Diptych varieties. I

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Abstract
We present a new class of affine Gorenstein 6-folds obtained by smoothing the
1-dimensional singular locus of a reducible affine toric surface; their existence is
established using explicit methods in toric geometry and serial use of Kustin–Miller
Gorenstein unprojection. These varieties have applications as key varieties in con-
structing other varieties, including local models of Mori flips of Type A.

We introduce a large class of remarkable 6-folds called diptych varieties. Each is an
affine 6-fold $V_{ABL M}$ constructed starting from two toric 4-fold panels $V_{AB} \cup V_{LM}$ hinged
along a reducible toric surface $T = V_{AB} \cap V_{LM}$ (compare the Wilton diptych [W]). The con-
struction depends on discrete toric data called a diptych of long rectangles, that describe
the monomial cone of the two toric panels $V_{AB}$ and $V_{LM}$. It is equivariant under a big
torus $\mathbb{T} = (\mathbb{G}_m)^4 = (\mathbb{C}^\times)^4$. Apart from easy initial cases, diptych varieties are indexed by
3 natural numbers $d, e, k$, or by a 2-step recurrent continued fraction $[d,e,d,\ldots,(d\text{ or } e)]$
to $k$ terms (Classification Theorem 3.3). Once this combinatorial data is set up, Main
Theorem 1.1 guarantees the existence of the diptych variety. The worked example 1.2
illustrates almost all the main features of our construction. This paper is backed up by a
website

http://www-staff.lboro.ac.uk/~magdb/aflip.html

that contains current drafts of Parts II–IV, together with computer algebra calculations,
links to other papers and further auxiliary material.

Diptych varieties $V_{ABL M}$ are designed for use as ambient spaces or key varieties in con-
structing other spaces, much as toric varieties. As discussed briefly in the final Section 6,
our main motivation is their relation with the “continued division” algorithm [M], that
Mori used to prove the existence of flips of Type A. Our work also overlaps with the more
recent Gross–Hacking–Keel deformations of cycles of planes [GHK] in some cases where
these lead to algebraic varieties.

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1 Introduction

This section gives rough statements of our main results and an outline plan of the paper.
The extended example of 1.2 illustrates all the main ideas. We write $\mathbb{A}^{n} = \mathbb{C}^{n}$ for affine
space, $\mathbb{G}_{m} = \mathbb{C}^{\times}$ for the multiplicative group and $T = (\mathbb{C}^{\times})^{4}$ for the 4-dimensional torus.
Our main interest is in varieties over $\mathbb{C}$, although in the final analysis, our diptych varieties
are defined as schemes over $\mathbb{Z}$.

1.1 Main results and overview of the paper

A tent is a reducible affine surface $T = S_{0} \cup S_{1} \cup S_{2} \cup S_{3}$ as in Figure 1.1. Its four irreducible
components are $S_{0}, S_{2} \cong \mathbb{A}^{2}$ and $S_{1}, S_{3}$ cyclic quotient singularities of type $\frac{1}{s}(\alpha, 1)$ and
$\frac{1}{s}(\beta, 1)$, where $r, \alpha$ are coprime natural numbers, and similarly for $s, \beta$. We glue the four
toric surfaces transversally along their toric strata, giving $T$ four 1-dimensional singular
axes of transverse ordinary double points; the two axes on $S_{2}$ are the top axes of $T$, and
the two on $S_{0}$ its bottom axes.

Section 2 recalls basic facts on toric geometry and studies certain deformations of tents.
Our first result is Theorem 2.10: an extension $T \subset V_{AB}$ of a tent $T$ to an affine toric 4-fold
$V_{AB}$ that smooths the top axes is determined by a matrix $\left( \frac{r}{s}, \frac{b}{a} \right) \in \text{SL}(2, \mathbb{Z})$ with $a, b \geq 0$
and $a \equiv \alpha \mod r, b \equiv \beta \mod s$. Corollary 2.8 gives an alternative statement in terms of
continued fraction expansions of $0$, obtained by concatenating with a 1 the expansions of
complementary fractions $\frac{r}{\beta}$ and $\frac{r}{r - \alpha}$. This is routine material in toric geometry, but the
basic results and detailed notation for the monomial cone $\sigma_{AB}$ introduced here are in use throughout the paper.

Section 3 treats our first substantial result, Classification Theorem 3.3, classifying diptychs of toric extensions $T \subset V_{AB}$ and $T \subset V_{LM}$ that smooth respectively the top and bottom axes of $T$. By Lemma 3.2, the numerical conditions on $T$ for the second smoothing to exist is a second matrix $( \frac{a}{b}, \frac{c}{d} ) \in SL(2,\mathbb{Z})$ with $ag \equiv 1 \mod r$ and $bh \equiv 1 \mod s$. Theorem 3.3 classifies all solutions to this problem: with simple initial exceptions, each corresponds to a 2-step recurrent continued fraction $[d, e, d, \ldots, (d \text{ or } e)]$. Theorem 3.3 is proved by a simple descent argument.

At this point we introduce a case division (we discuss the necessity for this briefly in Section 6). The main case is $d, e \geq 2$ and $de > 4$; we concentrate our efforts primarily on this case in the rest of the current paper. The other cases involve some new features, and their proofs require minor modifications; they are as follows:

1. $de \leq 3$. This involves only a small number of quite small cases, and we deal with them in an appendix to [BR2].

2. The cases $d = e = 2$ and $d = 1, e = 4$ are treated in [BR2]. There are two infinite series of varieties with a convincing standard quasihomogeneous structure.

3. $d$ or $e = 1$ and $de > 4$. This case requires a proof that is basically on the same scale as the main case; we relegate the details to [BR3] to avoid excessive repetition, bulky notation, and many case divisions.

Diptychs serve as the input to our Main Theorem, the existence of diptych varieties:

**Theorem 1.1** A diptych of 4-fold toric panels $T \subset V_{AB}$ and $T \subset V_{LM}$ that smooth respectively the top and bottom axes of $T$ extends to a 6-fold $V_{ABLM}$:

$$T \subset V_{AB} \quad \cap \quad V_{LM} \subset V_{ABLM}$$

The diptych variety $V_{ABLM}$ is an affine variety with an action of the torus $\mathbb{T} = (\mathbb{G}_m)^4$. It has a regular sequence $A, B, L, M$ consisting of eigenfunctions of the $\mathbb{T}$-action such that $V_{AB}$ and $V_{LM}$ are the sections given by $L = M = 0$ and $A = B = 0$, with $T$ their intersection $A = B = L = M = 0$.

It follows that $V_{ABLM}$ is a Gorenstein affine 6-fold and is a flat 4-parameter deformation of the tent $T$. The $\mathbb{T}$-action restricts to the big torus of both 4-fold panels $V_{AB}$ and $V_{LM}$; the original tent $T$ is a union of toric strata in each, with the $\mathbb{T}$-action inducing the natural $(\mathbb{G}_m)^2$ action on each of its four toric components.

Section 4 lays the groundwork for the proof in Section 5. The main idea is to exploit the relation between the monomial lattices and the monomial cones of the two different toric
varieties \( V_{AB} \) and \( V_{LM} \) to deduce important consequences for monomials in the coordinate ring of the diptych variety \( V_{ABLM} \). Our proof of Main Theorem 1.1 in Section 5 makes essential use of convexity properties of these monomials (illustrated in the Pretty Polytope of Figure 4.1) and congruence properties (the Padded Cell of Figure 4.3).

Section 5 proves Theorem 1.1 in the main case by serial unprojection. We start from two equations defining a codimension 2 complete intersection \( V_0 \subset \mathbb{A}^8(x_0, x_1, y_0, y_1, A, B, L, M) \), and adjoin the remaining variables one at a time by unprojection \( V_{r+1} \to V_r \). Section 5.2 determines the unprojection order in which we must adjoin the variables \( x_2, \ldots, y_r \). It is inverse to the order of elimination (or projection) of variables from the toric panel \( V_{AB} \), corresponding to the concatenated continued fraction \([a_2, \ldots, a_k, b_1, \ldots, b_1] = 0\). Serial use of the Kustin–Miller unprojection theorem of [PR] provides most of what we need.

Extended Example 1.2 is the case corresponding to the recurrent continued fraction \([2, 4, 2]\) or the expansion of zero \([4, 2, 1, 3, 2, 2] = 0\). We use a beautiful trick with Pfaffians to compute the sequence of unprojection variables \([x_2, y_2, y_3, x_3, y_4]\) as rational functions with specified poles, the geometric interpretation of Kustin–Miller unprojection. This example illustrates all but one of the main points, and exemplifies our strategy of handling a diptych variety \( V_{ABLM} \) as an explicit object, but without necessarily writing down all the relations for its coordinate ring, much as for a toric variety.

The extended example glosses over one logical point that is the key issue for most of Sections 4–5. Each step \( V_{r+1} \to V_r \) of the induction must set up a new unprojection divisor \( D_r \subset V_r \). The divisor \( D_r \) itself is the product of a monomial curve \( A^r B^3 = 0 \) with an affine space \( \mathbb{A}^4(x_1, y_1, L, M) \), but we still have to prove it is a subscheme of \( V_r \).

### 1.2 Extended example

#### 1.2.1 Background and notation

For \( r > 0 \) and \( a \) coprime to \( r \), we write \( \frac{1}{r}(1, a) \) for the action of \( \mathbb{Z}/r \) on \( \mathbb{A}^2 \) given by 
\[
(u, v) \mapsto (\varepsilon u, \varepsilon^a v)
\]
where \( \varepsilon = \exp \frac{2\pi i}{r} \in \mathbb{C} \) is a chosen primitive \( r \)th root of 1. We use the same notation for the cyclic quotient singularity \( \mathbb{A}^2/(\mathbb{Z}/r) = \text{Spec} \mathbb{C}[u, v]^{\mathbb{Z}/r} \). We focus here on concrete cases, starting with \( \frac{1}{r}(1, 2) \); the ring of invariants \( \mathbb{C}[u, v]^{\mathbb{Z}/r} \) is generated by the monomials

\[
y_0 = u^7, \quad y_1 = u^5 v, \quad y_2 = u^3 v^2, \quad y_3 = u v^3, \quad y_4 = v^7,
\]

with relations between them determined by the *tag equations*

\[
y_0 y_2 = y_1^2, \quad y_1 y_3 = y_2^2, \quad y_2 y_4 = y_3^2.
\]

These are of the general form \( v_i^{-1} v_{i+1} = v_i^{a_i} \) for any 3 consecutive monomials \( v_{i-1}, v_i, v_{i+1} \) on the Newton boundary. The exponents or *tags* \( a_i \) are the entries in the Jung–Hirzebruch continued fraction expansion of \( \frac{r}{r-a} \); here \( \frac{7}{7-2} = \frac{3}{2} = [2, 2, 3] \). The quotient \( \mathbb{A}^2 \to S \subset \mathbb{A}^5(y_0, \ldots, 4) \) is thus the morphism \( (u, v) \mapsto (y_0, \ldots, 4) \), and the image \( S \) is uniquely determined by (1.3): the complete intersection (1.3) consists of \( S \) plus the \((y_0, y_4)\)-plane with a “fat”

4
nonreduced structure. To see actual generators of the ideal \( I_S \) we also need the “long equations” \( y_0 y_3 = y_1 y_2, y_1 y_4 = y_2 y_3^2 \) and \( y_0 y_4 = y_1 y_3^2 \), that derive from (1.3) using easy syzygy manipulations. In what follows, we write \( S = S_3 \) for the quotient \( \frac{1}{7}(1, 2) \).

In the same way, the quotient singularity \( \frac{1}{7}(1, 3) \) is

\[
S_1 \subset \mathbb{A}^4_{(x_0, x_1, x_2, x_3)} \quad \text{given by} \quad x_0 x_2 = x_1^2, \quad x_1 x_3 = x_2^4 \quad (1.4)
\]

with \([2, 4] = 2 - \frac{1}{4} = \frac{7}{4} \).

1.2.2 The tent \( T \)

The starting point for our example is the reducible affine surface or tent of Figure 1.1 (with \( k = 3, l = 4 \) and \( k + l + 2 = 9 \) in our case). It consists of a cycle of 4 components,

\[
\begin{array}{c}
S_0 \\
S_1 \\
S_2 \\
S_3
\end{array}
\]

Figure 1.1: The tent \( T = S_0 \cup S_1 \cup S_2 \cup S_3 \subset \mathbb{A}^{k+l+2}_{(x_0...k,y_0...l)} \) is obtained by glueing \( S_0 \cup S_1 \) transversally along the \( x_0 \)-axis, \( S_1 \cup S_2 \) along the \( x_k \)-axis, \( S_2 \cup S_3 \) along the \( y_l \)-axis, and \( S_3 \cup S_0 \) along the \( y_0 \)-axis

with vertical sides the surface quotient singularities \( S_1 \subset \mathbb{A}^4_{(x_0...k)} \) and \( S_3 \subset \mathbb{A}^5_{(y_0...l)} \) of types \( \frac{1}{7}(1, 3) \) and \( \frac{1}{7}(1, 2) \) as just described, and top and bottom the coordinate planes \( S_2 = \mathbb{A}^2_{(x_k,y_l)} \) and \( S_0 = \mathbb{A}^2_{(x_0,y_0)} \). In equations, \( T \subset \mathbb{A}^9 \) is the reducible variety defined by

\[
I_{S_1}, I_{S_3} \quad \text{and} \quad x_i y_j = 0 \quad \text{for all} \quad i, j \quad \text{with} \quad (i, j) \neq (0, 0), (k, l). \quad (1.5)
\]

1.2.3 First toric extension \( T \subset V_{AB} \)

We now seek to embed \( T \) into a toric variety \( V \) (irreducible and normal) so that \( T \) is both a regular section of \( V \) and a union of toric strata.

