(G_{A_i}) - T_{A_iA_j}(G) = \left(\frac{1}{2} - \frac{1}{4} \right) |V_{1,1/2}| - \left(\frac{1}{4} - \frac{1}{8} \right) |V_{1/2,1}| = \frac{1}{4}|V_{1,1/2}| - \frac{1}{8}|V_{1/2,1}|.$$

Therefore, it suffices to show $2|V_{1,1/2}| \geq |V_{1/2,1}|$. For the sake of contradiction, assume

$$|V_{1/2,1}| > 2|V_{1,1/2}|. \qquad (17)$$

We will show that (17) contradicts the extremality of G. Note that

$$k_3(G_{A_j}) = k_3(G') + (|A_i| + |A_j|) \cdot T^o_{A_j}(G) + |A_i| \cdot |A_j| \cdot T_{A_iA_j}(G_{A_j}). \qquad (18)$$

By (15), (16), (18), and the extremality of G we have

$$k_3(G_{A_i}) \leq k_3(G) \Leftrightarrow \left(\frac{1}{4}|V_{1,1/2}| - \frac{1}{8}|V_{1/2,1}| \right) |A_i| \cdot |A_j| + |A_j| \cdot T^o_{A_i}(G) \leq |A_j| \cdot T^o_{A_j}(G), \qquad (19)$$

$$k_3(G_{A_j}) \leq k_3(G) \Leftrightarrow \left(\frac{1}{4}|V_{1/2,1}| - \frac{1}{8}|V_{1,1/2}| \right) |A_i| \cdot |A_j| + |A_i| \cdot T^o_{A_j}(G) \leq |A_i| \cdot T^o_{A_i}(G). \qquad (20)$$

Then (19) and (20) imply

$$\begin{aligned} &\left(\frac{1}{4}|V_{1,1/2}| - \frac{1}{8}|V_{1/2,1}| \right) |A_i| + T^o_{A_i}(G) \leq T^o_{A_j}(G), \\ &\left(\frac{1}{4}|V_{1/2,1}| - \frac{1}{8}|V_{1,1/2}| \right) |A_j| + T^o_{A_j}(G) \leq T^o_{A_i}(G). \end{aligned}$$

Therefore,

$$\begin{aligned}
&\left(\frac{1}{4}|V_{1/2,1}| - \frac{1}{8}|V_{1,1/2}|\right)|A_j| \le T^o_{A_i}(G) - T^o_{A_j}(G) \le \left(\frac{1}{8}|V_{1/2,1}| - \frac{1}{4}|V_{1,1/2}|\right)|A_i| \\
\Rightarrow\ & \frac{1}{8}|V_{1/2,1}|(2|A_j| - |A_i|) \le \frac{1}{8}|V_{1,1/2}|(|A_j| - 2|A_i|) \overset{(17)}{<} \frac{1}{16}|V_{1/2,1}|(|A_j| - 2|A_i|) \\
\Rightarrow\ & 4|A_j| - 2|A_i| < |A_j| - 2|A_i| \quad \Rightarrow \quad 4|A_j| < |A_j|,
\end{aligned}$$

a contradiction.

For (ii), by the extremality of G, it suffices to show that $k_3(G_{A_i}) + k_3(G_{A_j}) \ge 2k_3(G)$. By (14), (15), (16) and (18), we have

$$k_3(G_{A_i}) + k_3(G_{A_j}) - 2k_3(G) = |A_i||A_j| \cdot (T_{A_iA_j}(G_{A_i}) + T_{A_iA_j}(G_{A_j}) - 2T_{A_iA_j}(G)).$$