One solution is the affine toric 4-fold \( V_{AB} \) with monomial cone schematically represented in Figure 2.1, our first long rectangle. It is a schematic representation of a cone \( \sigma(V_{AB}) \), the Newton polygon of \( V_{AB} \) in the monomial lattice \( \mathbb{M} = \mathbb{Z}^4 \). We read \( \sigma(V_{AB}) \) and the toric variety \( V_{AB} \) automatically from the figure as follows: the dots around the boundary (clockwise from bottom left) are the generators \( x_0...k, y_l...0 \); the two remaining generators \( A, B \) are shown as annotations at the top corners. We also draw them in their correct geometric position in the 4-dimensional lattice \( \mathbb{M} \) in Figure 2.3, but this long rectangle shorthand is usually more convenient. The relations (1.3) and (1.4) continue to
Figure 1.2: The long rectangle for $V_{AB}$

hold, as represented by the tags down the long sides. These constrain $x_{0..k}$ to a plane face of $\sigma(V_{AB})$, and in that plane they generate the Newton boundary of $\frac{1}{2}(1,4)$; ditto $y_{0..l}$. The new ingredients are the tags and annotations $^42,^1B$ at the top corners, that say how we intend to deform the reducible equations $x_2y_4 = 0$ and $x_3y_3 = 0$ for $T$ appearing in (1.5) to usual binomial equations of toric geometry:

$$x_2y_4 = x_3^2A, \quad x_3y_3 = y_4B. \quad (1.6)$$

We view $A$ and $B$ as deformation parameters, and interpret (1.6) as smoothing the reducible double locus along the $x_3$- and $y_4$-axes, the top corners $S_1 \cap S_2$ and $S_2 \cap S_3$ of Figure 1.1.

On the other hand, equations (1.6) and the original tag equations (1.3–1.4) now completely determine the cone $\sigma(V_{AB})$ in a monomial lattice $\mathbb{M} = \mathbb{Z}^4$. Indeed, $x_3, y_4, A, B$ is a $\mathbb{Z}$-basis of $\mathbb{M}$, and the remaining generators $x_2, \ldots, x_0, y_3, \ldots, y_0$ are Laurent monomials in this basis, obtained by continued division from (1.6) together with (1.3–1.4):

$$x_2 = x_3^2(Ay_4^{-1}), \quad y_3 = y_4(Bx_3^{-1}),$$
$$x_1 = x_3^3(Ay_4^{-1})^4, \quad y_2 = y_4(Bx_3^{-1})^3,$$
$$x_0 = x_3^4(Ay_4^{-1})^7, \quad y_0 = y_4(Bx_3^{-1})^7. \quad (1.7)$$

A rational polyhedral cone $\sigma$ in the monomial lattice $\mathbb{M}$ defines a irreducible, normal toric variety $V_{\mathbb{M},\sigma} = \text{Spec} \mathbb{C}[\mathbb{M} \cap \sigma]$. We claim more: our monomials $x_{0..3}, y_{0..4}, A, B$ in $\mathbb{M}$ generate $\mathbb{M} \cap \sigma_{AB}$, and the resulting toric variety $V_{AB} = \text{Spec} \mathbb{C}[\mathbb{M} \cap \sigma_{AB}]$ is a flat deformation of $T$. When we say deformation, we mean the total space of the deformation; in fact, $A, B$ define a flat morphism $V_{AB} \to \mathbb{A}^2_{(A,B)}$ with fibre $T : (A = B = 0)$ over 0, although this morphism does not figure prominently in our considerations.

The relations satisfied by our monomials come implicitly from their inclusion in $\mathbb{M}$. We are usually not interested in writing them all out, but we want to find enough equations to justify our claim. By substituting from (1.7), we find the relation

$$x_1y_0 = A^4B^7 \quad (1.8)$$

that deforms the original equation $x_1y_0 = 0$ in $T$; this is the corner tag $(0)$ of Figure 1.2, indicating a tag equation at $x_0$, with tag 0 derived from the other tags (the annotation
\( A^4B^7 \) is left implicit). We view it as a partial smoothing of the reducible double locus of \( T \) along the \( x_0^- \)–axis – (1.8) of course defines a normal hypersurface in \( \mathbb{A}^4_{(x_1,y_0,A,B)} \).

Now, how does the relation \( x_0y_1 = 0 \) deform? From (1.7) we write out \( x_0y_1 = x_3^7y_1^{-4}A^7B^5 \), hence

\[
x_0y_1 = y_0^{-1}A^7B^{12} \quad \text{or} \quad x_0y_1 = x_1A^3B^5. \tag{1.9}
\]

The first equality is a tag equation for \( y_0 \), with negative tag \(-1\); this is the \((-1)\) at the bottom right of Figure 1.2. Along the \( y_0^- \)-axis of \( T \), where \( y_0 \neq 0 \), (1.9) ensures that the \( A,B \) deformation is also a partial smoothing of the singularity, making it irreducible and normal. However, (1.9) with its negative tag is anomalous in that it is not a polynomial equation, so we are not really allowed to use it as a generator of the ideal of the affine variety \( V_{AB} \). We thus replace it by the second expression, which in view of (1.8) is equivalent to it where \( y_0 \neq 0 \). The relation \( x_0y_1 = x_1A^3B^5 \) is also anomalous as a tag equation for \( y_0 \), since it involves the “opposite” generator \( x_1 \) in place of \( y_0 \). Now the equations of \( V_{AB} \) include (1.8–1.9); these define an irreducible normal complete intersection in \( \mathbb{A}^6_{(x_0,x_1,y_0,y_1,A,B)} \).

Since \( V_{AB} \) is a toric 4-fold, it is Cohen–Macaulay; we see in Lemma 2.3 that it is also Gorenstein. (Or one checks directly from the description above that the semigroup ideal of interior monomials of \( \sigma(V_{AB}) \) is generated by \( AB \); compare 2.3 and Figure 2.3.) One checks that the locus \((A = B = 0)\) inside \( V_{AB} \) equals \( T \) at the general point of each component, and in particular each component is 2-dimensional. Therefore \( A,B \) is a regular sequence and \( T \subset V_{AB} \) is a flat deformation.

### 1.2.4 Conclusion

In this example we found the \( A,B \) deformation \( T \subset V_{AB} \) in a more-or-less inevitable way starting from the new tag equations \( x_2y_4 = x_3^2A \) and \( x_3y_3 = y_1B \), that naturally smooth the double locus of \( T \) along the \( x_3^- \)– and \( y_1^- \)-axes. After a monomial calculation that is birationally forced, our rectangle closed up neatly to give the tag equations (1.8–1.9), so that this deformation also leads to partial smoothings of the \( x_0 \) and \( y_0^- \)-axes, giving an irreducible and normal variety \( V_{AB} \) such that \( A = B = 0 \) contains \( S_0 \) as a reduced component. Corollary 2.8 explains that this miracle works precisely because the concatenation \([4,2,1,3,2,2]\) is a continued fraction expansion of \( 0 \). These numbers are the tags at \( x_2, x_3, y_4, \ldots, y_1 \); the asymmetry \((x_1 \text{ omitted but } y_1 \text{ included})\) is significant, and relates to the anomalous tag equations (1.9).

### 1.2.5 Second toric extension \( T \subset V_{LM} \)

As hinted above, \( T \) has more than one deformation to a toric 4-fold. We now write down the second long rectangle Figure 1.3 and the resulting deformation \( T \subset V_{LM} \). The calculations are just as for \( V_{AB} \), except that we start from the bottom and work up. Hindsight based on Corollary 2.8 and \([3,2,2,1,4,2] = 0\) tells us that this will work. The new tag equations that smooth out the \( x_0^- \)– and \( y_0^- \)-axes of \( T \) are represented by the \( LA,1M \).
at the bottom:
\[ x_1 y_0 = x_0^4 L, \quad x_0 y_1 = y_0 M. \]  
\( (1.10) \)

This time, \( x_0, y_0, L, M \) base the monomial lattice and \( (1.10) \) together with \( (1.3–1.4) \) give the remaining variables as Laurent monomials:

\[
\begin{align*}
x_1 &= x_0^4 (Ly_0^{-1}), & y_1 &= y_0 (Mx_0^{-1}), \\
x_2 &= x_0^7 (Ly_0^{-1})^2, & y_2 &= y_0 (Mx_0^{-1})^2, \\
x_3 &= x_0^{14} (Ly_0^{-1})^7, & y_3 &= y_0^2 (Mx_0^{-1})^3, \\
x_4 &= x_0^{28} (Ly_0^{-1})^{14}. & y_4 &= y_0^4 (Mx_0^{-1})^7.
\end{align*}
\]
\( (1.11) \)

As before, we deduce the tag equations for \( x_3 \) and \( y_4 \):

\[
x_2 y_4 = L^2 M^7, \quad x_3 y_3 = y_4^3 L^7 M^{27} = x_2^3 L M^3.
\]
\( (1.12) \)

The latter is anomalous as before: the partial smoothing along the \( y_4 \)-axis is specified either by the Laurent monomial \( y_4^3 \) or by a polynomial equation \( x_2^3 \) in the “opposite” variable \( x_2 \).

### 1.2.6 The 6-fold \( V_{ABL} \)

We now have two deformations \( T \subset V_{AB} \) and \( T \subset V_{LM} \) of our tent \( T \) to toric 4-folds; we call this a diptych of toric deformations. The two panels are quite different: \( V_{AB} \) is smooth along the \( x_3 \)- and \( y_4 \)-axes by \( (1.6) \), but has hypersurface singularities along the \( x_0 \)- and \( y_0 \)-axes of transverse type \( x_1 y_0 = A^4 B^7 \) and \( x_0 y_1 = y_0^{-1} A^7 B^{12} \) by \( (1.8) \) and \( (1.9) \). In contrast, \( V_{LM} \) smooths the \( x_0 \)- and \( y_0 \)-axes by \( (1.10) \), but leaves the \( x_3 \)- and \( y_4 \)-axes with the transverse hypersurface singularities \( x_2 y_4 = L^2 M^7 \) and \( x_3 y_3 = y_4^{-3} L^7 M^{27} \) of \( (1.12) \).

Theorem 1.1 now asserts that these two toric panels fit together in a 4-parameter deformation \( T \subset V_{ABL} \):

\[
\begin{align*}
T \subset & \quad V_{AB} \\
\cap & \quad \cap \\
V_{LM} \subset & \quad V_{ABL}\end{align*}
\]
\( (1.13) \)
More precisely, we build an affine 6-fold $V_{ABLM}$ with a regular sequence $A, B, L, M$ such that the section $L = M = 0$ is $V_{AB}$ and $A = B = 0$ is $V_{LM}$. The idea is amazingly naive: starting at the top, we simply merge the tag equations (1.6) and (1.12) for $x_3$ and $y_4$ from $V_{AB}$ and $V_{LM}$, obtaining $W \subset \mathbb{A}^8_{(x_2, x_3, y_4, y_3, A, B, L, M)}$ defined by

$$x_2y_4 = x_3^2 A + L^2 M^7, \quad x_3y_3 = y_4 B + x_2^3 LM^3.$$  \hspace{1cm} (1.14)

It is a codimension 2 complete intersection, $A, B, L, M$ is a regular sequence for $W$, and the section $L = M = 0$ is birational to $V_{AB}$ by the Laurent monomial argument of (1.7).

The plan is now to adjoin $x_1, x_0, y_2, y_1, y_0$ as rational functions on $W$, so $V_{ABLM}$ will be birational to $W$. In commutative algebra terms, the coordinate ring of $V_{ABLM}$ is constructed from the complete intersection (1.14) by serial unprojection. We run through the construction as a pleasant narrative; the reasons it all works include some detailed tricks that we explain later when we treat the material more formally. Suffice it to say that we add the new variables $x_1, x_0, y_2, y_1, y_0$ one at a time, and in that order. Adding them in a different order does not work.

**1.2.7 First pentagram**

We construct $x_1$ as a rational function on $W$ (1.14) with divisor of poles the codimension 3 complete intersection

$$D : (x_3 = y_4 = LM^3 = 0) \subset W;$$ \hspace{1cm} (1.15)

where $LM^3$ is the hcf of the two terms $L^2 M^7$ and $x_2^3 LM^3$ in (1.14). The new variable $x_1$ appears in three equations

$$x_1 x_3 = \ldots, \quad x_1 y_4 = \ldots, \quad x_1 LM^3 = \ldots,$$ \hspace{1cm} (1.16)

that express the rational function $x_1$ as a homomorphism $\mathcal{I}_D \to \mathcal{O}_W$. More intrinsically, $x_1$ is an unprojection variable $x_1 \in \mathcal{H}om(\mathcal{I}_D, \omega_W)$ with Poincaré residue a basis of $\omega_D \cong \mathcal{O}_D$; see [PR] and [Ki] for the theory and practice of unprojection. In our calculation we take as input the equations (1.14) and (1.15) of $W$ and $D$, and use them to fix up a $5 \times 5$ skew matrix $A = \{a_{ij}\}$ whose five $4 \times 4$ Pfaffians are the two input equations (1.14) and the three new unprojection equations (1.16) for $x_1$. This calculation is repeated serially in what follows, and we make it systematic with magic pentagrams:

$$\begin{pmatrix}
  y_3 & x_3^3 & -B & -x_1 \\
  y_2 & LM^3 & -x_3 A \\
  x_3 & LM^4 \\
  x_2 &
\end{pmatrix}$$ \hspace{1cm} (1.17)

$$\begin{array}{cccc}
23.45 & x_2y_4 = x_3^2 A + L^2 M^7, & 12.34 & x_3y_3 = y_4 B + x_2^3 LM^3, \\
12.34 & x_3y_3 = y_4 B + x_2^3 LM^3, & 12.35 & x_1y_4 = x_3^2 x_3 A + y_3 LM^4, \\
& & 13.45 & x_1 x_3 = x_2^4 + BLM^4, \\
& & & 12.45 & x_2 y_4 = x_3 AB + x_1 LM^3.
\end{array}$$
The array is a skew $5 \times 5$ matrix $A = \{a_{ij}\}$; we only write the 10 upper-triangular entries $a_{12} = y_3, \ldots, a_{15} = -x_1$, etc. Its $4 \times 4$ Pfaffians are

$$\operatorname{Pf}_{i j, k l} = a_{i j} a_{k l} - a_{i k} a_{j l} + a_{i l} a_{j k} \quad \text{for any distinct } i, j, k, l$$  \hspace{1em} (1.18)

(as with minors and cofactors, with an overall choice of ±1; in long calculations we abbreviate $\operatorname{Pf}_{i j, k l}$ to $i j, k l$). In (1.17), viewing $y_3, y_1, x_3, x_2$ and the two equations $x_2 y_4 = \cdots, x_3 y_3 = \cdots$ as given, we seek to add $x_1$ and three new equations $x_1 x_3 = \cdots, x_1 y_4 = \cdots$ and $x_2 y_1 = \cdots + x_1 L M^3$. These trinomial equations play a role for $V_{A B L M}$ similar to the binomial tag equations $v_{i-1} v_{i+1} = v_i^{i+1}$ for the cyclic quotient singularities $S_1$ and the tent $T$.

The array is written out automatically from the pentagram and the given equations (1.14): we write the given variables $y_3, y_4, x_3, x_2$ down the superdiagonal, the new unprojection variable $x_1$ in the top right, and the given $L M^3 = \text{hcf}(L^2 M^7, x_3^3 L M^3)$ as the entry $a_{24}$. Requiring $\operatorname{Pf}_{12,34}$ and $\operatorname{Pf}_{23,45}$ to give (1.14) determines the remaining entries. The output is the three equations involving $x_1$ as the three remaining Pfaffians in (1.17).

### 1.2.8 Serial pentagrams

The remaining variables $x_0, y_2, y_1, y_0$ are adjoined likewise to give the codimension 7 variety $V_{A B L M}$ (see Section 5 for a formal treatment). We write out the calculations without further comment for your delight.

\[
\begin{align*}
\text{23.45} & \quad x_1 x_3 = x_2^3 + B L M^4, \\
\text{12.34} & \quad x_2 y_3 = A B x_3 + L M^3 x_1,
\end{align*}
\]

\[
\begin{pmatrix}
y_3 & x_1 & -A B & -x_0 \\
x_3 & L M^3 & -x_2^3 & \text{BM} \\
x_2 & \text{BM} & x_1 & \text{C}
\end{pmatrix}
\]

\[
\begin{align*}
\text{23.45} & \quad x_0 x_2 = x_1^2 + A B^2 M, \\
\text{12.34} & \quad x_1 y_3 = A B x_3^2 + L M^3 x_0,
\end{align*}
\]

\[
\begin{pmatrix}
y_3 & L M^2 x_0 & -A B x_2^2 & -y_2 \\
x_2 & M & -x_1 & A B^2 \\
x_1 & A B^2 & x_0 & \text{C}
\end{pmatrix}
\]

\[
\begin{align*}
\text{12.35} & \quad x_2 y_2 = A B^2 y_3 + L M^2 x_0 x_1, \\
\text{13.45} & \quad x_1 y_2 = A^2 B^3 x_2^2 + L M^2 x_0^2, \\
\text{12.45} & \quad x_0 y_3 = A B x_1 x_2^2 + M y_2.
\end{align*}
\]
23.45 \( x_0 x_2 = x_1^2 + AB^2 M \),
12.34 \( x_1 y_2 = A^2 B^3 x_2 + LM^2 x_0^2 \),

\[
\begin{pmatrix}
  y_2 & LM x_0^3 \\
  x_2 & -A^2 B^2 x_2 \\
  x_1 & -x_1 \\
  x_0 & AB^2
\end{pmatrix}.
\]

12.35 \( x_2 y_1 = AB^2 y_2 + LM x_0^2 x_1 \),
13.45 \( x_1 y_1 = A^3 B^3 x_2 + LM x_0^3 \),
12.45 \( x_0 y_2 = A^2 B^3 x_1 x_2 + M y_1 \),

\[
\begin{pmatrix}
  y_1 & L x_0^3 \\
  x_2 & -A^3 B^3 \\
  x_1 & -x_1 \\
  x_0 & AB^2
\end{pmatrix}.
\]

The final two equations \( x_1 y_0 = \cdots \), and \( x_0 y_1 = \cdots \) merge the tag equations (1.8–1.9) and (1.10) for \( x_0 \) and \( y_0 \) at the bottom of the two long rectangles in exactly the same way as (1.14) merged the tag equations at the top. In other words, the whole calculation could have been done starting with these two equations and working up – if you liked the puzzle, you will enjoy turning it upside down and doing it all over again.

2 Toric partial smoothings of tents

This chapter centres around the combinatorics of continued fractions. After recalling standard facts, we define a tent \( T \), and, under appropriate assumptions, construct a toric extension \( T \subset V_{AB} \) that smooths its top two axes. The toric variety \( T \subset V_{AB} \) can be treated in terms of a matrix \( (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \in \text{SL}(2, \mathbb{Z}) \), or equivalently, in terms of a certain continued fraction expansion of 0. We use the latter treatment in 5.2 to understand \( V_{AB} \) by a sequence of Gorenstein projections.

2.1 Jung–Hirzebruch continued fractions

A continued fraction expansion is a formal expression

\[
[c_1, \ldots, c_n] = c_1 - 1/(c_2 - 1/(c_3 - \cdots - 1/c_n) \cdots)
\]

\[
= c_1 - \frac{1}{c_2 - \frac{1}{\cdots - \frac{1}{c_n}}}
= c_1 - \frac{1}{[c_2, \ldots, c_n]} \quad (2.1)
\]

The entries \( c_i \) are called tags. If \( c_1, \ldots, c_n \) are integers, the righthand side is a rational number, provided that the expression makes sense, that is, division by zero does not occur. (The notation is explained in Riemenschneider [R] §3, pp. 220–3.)
The next proposition discusses four aspects of continued fractions. We spell out this material, because we use it often and with large multiplicity in what follows: we invert continued fractions and pass to complementary fractions, we “top and tail” them by cutting off a tag at one end and adding one at the other, say:

\[ [a_0, \ldots, a_{k-1}] \mapsto [a_k, a_{k-1}, \ldots, a_1], \text{ etc.,} \quad (2.2) \]

and we concatenate the resulting fractions.

**Proposition 2.1 (a) Factoring a matrix:** The formal identity

\[
\begin{pmatrix}
0 & 1 \\
-1 & c_1
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-1 & c_2
\end{pmatrix}
\cdots
\begin{pmatrix}
0 & 1 \\
-1 & c_n
\end{pmatrix}
= \begin{pmatrix}
-q' & q \\
-p' & p
\end{pmatrix}. \quad (2.3)
\]

holds in indeterminates or variables \( c_1, \ldots, c_n \), where \( p, q, p', q' \) are polynomials, the numerators and denominators of \( p/q = [c_1, \ldots, c_n] \) and \( p'/q' = [c_1, \ldots, c_{n-1}] \). (No cancellation occurs in the fraction \( p/q \), whatever the nature or values of the quantities \( c_i \), because \( p \) and \( q \) satisfy an hcf identity \( ap + bq = 1 \).) The fraction \( p'/q' \) is the first convergent of \( p/q \).

(b) **Blowdown:** \([c_1, \ldots, c_{n-1}, 1] = [c_1, \ldots, c_{n-1} - 1] \) and

\[
[c_1, \ldots, c_{i-1}, 1, c_{i+1}, \ldots, c_n] = [c_1, \ldots, c_{i-1}, 1, c_{i+1} - 1, \ldots, c_n]. \quad (2.4)
\]

This is just the identity \( \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1/b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1/a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1/b \end{pmatrix} \).  

Two notions of “inverse” of a continued fraction play a role in our theory:

(c) **Reciprocal:** \([c_1, \ldots, c_n] = p/q \) and its reciprocal continued fraction

\[
[c_n, \ldots, c_1] = p/q^* \quad (2.5)
\]

share the same numerator \( p \), and their denominators are inverse modulo \( p \). More precisely, there is a formal identity

\[
qq^* = N(c_2, \ldots, c_{n-1}) \cdot p + 1, \quad (2.6)
\]

where \( N(c_2, \ldots, c_{n-1}) \) is the numerator of \([c_2, \ldots, c_{n-1}] \). In particular, if \( c_i \in \mathbb{Z} \) and the expressions are meaningful then \([c_n, \ldots, c_1] = p/q^* \), where \( qq^* \equiv 1 \mod p \). See (2.11) for what this means in our context.

(d) **Complement:** Let \( p/q = [c_1, \ldots, c_n] \) with \( c_i \in \mathbb{Z} \) and \( c_i \geq 2 \). Then the complementary continued fraction is \([b_1, \ldots, b_m] = p/(p - q)\), and satisfies

\[
[c_n, \ldots, c_1, 1, b_1, \ldots, b_m] = 0. \quad (2.7)
\]

Moreover, serial blowdown reduces the expansion to \([1, 1] = [0] = 0\); in particular, \( \sum (c_i - 1) = \sum (b_j - 1) \), and one of \( b_1, c_1 \leq 2 \). For example,

\[
[4, 2, 1, 3, 2, 2] = [4, 1, 2, 2, 2] = [3, 1, 2, 2] = [2, 1, 2] = [1, 1] = 0. \quad (2.8)
\]
Remark 2.2 Traditionally, one uses Jung–Hirzebruch continued fractions to write a fraction \( \frac{r}{a} \) with \( r > a \geq 1 \) and \( a, r \) coprime integers as

\[
\frac{r}{a} = [b_1, \ldots, b_{n-1}] = b_1 - \frac{1}{b_2 - \ldots}
\]

Then \( b_1 \) is the round-up \( b_1 = \lceil \frac{r}{a} \rceil \), and is \( \geq 2 \), because \( \frac{r}{a} > 1 \), and for the same reason all subsequent \( b_i \geq 2 \) (to the end of the algorithm). Here we do something slightly bigger, with \( a \geq 1 \), but \( r \in \mathbb{Z} \) any integer coprime to \( a \): for example, \( \frac{-24}{7} = -3 - \frac{3}{2} = [-3, 3, 2, 2] \). This means that \( b_1 = \lceil \frac{r}{a} \rceil \in \mathbb{Z} ; \) however, from the second step onwards and to the end of the algorithm, \( 1/(b_1 - \frac{r}{a}) > 1 \) is a conventional fraction, so that \( b_i \geq 2 \) for each \( i \) with \( 2 \leq i \leq n - 1 \).

In traditional use, (2.3) identifies 3 types of data: a rational fraction \( p/q > 1 \), a continued fraction \( [c_1, \ldots, c_n] \) with all \( c_i \geq 2 \), and a matrix \( \left( \begin{smallmatrix} -q & p \\ -p' & p \end{smallmatrix} \right) \in \text{SL}(2, \mathbb{Z}) \) with \( p > q > 0 \). However, we relax these restrictions, considering things like \( [5, 1, 3] = 5 - \frac{3}{2} = \frac{7}{2} = [4, 2] \) (a blowdown) or \( [2, 0, 2] = 4 \), with

\[
\left( \begin{array}{cc} 0 & 1 \\ -1 & 2 \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ -1 & 2 \end{array} \right) = \left( \begin{array}{cc} 0 & -1 \\ -1 & 2 \end{array} \right) = \left( \begin{array}{cc} 0 & -1 \\ 1 & -4 \end{array} \right).
\]

The matrix product (2.3) is meaningful even when (2.1) involves division by zero. More generally, the sequence of integer tags \( [c_1, \ldots, c_n] \) contains more information than the matrix (2.3), which contains more information than the fraction \( \frac{r}{a} \) while \( \left( \begin{smallmatrix} -q & p \\ -p' & p \end{smallmatrix} \right) \in \text{SL}(2, \mathbb{Z}) \), the fraction \( \frac{t}{q} \) (when defined) is its image in the quotient group \( \text{PSL}(2, \mathbb{Z}) \), whereas the expression \( [c_1, \ldots, c_n] \) is a lift to the “universal cover” of \( \text{SL}(2, \mathbb{Z}) \) inside the universal cover of \( \text{SL}(2, \mathbb{R}) \), keeping track of winding number. For example, \( [0, 0, 0, 0] \) is the composite of 4 rotations by \( \pi/2 \), or \( \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \)^4 = id. Running around one of our long rectangles below always gives winding number 1.