It is left to show that $T_{A_iA_j}(G_{A_i}) + T_{A_iA_j}(G_{A_j}) - 2T_{A_iA_j}(G) \ge 0$. Indeed,

$$\begin{aligned}
&T_{A_iA_j}(G_{A_i}) + T_{A_iA_j}(G_{A_j}) - 2T_{A_iA_j}(G) \\
&= \frac{1}{2}\sum_{\substack{1\le k\le m,\\ k\ne i,j}} w(A_iA_k)^2|A_k| + \frac{1}{2}\sum_{\substack{1\le k\le m,\\ k\ne i,j}} w(A_jA_k)^2|A_k| - 2\cdot\frac{1}{2}\sum_{\substack{1\le k\le m,\\ k\ne i,j}} w(A_iA_k)w(A_jA_k)|A_k| \\
&= \frac{1}{2}\sum_{\substack{1\le k\le m,\\ k\ne i,j}} \left(w(A_iA_k) - w(A_jA_k)\right)^2|A_k| \ge 0.
\end{aligned}$$

□

Denote by S_2 the following process: let σ be the lexicographical ordering of $\binom{[m]}{2}$ and apply $S_2(i,j)$, according to σ, for all pairs (i,j) with $w(A_iA_j) = 1/2$. By Claim 4.2, S_2 is finite and keeps the number of triangles.

Claim 4.3. The operation S_2 does not change the weighted clique number of G.

Proof. Let (X,Y) be one of the largest weighted complete subgraphs of G of size ℓ. Note that $|Y|$ is still m. Also, since we only repeat this operation for vertices x and y with $w(xy) = 1/2$, the operation is not changing $|X|$ either. Hence, after repeated applications of this operation, the weighted clique number of G will not change. □

After applying S_2, we have an equivalence relation on classes $A_1,\dots,A_m$, which naturally extends to $V(G)$. To be precise, denote by $\mathcal{B} = \{B_1,\dots,B_{m'}\}$ the equivalence classes of this relation, i.e. two vertices u and v are in the same class if and only if $uv \notin G_1$. Then, the $\mathcal{A}$-partition is a refinement of the $\mathcal{B}$-partition. More importantly, the size of the largest weighted complete subgraph is $m+m'$.

We will next show that we can perform some transformations (Claims 4.4 and 4.5) to get a more structured graph (as those in Construction 2.2) without increasing the weighted clique number and decreasing the number of triangles.

Claim 4.4. Each B_i contains at most two A_j's.

Proof. Let us assume that B_1 contains k A_j's, $A_1, \ldots, A_k$, where $k \geq 3$. Denote by U the vertex set of B_1 and write $u = |U|$. Note that the edges between two B_i's always have weight 1 and the edges inside an A_i have weight 0 and all the other edges have weight 1/2. We will divide the proof into three cases depending on the value of k. In each case, we will modify B_1 by splitting it into multiple parts. This modification will only change the weight of the edges with both ends in U and also the equivalence classes $\mathcal{A}$ and $\mathcal{B}$. Then we need to prove that the weighted clique number did not increase, and the number of triangles increases. For the latter, since the weight of the edges with at least one end in $V \setminus U$ remain the same, we only need to show that the number of triangles with two or three vertices in U did not decrease. Therefore, it suffices to show that both $e(U)$ and $T(U)$ increase.

Case 1: Assume $k \geq 5$, which implies $u \geq 5$. We will split vertices in U into three parts, B_{11}, B_{12} and B_{13}, such that $|B_{11}| \leq |B_{12}| \leq |B_{13}| \leq |B_{11}| + 1$. Also, define $A_i = B_{1i}$ for all $1 \leq i \leq 3$. For every $u \in U$ and $v \in V \setminus U$, we will not change $w(uv)$. For all vertices $u, u' \in U$ if they belong to the same B_{1i}, let $w(uu') = 0$, otherwise let $w(uu') = 1$. The equivalence classes $\mathcal{A}$ and $\mathcal{B}$ will change to $\{A_1, A_2, A_3, A_{k+1}, \ldots, A_m\}$ and $\{B_{11}, B_{12}, B_{13}, B_2, \ldots, B_{m'}\}$. Since $k \geq 5$, the number of classes in the $\mathcal{A}$ partition decreased by at least two and the number of classes in the $\mathcal{B}$ partition increased by exactly 2, hence, the weighted clique number of G will not increase. Now, we only need to show that the number of triangles in the graph G increases.