Notation for the quotient \( \frac{1}{r}(\alpha, 1) \) As in Example 1.2, for \( r \geq 1 \) and \( 0 < \alpha \leq r \) coprime to \( r \) we write \( \frac{1}{r}(\alpha, 1) \) for the \( \mathbb{Z}/r \) action on \( \mathbb{A}^2_{(u,v)} \) given by \( (u, v) \mapsto (e^\alpha u, e^\alpha v) \), and for the quotient \( S = \mathbb{A}^2/\frac{1}{r}(\alpha, 1) \) by this action. We allow \( \mathbb{A}^2 \) as the case \( r = 1 \), without worrying unduly about the value of \( \alpha \) (of course, \( \alpha = 0 \)); it corresponds to the identity matrix or the empty continued fraction \( [0] \). The lattice \( \Lambda \) of invariant Laurent monomials consists of \( u^iv^j \) with \( ai + j \equiv 0 \mod r \); it is a lattice \( \Lambda \cong \mathbb{Z}^2 \), but with no preferred basis. The coordinate ring of \( S \), based by \( \mathbb{Z}/r \)-invariant monomials, is minimally generated by monomials on the Newton boundary of the positive quadrant \( \sigma \subset \mathbb{R}_+^2 \). Setting \( 0 < \beta \leq r \) with \( \alpha \beta = -1 \mod r \), these monomials are \( x_0 = u^r \), \( x_1 = u^\beta v \), etc. Either continued fraction \( \frac{r}{\beta} = [a_1, \ldots, a_k-1] \) or \( \frac{r}{a_\alpha} = [a_{k-1}, \ldots, a_1] \) provides the generators \( x_0, \ldots, k \) and the tag equations holding between them:

\[
x_{i-1}x_{i+1} = x_i^{a_i} \quad \text{for } i = 1, \ldots, k - 1.
\]
In particular,
\[
\frac{r}{\beta} = [a_1, \ldots, a_{k-1}] \quad \mapsto \quad x_0 = u^r, \ x_1 = u^\beta v, \ x_2 = x_0^{a_1} x_0^{-1}, \ldots
\]
\[
\frac{r}{r - \alpha} = [a_{k-1}, \ldots, a_1] \quad \mapsto \quad x_k = v^r, \ x_{k-1} = u v^{r - \alpha}, \ldots.
\]

The tag equations (2.10) determine \( S \) completely: they express any \( x_j \) as a Laurent monomial in any two consecutive monomials \( x_i, x_{i+1} \). The complete intersection in \( \mathbb{A}^{k+1}_{(x_0, \ldots, x_k)} \) given by (2.10) is \( S \) plus \( \mathbb{A}^2_{(x_0, x_k)} \) (usually with a nonreduced structure). The other generators of \( I_S \) are “long equations” \( x_j x_j = \text{monomial} \) for \( |i - j| > 2 \), that can be deduced from (2.10) via syzygies.

2.2 Tents and fans

A tent \( T = S_0 \cup S_1 \cup S_2 \cup S_3 \subset \mathbb{A}^{k+l+2}_{(x_0, \ldots, x_k, y_0, \ldots, l)} \) is the union of the four affine toric surfaces of Figure 1.1, with horizontal sides \( S_0 = \mathbb{A}^2_{(x_0, y_0)} \) and \( S_2 = \mathbb{A}^2_{(x_k, y_l)} \) and vertical sides the cyclic quotient singularities

- \( S_1 = \frac{1}{r} (\alpha, 1) \) with coordinates \( x_{k-0} \) from \( \frac{r}{r - \alpha} = [a_{k-1}, \ldots, a_1], \) and
- \( S_3 = \frac{1}{s} (\beta, 1) \) with coordinates \( y_{l-0} \) from \( \frac{s}{s - \beta} = [b_{l-1}, \ldots, b_1], \)

where \( \alpha \leq r \) are coprime natural numbers, and similarly for \( \beta \leq s \) (there are no other conditions on \( \alpha, \beta \) at this stage). The coordinates \( x_{0, \ldots, k}, y_{0, \ldots, l} \) of the ambient space \( \mathbb{A}^{k+l+2} \) and the equations for \( T \) are shown schematically in Figure 2.1; once we have added corner tags in 2.2.2 and annotations in 2.3, we refer to such arrays as long rectangles, and use them as a shorthand for certain toric 4-folds. The components glue transversally along their toric strata (= coordinate axes), giving \( T \) four singular axes of transverse ordinary double points; the two axes on \( S_2 \) are the top axes, and the two on \( S_0 \) the bottom axes.

![Figure 2.1: Coordinates and tags for a tent T](image)

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2.2.1 Tents without embeddings

Our definition expresses $T$ embedded in $\mathbb{A}^{k+l+2}$ by explicit coordinates; its ideal $I_T$ is generated by $I_{S_1}$ and $I_{S_2}$, determined by the tags down the sides as in 2.1, together with the cross-equations $x_i y_j = 0$ for all pairs $(i,j) \neq (0,0),(k,l)$.

However, $T$ can be viewed abstractly as an identification scheme as studied more generally in Reid [dP]: write $\Gamma'_i \cup \Gamma''_i$ for the toric 1-strata of the $S_i$ and $C = \bigsqcup_{i=1}^4 (\Gamma'_i \cup \Gamma''_i)$. Let $D$ be the four axes $\mathbb{A}^1$ with coordinates $x_0, x_k, y_0, y_0$ glued transversally at a common origin (as coordinate axes in $\mathbb{A}^4$); write $\varphi : C \to D$ for the morphism given by $x_0$ on the $x_0$-axes of $S_0$ and $S_1$, and so on, to perform the identifications of Figure 1.1. Then

$$T = (S_0 \sqcup S_1 \sqcup S_2 \sqcup S_3)/\varphi. \quad \text{(2.12)}$$

There are no parameters or moduli in this glueing.

**Lemma 2.3** Let $T$ be the tent as above. Then $T$ is a Gorenstein scheme. Moreover, $T$ has an action of $(\mathbb{G}_m)^4$ that restricts to the toric structure on each component.

**Proof** We use elementary results of [dP], Section 2. $T$ is Cohen–Macaulay because all the glueing happens in codimension 1 ([dP], 2.2). We prove it is Gorenstein using the criterion of [dP], Corollary 2.8.

Each component $S_i$ is a toric surface; on each, choose a $\mathbb{Z}$-basis $m_1, m_2$ for the monomial lattice, oriented clockwise (e.g., on $S_1$, take $x_0, x_1$ or $x_{k-1}, x_k$; on $S_2$, take $x_k, y_0$). The 2-form $s = \frac{d m_1}{m_1} \wedge \frac{d m_2}{m_2} \in \Omega^2_T$ on the big torus is a basis for $\Omega^2_T$, is defined over $\mathbb{Z}$, independent of the choice of oriented basis, and has log poles along each stratum of $S$, with residue along each stratum $\mathbb{A}^1$ equal to $\pm$ times the natural basis $\frac{d m_i}{m_i}$ of $\Omega^1_T$. We take this basis element $s$ on each component. Under the identification $\varphi : C \to D$ of the double locus, over the general point of each component of $D$, the residues from the two components are $\pm \frac{d m_i}{m_i}$, and therefore cancel out; thus $s$ satisfies the conditions of [dP], Corollary 2.8.ii and is a basis of the dualising sheaf $\omega_T$.

Each component of $T$ is a toric variety, so $(\mathbb{G}_m)^8$ acts on the disjoint union of the components. Each glueing imposes one linear condition on the action; we think of $T_{S_0} = (\mathbb{G}_m)^2 = \{ (\lambda_0,1,1,\lambda_3) \}$ as the big torus of $S_0$ and $T_{S_1} = \{ (\lambda_0,\lambda_1,1,1) \}$ that of $S_1$, etc.

Q.E.D.

2.2.2 The fan $\Phi(\begin{pmatrix} b & a \\ b & a \end{pmatrix})$ in the plane given by $(\begin{pmatrix} b & a \\ b & a \end{pmatrix}) \in \text{SL}(2,\mathbb{Z})$

Jung–Hirzebruch continued fractions factor a base change in $\text{SL}(2,\mathbb{Z})$ into elementary moves (Proposition 2.1(a)); in our case, the base change goes from the monomials $x_0, y_0$ at the bottom of our long rectangle to $x_k, y_1$ at the top (up to sign and orientation). 2.3 constructs the toric variety $V_{\mathcal{A}B}$ and the first extension $T \subset V_{\mathcal{A}B}$ generalising (1.7), using a matrix in $\text{SL}(2,\mathbb{Z})$ to generate the monomial cone $\sigma_{\mathcal{A}B}$ of Figure 2.3 in the 4-dimensional lattice $\mathbb{M} = \mathbb{Z}^4$. 

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We start by analysing the combinatorics of this construction in a stripped-down 2-dimensional setting $\overline{\mathbb{M}} = \mathbb{Z}^2$. Consider two oriented bases $x_0, y_0$ and $\eta, \xi$ of $\overline{\mathbb{M}}$ related by inverse base changes

$$x_0 = \eta^{-r}\xi^a, \quad y_0 = \eta^b\xi^{-s} \quad \text{and} \quad \eta = x_0^{-s}y_0^{-a}, \quad \xi = x_0^{-b}y_0^{-r}. \quad (2.13)$$

Here $r, s, a, b \geq 0$ are integers with $rs - ab = 1$, so

$$\begin{pmatrix} r & a \\ b & s \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} s & -a \\ -b & r \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \quad (2.14)$$

are a pair of inverse elements. (If $a$ or $b = 0$ then $r = s = 1$, and one or two points in what follows need minor restatement. Rather than do that systematically, it is easier simply to list all these initial cases, as in 2.2.4.)

The vectors $x_0, y_0, \eta, \xi$ subdivide the plane $\overline{\mathbb{M}}_\mathbb{R}$ into the fan $\Phi(\begin{pmatrix} r & a \\ b & s \end{pmatrix})$ of Figure 2.2.a consisting of 4 cones $\langle x_0, y_0 \rangle, \langle x_0, \xi \rangle, \langle \xi, \eta \rangle, \langle y_0, \eta \rangle$. It determines a tent $T$, with coordinate ring generated by the 4 monomial cones and related by $m_1m_2 = 0$ if $m_1, m_2$ are not in a common cone. The next lemma computes the affine toric surfaces that make up the tent $T$ corresponding to $\Phi(\begin{pmatrix} r & a \\ b & s \end{pmatrix})$; compare with the first long rectangle of Example 1.2 for which $\begin{pmatrix} r & a \\ b & s \end{pmatrix} = (\begin{pmatrix} 4 & 12 \\ 7 & 7 \end{pmatrix})$.

**Lemma 2.4** Suppose that $r, s, a, b \geq 1$. Consider the cone $\langle x_0, \xi \rangle$ (marked $S_1$ in Figure 2.2(a)). The lattice $\overline{\mathbb{M}}$ is generated by the monomials $x_0, \xi$ together with either of

$$y_0^{-1} = (x_0^b\xi)^{1/r} \quad \text{or} \quad \eta = (x_0^{-1}\xi^a)^{1/r}.$$  

Therefore $\langle x_0, \xi \rangle$ is the monomial cone $\frac{1}{r}(\alpha, 1)$ or $\frac{1}{r}(1, r - \beta)$, where $\alpha$ is the least residue of $a$ mod $r$, and $\beta$ that of $b$ (note that $rs - ab = 1$ implies $\alpha$ and $r - \beta$ are inverse mod $r$).

- $y_4 = \eta$

- $y_3$

- $y_2$

- $y_1$

- $x_0$

- $x_1$

- $x_2$

- $x_3 = \xi$

- $S_1$

- $S_3$

- $S_2$

- $x_0 \bullet$

- $y_0 \bullet$

- $x_1 \circ$

- $y_0 \bullet$

**Figure 2.2:** The fan $\Phi(\begin{pmatrix} r & a \\ b & s \end{pmatrix}) \in \text{SL}(2, \mathbb{Z})$ defined by $x_0, y_0, \eta, \xi$
Write \(x_0, x_1, \ldots, x_{k-1}, x_k = \xi\) for the successive monomials along the Newton boundary of \((x_0, \xi)\). The number \(k\) and the monomials themselves come from factoring the base change (2.13) into elementary moves:

\[
\begin{pmatrix}
-r & a \\
b & -s
\end{pmatrix}
= \begin{pmatrix}
0 & 1 \\
-1 & a_0
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-1 & a_1
\end{pmatrix}
\cdots
\begin{pmatrix}
0 & 1 \\
-1 & a_{k-1}
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-1 & a_k
\end{pmatrix},
\] (2.15)

in which each of \(a_1, \ldots, a_{k-1} \geq 2\). More concretely, they are given by the continued fraction expansions

\[
[a_0, a_1, \ldots, a_{k-1}] = \frac{-b}{r} \quad \text{and} \quad [a_k, \ldots, a_1] = \frac{a}{r}
\] (2.16)

by either of the following constructions:

(1) From the bottom, \(x_0\) is given, and \(x_1 = (x_0^\beta \xi)^{1/r}\), where \(\beta\) is the least residue of \(b\) mod \(r\). Thus \(a_0 = \lfloor -\frac{b}{r} \rfloor = \frac{-b + \beta}{r} \leq 0\) and

\[
x_1 = (x_0^\beta \xi)^{1/r} = y_0^{-1} x_0^{a_0}, \quad \text{that is,} \quad x_1 y_0 = x_0^{a_0}.
\] (2.17)

If \(\beta = 0\) then \(r\) divides \(b\), whereas \(rs - ab = 1\) implies that \(r, b\) are coprime; thus \(r = 1\), so that \(k = 1\) and \(x_1 = \xi\). Otherwise \(x_2, \ldots, x_k\) are determined as usual by tag equations

\[
x_{i-1} x_{i+1} = x_i^{a_i} \quad \text{for} \quad i = 1, \ldots, k - 1,
\]

where \([a_1, \ldots, a_{k-1}] = \frac{\xi}{\beta}\) (see Remark 2.2).