$$\text{before:}\quad e(U) \leq \binom{k}{2}\frac{u^2}{k^2}\cdot\frac{1}{2} < \frac{u^2}{4},$$
$$\text{after:}\quad e(U) = \begin{cases} 3\cdot\frac{u^2}{9} = \frac{u^2}{3} & \text{if } u \equiv 0 \pmod 3, \\ \frac{(u-1)^2}{9} + 2\cdot\frac{(u-1)(u+2)}{9} = \frac{u^2-1}{3} & \text{if } u \equiv 1 \pmod 3, \\ \frac{(u+1)^2}{9} + 2\cdot\frac{(u-2)(u+1)}{9} = \frac{u^2-1}{3} & \text{if } u \equiv 2 \pmod 3. \end{cases}$$

Therefore $e(U)$ increases for $u \geq 2$. Now, for $T(U)$ we have

$$\text{before:}\quad T(U) \leq \binom{k}{3}\frac{u^3}{k^3}\cdot\frac{1}{8} \leq \frac{u^3}{48},$$
$$\text{after:}\quad T(U) = \begin{cases} \frac{u^3}{27} & \text{if } u \equiv 0 \pmod 3, \\ \frac{(u-1)(u-1)(u+2)}{27} & \text{if } u \equiv 1 \pmod 3, \\ \frac{(u-2)(u+1)(u+1)}{27} & \text{if } u \equiv 2 \pmod 3, \end{cases}$$

which means that $T(U)$ increases if $u \geq 3$.

Case 2: Assume $k = 4$ which implies $u \geq 4$. Let us split vertices in U into three parts A_1, A_2 and A_3, such that $|A_1| \leq |A_2| \leq |A_3| \leq |A_1| + 1$. Also let $B_{11} = A_1 \cup A_2$ and $B_{12} = A_3$. For all vertices $u, u' \in U$ if they are in different B_{1i}'s then $w(uu') = 1$. If they are both in B_{11} but in different A_i's then $w(uu') = 1/2$, and $w(uu') = 0$ if they are in the same A_i. The equivalence classes $\mathcal{A}$ and $\mathcal{B}$ will change to $\{A_1, A_2, A_3, A_5, \ldots, A_m\}$ and $\{B_{11}, B_{12}, B_2, \ldots, B_{m'}\}$. Notice that the number of classes in the $\mathcal{A}$ partition decreased by one and the number of classes in $\mathcal{B}$ increased by one, hence, the weighted clique number of G will not change. For $e(U)$:

$$\text{before:}\quad e(U) \le \binom{4}{2}\frac{u^2}{16}\cdot\frac{1}{2} = \frac{3u^2}{16},$$
$$\text{after:}\quad e(U) = \begin{cases} \frac{1}{2}\cdot\frac{u^2}{9} + 2\cdot\frac{u^2}{9} & \text{if } u \equiv 0 \pmod 3,\\ \frac{1}{2}\cdot\frac{(u-1)(u-1)}{9} + 2\cdot\frac{(u-1)(u+2)}{9} & \text{if } u \equiv 1 \pmod 3,\\ \frac{1}{2}\cdot\frac{(u-2)(u+1)}{9} + \frac{(u-2)(u+1)}{9} + \frac{(u+1)(u+1)}{9} & \text{if } u \equiv 2 \pmod 3.\end{cases}$$

For $u \ge 2$, $e(U)$ increases. We also need to show that $T(U)$ increases:

$$\text{before:}\quad T(U) \le 4\cdot\frac{u^3}{4^3}\cdot\frac{1}{8} = \frac{u^3}{4^3}\cdot\frac{1}{2},$$
$$\text{after:}\quad T(U) = \begin{cases} \frac{1}{2}\cdot\frac{u^3}{27} & \text{if } u \equiv 0 \pmod 3,\\ \frac{1}{2}\cdot\frac{(u-1)(u-1)(u+2)}{27} & \text{if } u \equiv 1 \pmod 3,\\ \frac{1}{2}\cdot\frac{(u-2)(u+1)(u+1)}{27} & \text{if } u \equiv 2 \pmod 3.\end{cases}$$

Therefore $T(U)$ will increase for $u \ge 3$.

Case 3: Assume $k = 3$, which implies $u \ge 3$. First, suppose that $u \le 12n/13$. Split vertices in U into two equal parts B_{11} and B_{12}. Also define $A_1 = B_{11}$ and $A_2 = B_{12}$. For all vertices $u, u' \in U$ if they are in different B_{1i}'s then set $w(uu') = 1$, and $w(uu') = 0$ otherwise. The equivalence classes $\mathcal{A}$ and $\mathcal{B}$ will change to $\{A_1, A_2, A_4, \ldots, A_m\}$ and $\{B_{11}, B_{12}, B_2, \ldots, B_{m'}\}$. Notice that the number of classes in the $\mathcal{A}$ partition decreased by one and the number of classes in the $\mathcal{B}$ partition increased by one, hence, the weighted clique number of G will not change. For the change on the number of triangles, we have

$$\text{before:}\quad T(U) + e(U)(n-u) \le \frac{u^3}{3^3}\cdot\frac{1}{2^3} + \frac{1}{2}\cdot\binom{3}{2}\frac{u^2}{9}(n-u),$$
$$\text{after:}\quad T(U) + e(U)(n-u) = \begin{cases} 0 + \frac{u^2}{4}\cdot(n-u) & \text{if } u \text{ is even},\\ 0 + \frac{(u-1)(u+1)}{4}\cdot(n-u) \ge \frac{u^2}{4.5}(n-u) & \text{if } u \text{ is odd}.\end{cases}$$

Since $u \le 12n/13$, we have

$$\frac{u^3}{3^3}\cdot\frac{1}{2^3} + \frac{3}{2}\cdot(n-u)\cdot\frac{u^2}{3^2} \le (n-u)\cdot\frac{u^2}{4.5} \quad\Leftrightarrow\quad \frac{u^3}{6^3} \le (n-u)\cdot\frac{u^2}{18} \quad\Leftrightarrow$$
$$\frac{u}{12} \le (n-u) \quad\Leftrightarrow\quad u \le \frac{12}{13}n.$$

We may now assume that $u > 12n/13$. Let U' be the vertex set of B_2 and $u' = |B_2|$. Since $u \ge 12n/13$ and $u' \le n/13$, we may assume B_2 contains at most two A_i's. Note that $u' \le u/12$. We split $U \cup U'$ into three classes of the same size, B_0, B_1 and B_2. Define $A_0 = B_0$, $A_1 = B_1$, and $A_2 = B_2$. For two vertices $u, u' \in U \cup U'$, if they belong to the same B_i then $w(uu') = 0$, otherwise $w(uu') = 1$. The equivalence classes $\mathcal{A}$ and $\mathcal{B}$ will change to $\{A_0, A_1, A_2, A_5, \ldots, A_m\}$ and $\{B_0, B_1, B_2, B_3, \ldots, B_{m'}\}$. Notice that the number of classes in $\mathcal{A}$ decreased by one and the number of classes in $\mathcal{B}$ increased by one, which implies that the weighted clique number of G will not change. We are left to show that this operation will increase $e(U \cup U')$ and $T(U \cup U')$:

$$\text{before:}\quad e(U \cup U') \le \frac{u^2}{3^2}\cdot\frac{3}{2} + uu' + \frac{u'^2}{8} \le \frac{u^2}{6} + \frac{u^2}{12} + \frac{u'^2}{8} = \frac{3u^2}{12} + \frac{u'^2}{8},$$

$$\text{after:}\quad e(U \cup U') = \begin{cases} 3\cdot\frac{(u+u')^2}{9} & \text{if } u+u' \equiv 0 \pmod 3,\\ \frac{(u+u'-1)(u+u'-1)}{9} + 2\cdot\frac{(u+u'-1)(u+u'+2)}{9} & \text{if } u+u' \equiv 1 \pmod 3,\\ 2\cdot\frac{(u+u'-2)(u+u'+1)}{9} + \frac{(u+u'+1)^2}{9} & \text{if } u+u' \equiv 2 \pmod 3. \end{cases}$$