(2) From the top, \(x_k = \xi\) is given; if \(r | a\) then, as before, \(r = 1\) and the only monomials are \(x_0, x_1 = \xi\). Otherwise, set \(x_{k-1} = (x_0^{\xi^{-a}})^{1/r}\), where \(\alpha\) is the least residue of \(a\) mod \(r\). Then \(r - \alpha = a_k r - a\) where \(a_k = \lceil \frac{a}{r} \rceil \geq 1\), and

\[
x_{k-1} = \xi^{a_k} (x_0^{\xi^{-a}})^{1/r} = x_k^{a_k \eta^{-1}} \quad \text{that is,} \quad x_{k-1} \eta = x_k^{a_k}.
\]

The remaining monomials are determined by

\[
x_{i-1} x_{i+1} = x_i^{a_i}, \quad \text{where} \quad [a_{k-1}, \ldots, a_1] = \frac{\xi}{r - a}.
\]

In the same way, the sequence \([b_0, b_1, \ldots, b_l]\) factors the inverse transformation of (2.13) into elementary moves:

\[
\begin{pmatrix}
-s & -a \\
-b & -r
\end{pmatrix}
= \begin{pmatrix}
0 & 1 \\
-1 & b_1
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-1 & b_{l-1}
\end{pmatrix}
\cdots
\begin{pmatrix}
0 & 1 \\
-1 & b_1
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-1 & b_0
\end{pmatrix},
\] (2.18)

More concretely, \((y_0, \eta)\) is the monomial cone \(\frac{1}{s}(b, 1)\) or \(\frac{1}{s}(1, -a)\), and the tags and monomials on the \(S_3\) side are \(b_0, \ldots, b_l\) and \(y_0, \ldots, y_k\), given by

\[
[b_0, b_1, \ldots, b_{l-1}] = \frac{-a}{s} \quad \text{and} \quad [b_1, \ldots, b_l] = \frac{b}{s}
\] (2.19)

and \(y_1 = x_0^{-1} y_0 = (y_0^\beta \eta)^{1/s}\) where \(\beta\) is the least residue of \(b\) mod \(s\).
Not every tent $T = S_0 \cup S_1 \cup S_2 \cup S_3$ is given by a fan $\Phi\left(\frac{r}{s}\right)$. Which are? And in how many ways? What extra data does the fan know about beyond $T$? The tent $T$ knows the 4 monomial cones up to SL(2, $\mathbb{Z}$) isomorphism, but does not know how they fit together in $\mathbb{Z}^2$; it knows the fractions $\frac{1}{r}(\alpha, 1)$ and $\frac{1}{s}(\beta, 1)$, but not the corner tags $a_0, b_0, a_k, b_l$.

**Corollary 2.5** The fan $\Phi\left(\frac{r}{s}\right)$ gives $T$ with $S_1 = \frac{1}{r}(\alpha, 1), S_3 = \frac{1}{s}(\beta, 1)$ by the construction of 2.2.2 if and only if $a \equiv \alpha \text{ mod } r$ and $b \equiv \beta \text{ mod } s$.

For fixed $T$, except for initial cases with $r = s = 1$ (see 2.2.4), there are 0, 1 or 2 matrices for which $\Phi\left(\frac{r}{s}\right)$ gives $T$:

- if neither $\alpha$ nor $\beta$ divides $rs - 1$, there are none;
- if $\alpha$ divides $rs - 1$ then $a = \alpha, b = (rs - 1)/\alpha$ provides a solution;
- similarly, if $\beta \mid (rs - 1)$ then $a = (rs - 1)/\beta, b = \beta$ provides a solution.

**Remark 2.6** Whereas Figure 2.2(a) sketches the division of the plane into 4 cones $\langle x_0, y_0 \rangle$, $\langle x_0, \xi \rangle$, $\langle \xi, \eta \rangle$, $\langle y_0, \eta \rangle$, 2.2(b) accurately plots the monomials in the case $\left(\frac{r}{s}\right) = \left(\frac{5}{7}\right)$, with tags $[a_0, \ldots, a_3] = [0, 4, 2, 4]$ at the $x_i$ and $[b_0, \ldots, b_1] = [-3, 3, 2, 2, 1]$ at the $y_i$. The comparison of the rich and messy reality of 2.2(b) with our square-cut projective pictures such as Figures 1.2–1.3 and 2.1 is startling but enlightening: it reveals, for example,

\begin{align*}
a_0 = 0 & \text{ at } x_0 \implies x_1, 0, y_0 \text{ are in arithmetic progression;} \\
b_4 = 1 & \text{ at } y_4 \implies 0x_3y_4y_3 \text{ is a parallelogram;} \\
b_0 = -3 & \text{ at } y_0 \implies 1 \text{ is in the affine convex hull } 1 \in \langle x_0, y_1, y_0 \rangle,
\end{align*}

and so on. The figure and its monomials have other convexity and collinearity properties to which we return later (compare the Scissors of Figure 4.2).

**2.2.3 Big end, little end, and attitude of a long rectangle**

In the SL(2, $\mathbb{Z}$) geometry of the plane, all basic cones are equivalent, so there is of course no notion of the size of an angle. Despite this, the bottom cone $\langle x_0, y_0 \rangle$ is clearly the big end of the fan $\Phi$ in Figure 2.2: if we view $\Phi$ as a pie chart, $\langle x_0, y_0 \rangle$ occupies the lion’s share of the plane, practically 50%. The issue is not size, but convexity. Our choice of signs in (2.13) is equivalent to

\[ -\langle \xi, \eta \rangle \subseteq \langle x_0, y_0 \rangle. \tag{2.20} \]

Even more holds: every monomial appearing as a minimal generator in the other cones has inverse in $\langle x_0, y_0 \rangle$.

Our choices in $\Phi$ have already decided that the bottom $S_0 = A^2_{\langle x_0, y_0 \rangle}$ is its big end and the top $S_2 = A^2_{\langle \xi, \eta \rangle}$ its little end. (The two players will swap ends for the second half of the game.) Once this choice is out of the way, there are still two dichotomies for the corner tags, forming a division into 4 cases, the attitude of the long rectangle and of the panel $V_{AB}$. Treating this carefully here will save many headaches later.
Corollary 2.7 Except for initial cases with \( r \) or \( s = 1 \) (see 2.2.4) \( r, s \neq a, b \) and

\[
r < a \iff b < s \quad \text{and} \quad r < b \iff a < s.
\]

The long rectangle \( \sigma_{AB} \) thus has attitude:

**Top tags:** either \( a_k \geq 2 \) and \( b_1 = 1 \) if \( r < a \) and \( b < s \); or \( a_k = 1 \) and \( b_1 \geq 2 \) if \( r > a \) and \( b > s \); and

**Bottom tags:** either \( a_0 \leq -1 \) and \( b_0 = 0 \) if \( r < b \) and \( a < s \); or \( a_0 = 0 \) and \( b_0 \leq -1 \) if \( r > b \) and \( a > s \).

Corollary 2.8 If \( a_0 < 0 \) and \( b_0 = 0 \) then \( [a_2, \ldots, a_k, b_1, \ldots, b_2, b_1] = 0 \). If \( a_0 = 0 \) and \( b_0 < 0 \) then \( [a_1, \ldots, a_k, b_1, \ldots, b_2] = 0 \).

Conversely, given \( \frac{r}{s}(\alpha, 1) \) and \( \frac{s}{r}(\beta, 1) \), the tent \( T \) is given by a fan \( \Phi(\frac{r}{s}(\alpha, 1)) \) with big end \( S_0 = A_2^{(x_0, y_0)} \) if and only if the continued fractions

\[
\frac{r}{r - \alpha} = [a_{k-1}, \ldots, a_1] \quad \text{and} \quad \frac{s}{s - \beta} = [b_{l-1}, \ldots, b_1]
\]

can be concatenated with \( a_k \) and \( b_l \) such that

\[
\text{either} \quad [a_2, \ldots, a_k, b_1, \ldots, b_2, b_1] = 0 \quad \text{or} \quad [a_1, a_2, \ldots, a_k, b_1, \ldots, b_2] = 0.
\]

**Proof** \( x_1 \) and \( y_0 \) are opposite vectors in Figure 2.2, so \( \langle x_1, x_2, \ldots, y_1, y_0 \rangle \) is a half-space with a basic subdivision. Q.E.D.

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2.2.4 Initial cases

We list here all the cases with \( r \) or \( s \leq 1 \), treating all cases with attitude not covered by Corollary 2.7.

\[
\begin{array}{c|cc|c}
(1 & 0 & 0) & 0 & 0 & x_{0}y_{1} = A, \\
0 & 1 & 0 & x_{1}y_{0} = B. \\
\end{array}
\]

\[
\begin{array}{c|cc|c}
(1 & 0 & b) & 0 & 0 & x_{0}y_{1} = x_{1} A, \\
b & 1 & -b & x_{1}y_{0} = B. \\
\end{array}
\]

\[
\begin{array}{c|cc|c}
(1 & a & 0) & 0 & a & x_{0}y_{1} = A, \\
a & 0 & -a & x_{1}y_{0} = y_{1} B. \\
\end{array}
\]

\[
\begin{array}{c|cc|c}
(1 & s & 1) & 1 & 1 & x_{1}y_{1} = x_{2} A, x_{2}y_{0} = y_{1} B, \\
s & 0 & -s & x_{0}x_{2} = x_{1}^{2}, \\
& 1 & -s - 1 & x_{1}y_{0} = AB, x_{0}y_{1} = x_{1}^{s-1} A. \\
\end{array}
\]

\[
\begin{array}{c|cc|c}
(r & 1 & 1) & 1 & 1 & x_{0}y_{1} = x_{2} A, x_{1}y_{1} = y_{1} B, \\
r & -(r - 1) & -1 & y_{0}y_{2} = y_{1}, \\
& 0 & -(s - 1) & x_{1}y_{0} = y_{1}^{-1} B, x_{0}y_{1} = AB. \\
\end{array}
\]

The cases with \( a \) or \( b = 1 \) and \( r, s \geq 2 \) are not exceptional; rather, they serve as the first regular example of our construction:

\[
\begin{array}{cc}
\left( \begin{array}{c}
1 \\
0
\end{array} \right) & 1 \\
\left( \begin{array}{cccc}
1 & r & s - 1 \\
0 & -r - s
\end{array} \right) & 2^{s-1} \\
\left( \begin{array}{cccc}
1 & r & 1 \\
0 & -(s - 1)
\end{array} \right)
\end{array}
\qquad
\begin{array}{cc}
\left( \begin{array}{c}
1 \\
0
\end{array} \right)
\end{array}
\]

2.3 Construction of \( T \subset V_{AB} \) from \( \left( \begin{array}{c}
\frac{r}{b} \\
\frac{s}{a}
\end{array} \right) \in \text{SL}(2, \mathbb{Z}) \)

To construct the deformation \( T \subset V_{AB} \), we pump up the fan \( \Phi(\frac{r}{b}, \frac{s}{a}) \) of 2.2.2 out of the plane \( \mathbb{M} \) to the cone \( \sigma_{AB} \) of Figure 2.3 in the 4-space of \( \mathbb{M} = \mathbb{Z}^{4} \), using the new variables \( A, B \) respectively to bend along the \( \xi \) and \( \eta \) axes. In more detail, consider the monomial lattice \( \mathbb{M} \cong \mathbb{Z}^{4} \) based by \( \xi, \eta, A, B \), and the cone \( \sigma_{AB} \) in \( \mathbb{M}_{\mathbb{R}} \) spanned by

\[
\xi, \eta, A, B \quad \text{together with} \quad x_{0} = (A\eta^{-1})\xi, \quad y_{0} = \eta^{b}(B\xi^{-1})^{s}
\]

(compare these with the equations of (2.13)). We draw \( \sigma_{AB} \) projectively, so that it has two quadrilateral faces \( \xi\eta Ax_{0} \) and \( \xi\eta y_{0} B \) (the “back”), and four triangles \( \xi Bx_{0}, Bx_{0}y_{0}, x_{0}y_{0}A \) and \( y_{0}A\eta \) (the “front”). The primitive vectors orthogonal to these faces are, in order,

\[
(0, 0, 0, 1) \quad (0, 0, 1, 0) \quad (0, 1, 1, 0) \quad (rb, rs, 1, 0) \quad (rs, as, 0, 1) \quad (1, 0, 0, 1).
\]
We get $\sigma_{AB}$ from the simplex $\langle \xi, \eta, x_0, y_0 \rangle$ by pulling out each of the two back faces to quadrilaterals, adding a vertex $A$ in the plane of $x_0, \xi, \eta$ such that $\xi, \eta, A$ is basic and the quadrilateral $x_0, \xi, \eta, A$ is convex as shown, and likewise for $B$ in the plane of $y_0, \eta, \xi$. The picture can be viewed from different perspectives (we use some below, see Figure 4.1), and trying to read metric properties from these can be misleading.

The dual cone $\sigma_{AB}^\vee$ is the convex hull of the orthogonal vectors (2.23). Since these are all in the hyperplane of weights $w$ with $w(AB) = 1$, the dotted line from $A$ to $B$ is interior to $\sigma_{AB}$, and $AB$ generates the ideal of interior monomials. This is Danilov’s criterion for the toric variety $V_{AB} = \text{Spec}(\mathbb{C}[\sigma_{AB} \cap M])$ to be Gorenstein. The unextended simplex $\langle \xi, \eta, x_0, y_0 \rangle$ itself does not in general determine $A, B$ or the matrixes (2.14).