Hence $e(U \cup U')$ is increasing for $u \ge 3$. We also have

$$\text{before:}\quad T(U \cup U') \le \left(\frac{u}{3}\right)^3\cdot\frac{1}{8} + \frac{3}{2}\cdot\frac{u^2}{3^2}\cdot u' + u\cdot\frac{u'^2}{8} \le \frac{u^3}{6^3} + \frac{u^3}{6\cdot 12} + \frac{u^3}{8\cdot 12^2} \le \frac{u^3}{51.5},$$

$$\text{after:}\quad T(U \cup U') = \begin{cases} \frac{(u+u')^3}{27} & \text{if } u+u' \equiv 0 \pmod 3,\\ \frac{(u+u'-1)(u+u'-1)(u+u'+2)}{27} & \text{if } u+u' \equiv 1 \pmod 3,\\ \frac{(u+u'-2)(u+u'+1)(u+u'+1)}{27} & \text{if } u+u' \equiv 2 \pmod 3. \end{cases}$$

Therefore $T(U \cup U')$ increases for $u + u' \ge 3$. □

Claim 4.5. There is at most one B_i that contains two A_j's.

Proof. Now, we know that no B_i contains three or more A_i's. Let us assume that $B_1 = A_1 \cup A_2$ and $B_2 = A_3 \cup A_4$. Denote by U the vertex set of $B_1 \cup B_2$, and write $u = |U|$. Since each A_i contains at least one vertex, we have that $u \ge 4$. We will split the vertices in U into three equal pieces, B_{11}, B_{12} and B_{13}, and redefine $A_1 = B_{11}$, $A_2 = B_{12}$, and $A_3 = B_{13}$. For two vertices $u, u' \in U$ if they are in two different B_{1i}'s then $w(uu') = 1$, otherwise $w(uu') = 0$. This operation will change $\mathcal{A}$ and $\mathcal{B}$ to $\{A_1, A_2, A_3, A_5, \dots, A_m\}$ and $\{B_{11}, B_{12}, B_{13}, B_3, \dots, B_{m'}\}$, therefore the weighted clique number does not change. We only need to show that $e(U)$ and $T(U)$ increase.

$$\text{before:}\quad e(U) \le \frac{u^2}{4^2} + \frac{u^2}{4} = \frac{5u^2}{16},$$

$$\text{after:}\quad e(U) = \begin{cases} 3\cdot\frac{u^2}{9} & \text{if } u \equiv 0 \pmod 3,\\ \frac{(u-1)^2}{9} + 2\cdot\frac{(u-1)(u+2)}{9} & \text{if } u \equiv 1 \pmod 3,\\ \frac{(u+1)^2}{9} + 2\cdot\frac{(u-2)(u+1)}{9} & \text{if } u \equiv 2 \pmod 3, \end{cases}$$

$$\text{before:}\quad T(U) \le \frac{u^2}{16}\cdot\frac{u}{2} = \frac{u^3}{32},$$

$$\text{after:}\quad T(U) = \begin{cases} \frac{u^3}{27} & \text{if } u \equiv 0 \pmod 3,\\ \frac{(u-1)^2(u+2)}{27} & \text{if } u \equiv 1 \pmod 3,\\ \frac{(u+1)^2(u-2)}{27} & \text{if } u \equiv 2 \pmod 3. \end{cases}$$

It can be easily checked that for $u \ge 4$, both $e(U)$ and $T(U)$ are not decreasing. □

Now, we will use the Claims 4.4 and 4.5 to complete the proof of Lemma 4.1. Let us assume that the extremal graph has partitions $\mathcal{A} = \{A_1, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, \ldots, B_{m'}\}$. Also, since $\mathcal{A}$ is a refinement of $\mathcal{B}$ and also by Claims 4.4 and 4.5, we have $m' \leq m \leq m' + 1$. When $t = 2\ell + 1$, the graph does not contain a weighted clique of size $2\ell + 1$, which implies $m + m' \leq 2\ell$. Therefore $m' = m = \ell$ will maximize the number of triangles. In particular, the extremal graph is an ℓ-partite graph with partite sets $B_1 \cup \ldots \cup B_\ell$, where $||B_i| - |B_j|| \leq 1$ for all $1 \leq i < j \leq \ell$. Define $A_i = B_i$ for all $1 \leq i \leq \ell$, and for two vertices u and v if they belong to two different B_i's then $w(uv) = 1$, otherwise $w(uv) = 0$.

When $t = 2\ell$, the graph does not contain a weighted clique of size 2ℓ which implies $m + m' \leq 2\ell - 1$. Therefore, in the extremal example, $m' = \ell - 1$ and $m = \ell$. Hence, the extremal example is an $(\ell - 1)$-partite graph, with partite sets $B_1 \cup \ldots \cup B_{\ell-1}$, and let $B_1 = A_1 \cup A_2$. Simple optimization shows that $|A_1| = |A_2|$, and, for all $2 \leq i \leq \ell - 1$, all the B_i's have the same size, i.e. $|B_1| = x$ and $|B_i| = (n-x)/(\ell - 2)$ for all $2 \leq i \leq \ell - 1$. Fix two vertices u and v, if they belong to two different B_i's then set $w(uv) = 1$. Otherwise, if they both belong to B_1 but to different A_i's then set $w(uv) = 1/2$, and $w(uv) = 0$ in all other cases. Now, we only need to maximize the number of triangles with respect to x, which is exactly the optimization in (4), showing that $T(G) \leq a_t n^3$. This completes the proof of Lemma 4.1. □

Proof of Theorem 1.4. For any given integer $t \geq 6$, let $\ell = \left\lfloor \frac{t}{2} \right\rfloor$. The lower bound comes from $\mathcal{H}(n, k)$ with $k = t$ in Construction 2.2. In this case, we solve an optimization problem to find the size of V_i's that maximizes the number of triangles, which is how a_ℓ is defined in (4).

For the upper bound, we will show that for any $\varepsilon > 0$, there exists $\delta > 0$ and $n_0 > 0$ such that the following holds for any $n \geq n_0$. Let G be an n-vertex K_t-free graph with $\alpha(G) \leq \delta n$. Then G has at most $(1+\varepsilon)a_\ell n^3$ triangles. Choose constants

$$0 \ll 1/n_0 \ll \delta \ll 1/M \ll \varepsilon_1 \ll \varepsilon < 1,$$

where δ and M resp. are the constants returned from Lemma 2.7 and Theorem 2.9 resp. with ε_1 playing the role of ε.

Let $R = R(\mathcal{C}, w)$ be the weighted cluster graph obtained from applying Theorem 2.9 to G with ε_1 playing the role of ε. By Lemma 2.7, we have that $R(\mathcal{C}, w)$ does not contain a weighted clique of size t. Then the upper bound follows from Lemma 4.1 and that

$$k_3(G) \leq T(R) \cdot \frac{n^3}{|R|^3} + \varepsilon n^3 \leq (1+\varepsilon)a_\ell n^3,$$

as desired, where the last term bounds the number of triangles in G that do not correspond to a triangle in R. □

5 Concluding remarks

In this paper, we study the Ramsey-Turán extensions of two special cases of classical problems. We determine $\mathrm{RF}(2, k, o(n))$, that is, the maximum number of 2-edge-colorings an

n-vertex graph with independence number $o(n)$ can have without a monochromatic K_k, and $\mathrm{RT}(K_3, K_t, o(n))$, the maximum number of triangles in an n-vertex K_t-free graph with $o(n)$ independence number.

5.1 3-edge-colorings

The Ramsey-Turán extension of the Erdős-Rothschild problem for more than 2 colors remains widely open. It is known [1] that $F(n, 3, k) = 3^{\mathrm{ex}(n,K_k)}$. It will be interesting to study for 3-edge-colorings, $\mathrm{RF}(3, k, o(n))$. The following determines the case when forbidding monochromatic triangles. We give here only a sketch of a proof.