The 2-faces $\langle x_0, \xi \rangle$ and $\langle y_0, \eta \rangle$ of the simplicial cone $\langle \xi, \eta, x_0, y_0 \rangle$ are also faces of $\sigma_{AB}$, basic in $M$ if and only if $r = 1$, respectively $s = 1$; they are the monomial cones of toric surfaces $S_1$ and $S_3$, and are determined exactly as in 2.2.2. The new feature is the relations (2.22) and their inverses

$$\eta = (A^r B^a x_0^{-1})^s y_0^{-a}, \quad \xi = x_0^{-b} (A^b B^s y_0^{-1})^r$$ \hspace{1cm} (2.24)

that determine tag relations at the corners. Indeed (2.22) and (2.24) give

$$\begin{align*}
(x_0^a \xi^{-a})^{1/r} &= A \eta^{-1} & (y_0 \eta^{-b})^{1/s} &= B \xi^{-1} \\
(x_0^b \xi^a)^{1/r} &= A^b B^s y_0^{-1} & (y_0^{-a} \eta)^{1/s} &= A^r B^a x_0^{-1} \in M.
\end{align*}$$ \hspace{1cm} (2.25)

and the analogue of Lemma 2.4 follows as in 2.2.2.

**Lemma 2.9** The face $\langle x_0, \xi \rangle$ spans a 2-dimensional vector space in $M_{2R}$, that intersects $M$ in the sublattice generated as a $\mathbb{Z}$-module by $x_0, \xi$ together with either of

$$(x_0^b \xi^a)^{1/r} = A^b B^s y_0^{-1} \quad \text{or} \quad (x_0^{-a} \xi^{-a})^{1/r} = A \eta^{-1}.$$
Write $x_0, x_1, \ldots, x_{k-1}, x_k = \xi$ for the successive monomials along the Newton boundary of $(x_0, \xi)$. The number $k$ and the monomials themselves come from either of the continued fraction expansions

\[ [a_0, a_1, \ldots, a_{k-1}] = \frac{-b}{r} \quad \text{and} \quad [a_k, \ldots, a_1] = \frac{a}{r} \]  

by the following constructions:

1. From the bottom, $x_0$ is given, and $x_1 = (x_0^\beta \xi)^{1/r}$, where $\beta$ is the least residue of $b$ modulo $r$. Thus $a_0 = \lceil \frac{-b}{r} \rceil = \frac{-b+\beta}{r} \leq 0$ and
\[ x_1 = (x_0^\beta \xi)^{1/r} = A^b B^ax_0^{a_0}y_0^{-1}, \quad \text{that is,} \quad x_1y_0 = A^b B^ax_0^{a_0}. \]

If $\beta = 0$ then $r$ divides $b$, whereas $rs - ab = 1$ implies that $r, b$ are coprime; thus $r = 1$, so that $k = 1$ and $x_1 = \xi$. Otherwise $x_2, \ldots, x_k$ are determined as usual by tag equations
\[ x_{i-1}x_{i+1} = x_i^{a_i} \quad \text{for} \quad i = 1, \ldots, k - 1, \]
where $[a_1, \ldots, a_{k-1}] = \frac{r}{\beta}$ (see Remark 2.2).

2. From the top, $x_k = \xi$ is given; if $r \mid a$ then, as before, $r = 1$ and the only monomials are $x_0, x_1 = \xi$. Otherwise, set $x_k^{-1} = (x_0^\alpha \xi^{-a})^{1/r}$, where $\alpha$ is the least residue of $\beta$ modulo $r$. Then $r - \alpha = a_k\beta - a$ where $a_k = \lceil \frac{a}{r} \rceil \geq 1$, and
\[ x_{k-1} = \xi^{a_k}(x_0^\alpha \xi^{-a})^{1/r} = x_k^{a_k} A^{\eta^{-1}} \quad \text{that is,} \quad x_{k-1}^{a_k} = Ax_k. \]

The remaining monomials are determined by
\[ x_{i-1}x_{i+1} = x_i^{a_i}, \quad \text{where} \quad [a_{k-1}, \ldots, a_1] = \frac{r}{r-\alpha}. \]

The ring $\mathbb{C}[\mathcal{M} \cap (x_0, \xi)] = \mathbb{C}[S_1]$ is isomorphic to the invariant ring of the cyclic quotient singularity $\frac{1}{r}(a, 1) \cong \frac{1}{r}(1, -b)$; here $ab = rs - 1$, so that $ab \equiv -1 \mod s$.

In the same way, $\langle y_0, \eta \rangle \cong \frac{1}{s}(b, 1) \cong \frac{1}{s}(1, -a)$ with initial monomials $y_1$ and $y_{l-1}$ determined by the corner tag equations
\[ x_0y_1 = A^b B^ax_0^{b_0} \quad \text{and} \quad \xi y_{l-1} = B^b \eta, \]
with $b_0 = \lceil \frac{-a}{s} \rceil \leq 0$ and $b_l = \lceil \frac{b}{s} \rceil \geq 1$, and the remaining monomials for $S_3$ are $y_0, y_1, \ldots, y_{l-1}, y_l$ tagged by
\[ [b_0, b_1, \ldots, b_{l-1}] = \frac{-a}{s} \quad \text{and} \quad [b_l, \ldots, b_1] = \frac{b}{s}. \]

In conclusion, the following theorem states the complete solution to toric deformation of tents that smooth the axes at one end.
Theorem 2.10 Let
\[ T = S_0 \cup S_1 \cup S_2 \cup S_3 \]
be a tent with two given cyclic quotient singularities in reduced form \( S_1 = \frac{1}{r}(\alpha, 1) \) and \( S_3 = \frac{1}{s}(\beta, 1) \). Then toric deformations \( T \subset V_{AB} \) that smooth the \( \xi \) and \( \eta \) axes correspond one-to-one with matrices
\[
\begin{pmatrix} r & a \\ b & s \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \quad \text{with} \quad a \equiv \alpha \mod r, \quad b \equiv \beta \mod s.
\]
Since \( ab = rs - 1 \) obviously implies that \( a < r \) or \( b < s \), this means that
- either \( a = \alpha | rs - 1 \) and \( b = \frac{rs - 1}{\alpha} \),
- or \( b = \beta | rs - 1 \) and \( a = \frac{rs - 1}{\beta} \).

There may be 0, 1 or 2 solutions.

3 Classification of diptychs

A diptych, for a tent \( T \), is a pair of toric deformations
\[ T \subset V_{AB} \quad \text{and} \quad T \subset V_{LM} \]
(the two panels of the diptych), in which the first smooths the top axes and the second smooths the bottom axes.

Our construction in 2.3 of \( T \subset V_{AB} \) is given, already at the level of \( T \), by the fan \( \Phi\left( \begin{pmatrix} r & a \\ b & s \end{pmatrix} \right) \) dividing the plane \( \overline{M} \) into the four cones of Figure 2.2. Its key properties are that its four cones give the four sides of \( T \), and the union of its three top cones is one step beyond convex; by this we mean that shaving either \( x_0 \) or \( y_0 \) off the two side cones makes the union of the three top cones convex, which we express by saying that the cone \( \langle x_0, y_0 \rangle \) corresponding to \( S_0 \) is the big end of the fan.

3.1 A second fan \( \Phi'\left( \begin{pmatrix} r & g \\ h & s \end{pmatrix} \right) \) and a second panel \( V_{LM} \)

For the right panel \( V_{LM} \) of our diptych, we need a second fan \( \Phi' \) in a plane \( \overline{M}' \) (not identified with \( \overline{M} \)), defining the same tent \( T \), but this time the big end of \( \Phi' \) is the top \( \langle \xi, \eta \rangle \) corresponding to \( S_2 \), and its little end the bottom \( \langle x_0, y_0 \rangle \) corresponding to \( S_0 \). For this, replace (2.13) with the base change
\[ x_0 = \eta^{-r} \xi^{-g}, \quad y_0 = \eta^{-h} \xi^{-s} \quad \text{and} \quad \eta = x_0^{-s} y_0^{-g}, \quad \xi = x_0^{-h} y_0^{-r} \quad \text{(3.1)} \]
based on the inverse pair \( (\begin{pmatrix} h & -r \\ g & -s \end{pmatrix}) \), with \( g, h \geq 0 \). As before, \( x_0, \xi, \eta, y_0 \) define a fan \( \Phi' \) of 4 cones, but with signs giving the inclusion \(- \langle x_0, y_0 \rangle \subseteq \langle \xi, \eta \rangle \) opposite to (2.20), so that \( \langle \xi, \eta \rangle \) is the big end.
Lemma 3.1 In $\Phi'$ the cone $\langle x_0, \xi \rangle$ corresponding is $\frac{1}{r}(1, h) \cong \frac{1}{r}(-g, 1)$; the cone $\langle y_0, \eta \rangle$ is $\frac{1}{s}(1, g) \cong \frac{1}{s}(-h, 1)$.

Hence $\Phi'$ defines the same tent $T$ as $\Phi$ of 2.2.2 if and only if $-g \equiv \alpha \mod r$ and $-h \equiv \beta \mod s$.

We say that $\Phi$ and $\Phi'$ related in this way are partners. 3.2 classifies all partner pairs. The analysis of the coordinate ring of $V_{AB}$ in Lemma 2.9 can be applied, with the ends exchanged, to $V_{LM}$ to prove immediately:

Lemma 3.2 From $V_{AB}$, the cone $\langle x_0, \xi \rangle$ is $\frac{1}{r}(a, 1) \cong \frac{1}{r}(1, -b)$ and from $V_{LM}$ it is $\frac{1}{s}(1, g) \cong \frac{1}{s}(-h, 1)$. The cone $\langle y_0, \eta \rangle$ is $\frac{1}{r}(b, 1) \cong \frac{1}{s}(1, -a)$ and also $\frac{1}{s}(1, h) \cong \frac{1}{s}(-g, 1)$. Therefore $ag \equiv 1 \mod r$ and $bh \equiv 1 \mod s$; together with $rs - ab = rs - gh = 1$, these imply that

$$a + h \equiv b + g \equiv 0 \mod r \text{ and } \mod s. \quad (3.2)$$

We draw the two monomial cones $\sigma_{AB}$ and $\sigma_{LM}$ together in Figure 4.1; it is easy to see that the union $\sigma_{AB} \cup \sigma_{LM}$ has convex hull a cone with a vertex.

As an example and sanity check, it is a fun exercise to run through $(\frac{7}{h} \frac{a}{s}) = (\frac{7}{4} \frac{12}{7})$ and $(\frac{g}{h} \frac{s}{s}) = (\frac{7}{2} \frac{24}{7})$ to recover the two long rectangles of Example 1.2.

### 3.2 Classification of partner pairs

Classifying all partner pairs $\Phi$, $\Phi'$ of fans is an elementary “infinite descent”.

**Rules of the game:** Given integers

$$r, s \geq 1, \quad a, b, g, h \geq 0, \quad \text{with } ab = gh = rs - 1$$

and $a + h \equiv b + g \equiv 0 \mod r$ and $\mod s. \quad (3.3)$

Use the congruences to define two integers $d \geq 1$ and $e \geq 1$:

$$a + h = ds \quad \text{and } \quad b + g = er. \quad (3.4)$$

**Theorem 3.3 (Classification Theorem I)** Each solution of (3.3–3.4) is one of the exceptional solutions (3.8) below, or is given either by

$$\begin{pmatrix} r \ a \\ b \ s \end{pmatrix} = \begin{pmatrix} d \ -1 \\ 1 \ 0 \end{pmatrix} \begin{pmatrix} e \ -1 \\ 1 \ 0 \end{pmatrix} \cdots \begin{pmatrix} e \ or \ d \ -1 \\ 1 \ 0 \end{pmatrix} \begin{pmatrix} 0 \ 1 \\ -1 \ 0 \end{pmatrix},$$

$$\begin{pmatrix} r \ g \\ h \ s \end{pmatrix} = \begin{pmatrix} 0 \ 1 \\ -1 \ 0 \end{pmatrix} \begin{pmatrix} 0 \ -1 \\ 1 \ d \ or \ e \end{pmatrix} \cdots \begin{pmatrix} 0 \ -1 \\ 1 \ d \end{pmatrix} \begin{pmatrix} 0 \ -1 \\ 1 \ e \end{pmatrix} \quad (3.5)$$

or the same with the two lefthand sides exchanged, or by

$$\begin{pmatrix} r \ a \\ b \ s \end{pmatrix} = \begin{pmatrix} 0 \ 1 \\ -1 \ d \end{pmatrix} \begin{pmatrix} 0 \ 1 \\ -1 \ e \ or \ d \end{pmatrix} \cdots \begin{pmatrix} 0 \ 1 \\ -1 \ e \ or \ d \end{pmatrix} \begin{pmatrix} 0 \ -1 \\ 1 \ 0 \end{pmatrix},$$

$$\begin{pmatrix} r \ g \\ h \ s \end{pmatrix} = \begin{pmatrix} 0 \ -1 \\ 1 \ 0 \end{pmatrix} \begin{pmatrix} 0 \ -1 \\ d \ or \ e \ 1 \end{pmatrix} \cdots \begin{pmatrix} 0 \ -1 \\ d \ 1 \end{pmatrix} \begin{pmatrix} e \ 1 \\ -1 \ 0 \end{pmatrix} \quad (3.6)$$
or the same with the two lefthand sides exchanged.

In each case, the values \(d, e \geq 1\) alternate, the two lines have the same number \(k + 1\) of factors for some \(k \geq 1\), and the values of \(d, e\) and \(k\) that are allowed are constrained only by the following table:

\[
\begin{array}{c|c|c|c|c}
\text{de} & 0 & 1 & 2 & 3 \geq 4 \\
\hline
k & 1 & \leq 2 & \leq 3 & \text{any}
\end{array}
\]  

(3.7)

**Exceptional solutions** The cases \(b = g = 0\) or \(a = h = 0\), the matrixes

\[
\begin{pmatrix} r & a \\ b & s \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} r & g \\ h & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix},
\]

(3.8)

for any \(a, g \geq 0\), or the same with both matrixes transposed.