Theorem 5.1. $\mathrm{RF}(3, 3, o(n)) = 2^{n^2/4+o(n^2)}$.

Sketch of a proof. (Lower bound) Let $G \in \mathcal{H}(n, 5)$ with $|V_1| = |V_2| = n/2$. Consider the following 3-edge-colorings. Color edges in $G[V_i]$, $i = 1, 2$, red and color the cross-edges $G[V_1, V_2]$ either green or blue.

(Upper bound) Let G be an extremal graph, $\phi : E(G) \to \{\phi_1, \phi_2, \phi_3\}$ be a 3-edge-coloring with no monochromatic K_3, and (A_1, A_2, A_3) be the partition obtained from Lemma 3.1 such that $\alpha(G_{\phi_i}[A_i]) = o(n)$. Let R^* be the multigraph by taking the union $\cup_{i=1}^{3} R_{\phi_i}$, where R_{ϕ_i} is the cluster graph in color ϕ_i. Denote by μ_i, $i = 1, 2, 3$, the edge-density of the subgraph of R^* induced by edges with multiplicity i. Note first that $\mu_3 = 0$, since otherwise a multiplicity-3 edge results in a weighted clique of size 3 in $\cap_i R_{\phi_i}$, contradicting to ϕ containing no monochromatic K_3. It suffices then to show $\mu_2 \le 1/2$. Notice that no ϕ_i-colored edge can have an endpoint in A_i, otherwise we have a ϕ_i-colored triangle. This implies that

(i) for every $i \neq j \in [3]$, all edges in $R^*[A_i, A_j]$ have multiplicity 1 with color ϕ_k, $k \neq i, j$;

(ii) for $i \in [3]$, all edges in $R^*[A_i]$ have multiplicity at most 2, colored in $\{j, k\} = [3] \setminus \{i\}$.

By (i), we only need to consider edges in $\cup_i R^*[A_i]$. By (ii), inside A_i, there is no color ϕ_i. This together with the observation that edges colored in $\{\phi_p, \phi_q\}$ for any $p \neq q \in [3]$ is triangle-free, we have $\mu_2 \le 1/2$ as desired. □

5.2 Generalized Ramsey-Turán for larger cliques

It seems plausible that for the general case $\mathrm{RT}(K_s, K_t, o(n))$, $t > s \ge 3$, some graph from the following construction has the maximum number of K_s.

Construction 5.2. Given $3 \le s < t \le 2s - 1$, denote by $\mathcal{H}(n, s, t)$ the family of n-vertex graphs G on vertex set $V_1 \cup \ldots \cup V_s$ obtained as follows. Let H be an extremal K_{t-s}-free graph on vertex set $[s]$. Make $[V_i, V_j]$ complete bipartite if $ij \in E(H)$; otherwise, put a copy of $\mathrm{BE}(V_i, V_j)$ if $ij \notin E(H)$. For every $i \in V(H)$ with $d_H(i) = s - 1$, put a $|V_i|$-vertex triangle-free graph with $o(|V_i|)$ independence number in V_i.

Note that all graphs G in the above construction are K_t-free and have $o(n)$ independence number. Indeed, since $G[V_i]$ is triangle-free for all i, in order to have a copy of K_t, there should be at least $t - s$ classes, $V_{j_1}, \ldots, V_{j_{t-s}}$, each containing 2 vertices that form a $K_{2(t-s)}$.

This would imply for every $1 \le p < q \le t-s$, $G[V_{j_p}, V_{j_q}]$ contains a K_4, which contradicts to H being K_{t-s}-free. It should also be noted that the sizes of V_i's need to be optimized.

Conjecture 5.3. Given integers $t > s \ge 3$, one of the extremal graphs for $\mathrm{RT}(K_s, K_t, o(n))$ lies in $\mathcal{H}(n, s, t)$ from Construction 5.2 when $t \le 2s-1$, and lies in $\mathcal{H}(n, k)$ with $k = t$ from Construction 2.2 when $t \ge 2s$.

5.3 Phase transition

For a given graph H and two functions $f(n) \le g(n)$, we say that the Ramsey-Turán function for H exhibits a *jump* or has a *phase transition* from $g(n)$ to $f(n)$ if

$$\limsup_{n\to\infty} \frac{\mathrm{RT}(n, H, f(n))}{n^2} < \liminf_{n\to\infty} \frac{\mathrm{RT}(n, H, g(n))}{n^2}.$$

Let $g_r(n) = n2^{-\omega(n)\log^{1-1/r} n}$. Balogh, Hu and Simonovits [3] showed that the Ramsey-Turán function for the even clique K_{2r} exhibits a jump from $o(n)$ to $g_r(n)$. A similar phenomenon happens in the more general setup.

Theorem 5.4. *1.* $\mathrm{RT}(K_3, K_5, g_3(n)) = o(n^3)$ *and* $\mathrm{RT}(K_3, K_6, g_3(n)) = o(n^3)$.

2. Odd cliques larger than 5 are stable: for every $\ell \ge 3$,

$$\mathrm{RT}(K_3, K_{2\ell+1}, g_{\ell+1}(n)) = (1 + o(1))\mathrm{RT}(K_3, K_{2\ell+1}, o(n)).$$

3. Even cliques always exhibit a jump: for every $\ell \ge 3$,

$$\mathrm{RT}(K_3, K_{2\ell+2}, g_{\ell+1}(n)) = (1 + o(1))\mathrm{RT}(K_3, K_{2\ell+1}, o(n)).$$

We will need a lemma by Balogh-Hu-Simonovits (Claim 6.1 in [3]).

Lemma 5.5. *Let G be an n-vertex graph with $\alpha(G) = g_q(n)$, where $g_q(n) = n2^{-w(n)\log^{1-1/q} n}$ and $\omega(n) \to \infty$ arbitrary slowly. If there exists a K_q in the cluster graph of G, then $K_{2q} \subseteq G$.*

Proof of Theorem 5.4. For (1), note that $\mathrm{RT}(K_3, K_5, g_3(n)) \le \mathrm{RT}(K_3, K_6, g_3(n))$. Therefore, we only need to prove $\mathrm{RT}(K_3, K_6, g_3(n)) = o(n^3)$. By Lemma 5.5, if G is an n-vertex K_6-free graph with $\alpha(G) \le g_3(n)$ then the cluster graph of G is K_3-free, which means that $k_3(G) = o(n^3)$.

For (2), note that $\mathrm{RT}(K_3, K_{2\ell+1}, g_{\ell+1}(n)) \le \mathrm{RT}(K_3, K_{2\ell+1}, o(n))$, hence, by Theorem 1.4, it is sufficient to prove $\mathrm{RT}(K_3, K_{2\ell+1}, g_{\ell+1}(n)) \ge (1 + o(1))\binom{\ell}{3}\left(\frac{n}{\ell}\right)^3$. Construction 2.2 shows that this inequality holds.

For (3), note that $\mathrm{RT}(K_3, K_{2\ell+1}, g_{\ell+1}(n)) \le \mathrm{RT}(K_3, K_{2\ell+2}, g_{\ell+1})$. Hence, using (2), we only need to show that $\mathrm{RT}(K_3, K_{2\ell+2}, g_{\ell+1}(n)) \le \mathrm{RT}(K_3, K_{2\ell+1}, o(n))$. By Lemma 5.5, if G is an n-vertex $K_{2\ell+2}$-free graph with $\alpha(G) \le g_{\ell+1}(n)$ then the cluster graph of G is $K_{\ell+1}$-free. Then, by the result of Erdős [10], among all $K_{\ell+1}$-free graphs the ℓ-partite Turán graph has the maximum number of triangles. Hence, we have

$$\mathrm{RT}(K_3, K_{2\ell+2}, g_{\ell+1}(n)) \le (1 + o(1))\binom{\ell}{3}\left(\frac{n}{\ell}\right)^3 = RT(K_3, K_{2\ell+1}, o(n)),$$

where the last equality is by Theorem 1.4. □

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