**Remark 3.4** (1) In the statement, exchanging the two lefthand sides amounts to exchanging the roles of the two long rectangles, so exchanges \(V_{AB}\) and \(V_{LM}\) in the diptych (and turns them upside down if one draws them as long rectangles). Whether the first or second factorisation occurs depends on the attitude of the long rectangles, which is determined by whether \(b < r\) or \(b > r\); this becomes clear in the proof.

(2) The computation of a pair of long rectangles from these two matrices is implicit from Lemma 2.4, but we spell it out. The tags on the long rectangle of \(V_{AB}\) are given by the tags of the continued fraction expansion

\[-b/r = [a_0, \ldots, a_{k-1}] \quad \text{and} \quad a/r = [a_k, \ldots, a_1].\]

If \(b < r\) and \(d, e \geq 2\), then the alternating \(d, e\) tags run up the lefthand side, and the first of these will be of the form \([0, d, e, d, \ldots]\). If either \(d = 1\) or \(e = 1\), the tags one computes are those after blowdown of the 1s, as in Proposition 2.1(b); one can reintroduce them by blowup as redundant generators to see the alternating \(d, e\) sequence. The tags down the righthand side are

\[-a/s = [b_0, \ldots, b_{l-1}] \quad \text{and} \quad b/s = [b_l, \ldots, b_1].\]

The tags on the long rectangle for \(V_{LM}\) are

\[g/r = [a'_0, a_1, \ldots, a_{k-1}], \quad -h/r = [a'_k, a_{k-1}, \ldots, a_1]\]

and

\[h/s = [b'_0, b_1, \ldots, b_{l-1}] \quad \text{and} \quad g/s = [b'_l, b_{l-1}, \ldots, b_1]\]

where all but the corner tags are of course common to both long rectangles.

(3) The exceptional cases correspond to the not-very-long rectangles and not-very-surprising diptych varieties:

\[
x_0y_1 = Ax_1^e + My_0^h \\
x_1y_0 = B + L
\]
and we do not mention them again.

(4) The cases $b = h = 0$ or $a = g = 0$ are regular solutions in Theorem 3.3 with $k = 1$
and (say) $d = a$, $e = g$:

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} d & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & e \end{pmatrix}. \quad (3.9)$$

They provide the endpoint of our infinite descent:

$$\begin{array}{cccc}
& a & 0 & -g \\
0 & -a & 0 & \end{array} \quad x_0 y_1 = Ax_1^m + M
\quad x_1 y_0 = B + Lx_0^n
$$

The equations can be used to eliminate variables $B$ and $M$, so the diptych varieties in
these cases are simply isomorphic to $\mathbb{C}^6$.

(5) The restriction on $k$ when $de \leq 3$ in (3.7) arises because the product in (3.6) no
longer satisfies $r, s, a, b \geq 0$ for bigger values of $k$. Thus

$$\begin{pmatrix} d & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} d & de - 1 \\ 1 & e \end{pmatrix}$$

has top right-hand entry $< 0$ for $de = 0$ and $k = 2$. For $de = 1, 2, 3$ and $k = 3, 4, 6$
respectively, the product of $k$ factors is $-1$:

$$\begin{pmatrix} d & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} d or e & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

so we are basically into elements of finite order in $\text{SL}(2, \mathbb{Z})$.

**Proof of the Classification Theorem** The following two operations preserve all the
equalities and congruences in the rules of the game while interchanging the roles of $e$ and $d$:

$$\begin{pmatrix} r & a \\ b & s \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & d \end{pmatrix} \begin{pmatrix} r & a \\ b & s \end{pmatrix} = \begin{pmatrix} b & s \\ db - r & h \end{pmatrix} \quad (3.10)$$

and (“its inverse with $d, e$ interchanged”)

$$\begin{pmatrix} r & a \\ b & s \end{pmatrix} \leftrightarrow \begin{pmatrix} e & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r & a \\ b & s \end{pmatrix} = \begin{pmatrix} g & ea - s \\ r & a \end{pmatrix} \quad (3.11)$$

Indeed, under operation (3.10) transforms the equalities for the sums of opposing off-
diagonal terms $a + h = ds$ and $b + g = er$ into

$$s + (eh - s) = eh \quad \text{and} \quad (db - r) + r = db.$$
The inequalities in the rules of the game need not be preserved, but their failure is a termination condition.

It turns out that a series of these operations (say using (3.10) to result in (3.5)) with alternating $e$, $d$ reduces to the initial case $(\begin{smallmatrix} 1 & e \\ 0 & 1 \end{smallmatrix})$, $(\begin{smallmatrix} 0 & d \\ 1 & 0 \end{smallmatrix})$ (or the other way round) and then down to $(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$, $(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix})$, so that inverting the procedure proves the theorem. The only point is to show that these operations, or combinations of them, decrease the entries of both matrixes; the claim then follows. When $d, e \geq 2$, which operation works is a matter of the attitude of the long rectangles; when $d$ or $e = 1$, either operation decreases some entries and increases others, but composing the two, in an order determined by attitude, decreases them all. We treat the attitude in terms of the relative sizes of $r, \ldots, h$.

Consider an initial pair

$$
\begin{pmatrix}
  r & a \\
  b & s
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix} r & g \\ h & s \end{pmatrix}
$$

satisfying the rules of the game.

The case $d \geq 2$ and $e \geq 2$ Suppose provisionally that $b < r$. We apply the reduction operation (3.10) to get

$$
\begin{pmatrix}
  r & a \\
  b & s
\end{pmatrix} \mapsto \begin{pmatrix} b & s \\ db - r & h \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} r & g \\ h & s \end{pmatrix} \mapsto \begin{pmatrix} b & r \\ eh - s & h \end{pmatrix}.
$$

We claim that every entry of the two resulting matrices is strictly smaller than the corresponding entry of the initial pair: this holds in the top left entry of either matrix by the case assumption.

Since $rs - ab = 1$, we get $s < a$. By (3.4), $b < r$ implies that $g > r$, and again it is immediate that $s > h$. It remains to consider the two larger entries in the bottom left of the pair.

To see that $eh - s < h$, it is enough to check $hb - 1 < hr$: indeed multiplying by $r$ and substituting for $er = b + g$ and $rs = gh - 1$ gives

$$
ehr - rs = h(b + g) - (gh + 1) = bh - 1 < hr.
$$

But the inequality $bh - 1 < hr$ holds by the initial assumption.

Similarly we check $db - r < b$ by observing that the equivalent inequality

$$
bh - 1 = b(a + h) - (ab + 1) = bds - rs < bs,
$$

holds since we already know that $h < s$.

The inequality $db - r < b$ also implies that the resulting matrices have the same attitude, so that if $b, h \geq 1$, the same operation (3.10) will be applied at the next step, but with $e, d$ exchanged, and the descent continues.

The termination condition is that $r = 0$ or $s = 0$, since the inequalities for $r, s$ are the only rules of the game that the reduction operation can break. In either case $ab = -1$, so that $b < r$ and its friends imply $(\begin{smallmatrix} b & s \\ a & h \end{smallmatrix}) = (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$ and $(\begin{smallmatrix} b & g \\ h & s \end{smallmatrix}) = (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$. Multiplying by the inverse matrices gives the factorisation (3.5).
Finally, notice that if instead we have \( b > r \), then we must have \( g < r \) (otherwise both \( a < s \) and \( h < s \), implying \( d = 1 \), contrary to the case assumptions), in which case the operation (3.11) performs the required reduction. This gives the factorisation (3.6).

**The case** \( d > 4 \) and \( e = 1 \)  

The definitions (3.4) imply that \( b < r \) and \( g < r \). Suppose provisionally that \( b < g \).

In this case we apply the reduction operation (3.10) twice, alternating \( d \) and \( e \), to see that it reduces the pair. Thus we compute a new pair

\[
\begin{pmatrix}
0 & 1 \\
-1 & e
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-1 & d
\end{pmatrix}
\begin{pmatrix}
r & a \\
b & s
\end{pmatrix}
= \begin{pmatrix}
db - r & h \\
(d - 1)b - r & h - s
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
r & g \\
h & s
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
d & 1 \\
-1 & 0
\end{pmatrix}
= \begin{pmatrix}
db - r & b \\
(d - 2)h - a & h - s
\end{pmatrix}.
\]

We start knowing \( b, g < r \), and so \( a, h > s \), together with the case assumption \( b < g \), or equivalently \( h < a \). So we have \( bh - 1 < gh - 1 = rs \). Substituting for \( h \) from (3.4) gives \( db - rs = db - ab - 1 < rs \), so \( db - r < r \). Similarly \( bh - 1 < gh - 1 = rs \), so substituting for \( b \) from (3.4) gives \( hr - rs = hr - gh - 1 < rs \), so \( h - s < s \).

The two longer inequalities remain: \( (d - 1)b - r < b \) and \( (d - 2)h - a < h \). For the first, note that \( bh - 1 < 2bs \) since \( h - s < s \). Substituting for \( h \) gives \( (d - 1)bs - rs < (d - 1)bs - ab - 1 = (ds - a)b - bs - 1 < bs \), and dividing by \( s \) concludes.

Substituting for \( h \) in \( h - s < s \) gives \( a > (d - 2)s \). Since \( rs - ab = 1, r/b = 1/(bs) + a/s > 1/(bs) + d - 2 > d - 2 \) and we have \( r > (d - 2)b \). Substituting for \( r \) now gives \( g > (d - 2)b \).

Since \( g/b = a/h \), we get \( (d - 2)h - a < h \) as required for the second longer inequality.

The same calculations show that \( db - r \geq 0 \), so that the analogue of the provisional supposition \( b < g \) holds again after the two reduction steps, and the descent continues unless we have reached a terminal stage where the inequalities don’t hold any more. (Since we jumped straight in with two reduction steps, we should also check whether the inequalities already fail after just one of the steps: by the same calculation, this would only happen if \( b = h = 0 \), in which case the theorem follows despite the fact that not all matrix entries reduce.)

Finally, if \( b > g \) then operation (3.11) applied twice makes the reduction following a similar analysis (in this case a terminal state cannot arise after just one of the steps).

**The case** \( d = 1 \), \( e > 4 \)  

The definitions (3.4) imply that \( a < s \) and \( h < s \). Suppose provisionally that \( h < a \).

In this case we apply the reduction operation (3.10) twice, alternating \( d \) and \( e \), to see that it reduces the pair. Thus we compute a new pair

\[
\begin{pmatrix}
0 & 1 \\
-1 & e
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
r & a \\
b & s
\end{pmatrix}
= \begin{pmatrix}
db - r & h \\
(d - 1)b - r & h - s
\end{pmatrix}
\]

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and

\[
\begin{pmatrix}
  r & g \\
  h & s
\end{pmatrix}
\begin{pmatrix}
  e & 1 \\
  -1 & 0
\end{pmatrix}
\begin{pmatrix}
  1 & 1 \\
  -1 & 0
\end{pmatrix}
= \begin{pmatrix}
  db - r & b \\
  (d - 2)h - a & h - s
\end{pmatrix}.
\]

The analysis is now virtually identical to the other cases, and we omit it.

4 Combining monomial cones $\sigma_{AB}$ and $\sigma_{LM}$

Here we spell out how the factorisations in the Classification Theorem 3.3 imply growth conditions and congruences on the generators of the varieties $V_{AB}$ and $V_{LM}$; these are the conditions (i)–(v) of Corollaries 4.2 and 4.7. From Corollary 4.2(ii) onwards we restrict to the case $d, e \geq 2$. In Section 5 we also impose $de > 4$, so that we are in the main case of the introduction 1.1. The other cases are treated in [BR2, BR3].

4.1 The Pretty Polytope $\Pi(d, e, k)$

![Figure 4.1: Pretty Polytope $\Pi$: Starting from simplex $ABLM$, pull out $x_0$ on plane $ABL$, etc., with crosspiece $x_0y_0$ on the edge $AB$ in ratio $1 : d$, and $x_ky_l$ on the edge $LM$ in ratio $1 : e$. $\Pi$ has 8 vertices and 12 triangular faces; $A, B, L, M$ have valency 5, and $x_0, y_0, x_k, y_l$ valency 4.](image)

All our varieties $T, V_{AB}, V_{LM}, V_{ABLM}$ are equivariant under the same torus $T = G_m^d$; write $M = \text{Hom}(T, G_m)$ for its character lattice, identified with the monomial lattice of both $V_{AB}$ and $V_{LM}$. The coordinate ring of $V_{ABLM}$ constructed in Section 5 is $M$-graded (that is, $T$-equivariant). Write $f \sim g$ to mean that $f$ and $g$ are eigenfunctions with the same $T$-weight or eigenvalue in $M$. This chapter mostly treats the $T$-weights of monomials; we mix additive and multiplicative notation, and sometimes write $\sim = f \overset{\tau}{\sim} g$, so that, for example, the first equation of (4.1) means $x_0 \overset{\tau}{\sim} L^{-1/d} A^7 B^5$.

The Pretty Polytope $\Pi$ of Figure 4.1 combines the two polytopes $\sigma_{AB}$ of 2.3 and $\sigma_{LM}$ of 3.1. While $V_{AB}$ and $V_{LM}$ each provided many possible $\mathbb{Z}$-bases of $M$, we use instead
the impartial $\mathbb{Q}$-basis $L, M, A, B$, writing out the $\mathbb{T}$-weights of $x_{0\ldots k}, y_{0\ldots l}$ as follows:

$$x_0 = (-\frac{1}{d}, 0, \gamma, \delta) \quad \text{and} \quad x_1 = (0, \frac{1}{e}, \alpha, \beta),$$

where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & e \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & e \end{pmatrix} \begin{pmatrix} -\frac{1}{d} & 0 \\ 0 & \frac{1}{e} \end{pmatrix} & \text{if } k \text{ is even} \\ \begin{pmatrix} 0 & 1 \\ -1 & e \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & d \end{pmatrix} \begin{pmatrix} -\frac{1}{e} & 0 \\ 0 & \frac{1}{d} \end{pmatrix} & \text{if } k \text{ is odd} \end{cases}$$

(\(k\) factors in each product). Compared to (3.6), we simply remove the first and last tags (\(d\) at \(x_k\) and 0 at \(x_0\)), and put in denominators \(d, e\) corresponding to the index of the sublattice \(\mathbb{M}' = \mathbb{Z} \cdot (L, M, A, B) \subset \mathbb{M}\) (see Corollary 4.7).

The impartial basis gives \(\mathbb{M}\) two projections

$$\pi_{AB}: \mathbb{M} \to \mathbb{Q}^2 \quad \text{and} \quad \pi_{LM}: \mathbb{M} \to \mathbb{Q}^2$$

that track the exponents of \(A, B\) and of \(L, M\). The image group \(\mathbb{Q}^2\) is partially ordered, and we write \(\pi_{LM}(m) \leq 0\) to mean that \(m \in \mathbb{M}\) has nonpositive \(L, M\) exponents, etc.

**Proposition 4.1** In the impartial basis \(L, M, A, B\), the monomials \(x_0, \ldots, y_l\) have \(\mathbb{T}\)-weights of the form (for even \(k\)):

$$\begin{align*}
x_0 &= (-\frac{1}{d}, 0, \gamma, \delta) & \cdots \\
x_1 &= (0, \frac{1}{e}, \alpha, \beta) \\
x_2 &= (\frac{1}{d}, 1, \cdots) \\
x_3 &= (1, d - \frac{1}{e}, \cdots)
\end{align*}$$

$$\begin{align*}
x_{k-2} &= (\cdots 0, \frac{1}{d}) \\
x_{k-1} &= (\alpha, \beta, 0, \frac{1}{e}) \\
x_k &= (\gamma, \delta, -\frac{1}{d}, 0)
\end{align*}$$

and

$$\begin{align*}
y_0 &= (0, -\frac{1}{e}, d\gamma - \alpha, d\delta - \beta) \\
y_1 &= (\frac{1}{d}, 1 - \frac{1}{e}, \cdots)
\end{align*}$$

$$\begin{align*}
\cdots \\
y_{j+1} &= b_jy_j - y_{j-1}
\end{align*}$$

$$\begin{align*}
\cdots \\
y_{\text{odd}} &= (d\gamma - \alpha, d\delta - \beta, 0, -\frac{1}{d})
\end{align*}$$

where the \(b_j\) in (4.5) are the tags at \(y_j\) (usually 2 or 3).

When \(k\) is odd, the top-to-bottom symmetry swaps \(d\) and \(e\). At the top, nothing changes (recall that we define \(\alpha, \beta, \gamma, \delta\) in \(x_1, x_0\) by the other choice in (4.2)); at the bottom we do \(d \leftrightarrow e\) and modify \(\alpha, \beta, \gamma, \delta\) accordingly, giving \(x_k = (\gamma', \delta', -\frac{1}{e}, 0)\) and \(y_l = (e\gamma' - \alpha', e\delta' - \beta', 0, -\frac{1}{d})\).
Proof The matrix product in (4.2) ensures that the $k-1$ changes of basis of the form $x_2 = x^d_1x^{-1}_0$, etc., take the last two entries $\left(\begin{array}{c} \alpha \\ \beta \\ \gamma \\ \delta \end{array}\right)$ of $x_1, x_0$ into the last two entries $\left(\begin{array}{c} -\frac{1}{d} \\ \frac{0}{e} \\ 1/e \\ 0 \end{array}\right)$ of $x_k, x_{k-1}$. The first two columns then just record known data from $V_{LM}$, and the last two from $V_{AB}$.

Q.E.D.

Corollary 4.2 (i) Except for the explicit $-\frac{1}{d}$ and $-\frac{1}{e}$ in $x_0, x_k, y_0, y_1$ at the four corners, all the entries are $\geq 0$.

(ii) (From here on, we assume $d, e \geq 2$.) The $L$ and $M$ exponents $\pi_{LM}(x_i)$ and $\pi_{LM}(y_j)$ increase monotonically with $i$ and $j$ (in fact, increase exponentially if $de > 4$, as illustrated in Figure 4.2), while $\pi_{AB}(x_i)$ and $\pi_{AB}(y_j)$ decrease.

(iii) No $x_{0\ldots k}$ or $y_{0\ldots l}$ is $\mathbb{T}$-equivalent to a monomial in the other variables (all the $x_i, y_j, A, B, L, M$).

For (iii), notice that the $x_i, y_j, A$ and $B$ are minimal generators of the coordinate ring of $V_{AB}$ by the results of 2.3. So it is impossible to write even the first two entries of $x_i$ or $y_j$ as a positive integral combination of the other variables.

Example 4.3 (Case $k = 2$) Then

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & e \end{pmatrix} \begin{pmatrix} -\frac{1}{d} & 0 \\ 0 & \frac{1}{e} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{e} \\ \frac{1}{d} & 1 \end{pmatrix}$$

The variables $x_{0\ldots 2}, y_{0\ldots d}$ are

$$\begin{align*}
x_0 &= (-\frac{1}{d}, 0, \frac{1}{d}, 1) \\
x_1 &= (0, \frac{1}{e}, 0, \frac{1}{e}) \\
x_2 &= (\frac{1}{d}, 1, -\frac{1}{d}, 0)
y_0 &= (0, -\frac{1}{e}, 1, d - \frac{1}{e}) \\
y_i &= (\frac{i}{d}, i - \frac{1}{e}, 1 - \frac{i}{d}, d - i - \frac{1}{e}) \\ &\quad \text{for } i = 0, \ldots, d \\
y_d &= (1, d - \frac{1}{e}, 0, -\frac{1}{e})
\end{align*}$$

Check top-to-bottom symmetry. Check the two tag equations at $x_0$:

$dx_0 + (1, 0, 0, 0) = x_1 + y_0$; and $0x_0 + (0, 0, 1, d) = (0, 0, 1, d)$

corresponding to the corner tag equations $x_1y_0 = x_0^2L$ in $V_{LM}$ and $x_1y_0 = ABd$ in $V_{AB}$.

Check the tag equations at $y_0$: $1y_0 + (0, 1, 0, 0) = x_0 + y_1$, and

$$(e - 1)x_1 + (0, 0, 1, d - 1) = (0, 1 - \frac{1}{e}, 1, d - \frac{1}{e})$$

corresponding to $x_0y_1 = y_0M$ in $V_{LM}$ and $x_0y_1 = x_1^{e-1}AB^{d-1}$ in $V_{AB}$.

Example 4.4 (Case $k = 3$) Then

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & e \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & d \end{pmatrix} \begin{pmatrix} -\frac{1}{e} & 0 \\ 0 & \frac{1}{d} \end{pmatrix} = \begin{pmatrix} \frac{1}{e} & 1 \\ 1 & e - \frac{1}{d} \end{pmatrix}$$

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So $x_0, y_0, d+e-2$ are

\[
x_0 = (-\frac{1}{d}, 0, 1, e - \frac{1}{d}) \quad y_0 = (0, -\frac{1}{e}, d - \frac{1}{e}, de - 2)
\]
\[
x_1 = (0, \frac{1}{e}, \frac{1}{e}, 1) \quad y_1 = (\frac{1}{d}, 1 - \frac{1}{e}, d - 1 - \frac{1}{e}, (d - 1)e - 2 + \frac{1}{d})
\]
\[
x_2 = (\frac{1}{d}, 1, 0, \frac{1}{d}) \quad \ldots
\]
\[
x_3 = (1, d - \frac{1}{e}, -\frac{1}{e}, 0) \quad y_i = (\frac{i}{d}, i - \frac{1}{e}, d - i - \frac{1}{e}, (d - i)e - 2 + \frac{1}{d})
\]
\[\text{for } i = 0, \ldots, d - 1\]
\[
y_{d-2} = (1 - \frac{2}{d}, d - 2 - \frac{1}{e}, 2 - \frac{1}{e}, 2e - 1 - \frac{2}{d})
\]
\[
y_{d-1} = (1 - \frac{1}{d}, d - 1 - \frac{1}{e}, 1 - \frac{1}{e}, e - 1 - \frac{1}{d})
\]
\[
y_d = (2 - \frac{1}{d}, 2d - 1 - \frac{2}{e}, 1 - \frac{2}{e}, e - 2 - \frac{1}{d})
\]
\[
\ldots
\]
\[
y_{d+e-3} = (e - 1 - \frac{1}{d}, d(e - 1) - 2 + \frac{1}{e}, 1 - \frac{1}{d})
\]
\[
y_{d+e-2} = (e - \frac{1}{d}, de - 2, 0, -\frac{1}{d})
\]

Same checks; note especially the effect of the tag 3 at $y_{d-1}$.

**Example 4.5 (Case $d = 4$, $e = 6$, $k = 6$)**

<table>
<thead>
<tr>
<th>$x$</th>
<th>$L$</th>
<th>$M$</th>
<th>$A$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0$</td>
<td>( -1/4, 0, 505/4, 483 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$x_1$</td>
<td>( 0, 1/6, 22, 505/6 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$x_2$</td>
<td>( 1/4, 1, 23/4, 22 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$x_3$</td>
<td>( 1, 23/6, 1, 23/6 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$x_4$</td>
<td>( 23/4, 22, 1/4, 1 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$x_5$</td>
<td>( 22, 505/6, 0, 1/6 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$x_6$</td>
<td>( 505/4, 483, -1/4, 0 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
and

\[
\begin{array}{cccc}
L & M & A & B \\
y_0 &=& (0 & -1/6 & 483 & 11087/6) \\
y_1 &=& (1/4 & 5/6 & 1427/4 & 8189/6) \\
y_2 &=& (1/2 & 11/6 & 461/2 & 5291/6) \\
y_3 &=& (3/4 & 17/6 & 417/4 & 2393/6) \\
y_4 &=& (7/4 & 20/3 & 329/4 & 944/3) \\
y_5 &=& (11/4 & 21/2 & 241/4 & 461/2) \\
y_6 &=& (15/4 & 43/3 & 153/4 & 433/3) \\
y_7 &=& (19/4 & 109/6 & 65/4 & 373/6) \\
y_8 &=& (21/2 & 241/6 & 21/2 & 241/6) \\
y_9 &=& (65/4 & 373/6 & 19/4 & 109/6) \\
y_{10} &=& (153/4 & 439/3 & 15/4 & 433/3) \\
y_{11} &=& (241/4 & 461/2 & 11/4 & 21/2) \\
y_{12} &=& (329/4 & 944/3 & 7/4 & 20/3) \\
y_{13} &=& (417/4 & 2393/6 & 3/4 & 17/6) \\
y_{14} &=& (461/2 & 5291/6 & 1/2 & 11/6) \\
y_{15} &=& (1427/4 & 8189/6 & 1/4 & 5/6) \\
y_{16} &=& (483 & 11087/6 & 0 & -1/6) \\
\end{array}
\]

(4.7)

We read this table in several ways. Omitting the \(A\) and \(B\) columns describes \(\sigma_{LM}\) in the impartial basis. Notice the tag equations

- **Bottom:** \(x_1 y_0 = x_0^4 L\) and \(x_0 y_1 = y_0 M\);
- **Sides:** \(x_0 x_2 = x_1^6\), \(x_1 x_3 = x_2^4\) and so on;
- **Top:** \(x_5 y_{16} = L^{305} M^{1932}\) and \(x_6 y_{15} = x_2^3 L^{373} M^{1427}\).

![Figure 4.2: Scissors (compare the dots of Figure 2.2.b). The exponents of \(L\) are in units of \(1/4\) and those of \(M\) in units of \(1/6\). The initial points are \(x_0 = (-1, 0)\), \(y_0 = (0, -1)\), \(x_1 = (0, 1)\), \(y_1 = (1, 5)\).](image-url)
Figure 4.2 plots the first two columns of (4.7) as “scissors” controlled by the points $x_0 = (-\frac{1}{d}, 0)$ and $y_0 = (0, -\frac{1}{e})$ and the origin $(0, 0)$ (implicit but crucial). To describe it in words, the sequence of $y_i$ starts from $y_0$ and tries to grow along the line $\Lambda$ of slope $1/[4, 6, 4, 6, \ldots] \approx 0.261387212$, without crossing it. It first tries $x_0$ (slope $-\infty$), then $x_1$ (slope 0) and $x_2$ (slope 1/4, so under $\Lambda$), then takes one step back to $y_1 = x_2 y_0$ (slope 3/10, so above $\Lambda$). Now $y_0, y_1, y_2, y_3, x_3$ is an arithmetic progression of length $5 = d + 1$ with increment $x_2$ (and $y_{i+1} = y_i x_2$), so $0y_0 y_1 x_2, 0y_1 y_2 x_2$, etc., are parallelograms); but $x_3$ (slope 6/23) is below $\Lambda$; so take one step back to $y_3$ and construct the next arithmetic progression $y_3, y_4, y_5, y_6, y_7, x_4$ of length $6 = e$ with increment $x_3$, and so on. Compare Figure 2.2, where the scissors were more open.

**Remark 4.6** The abstract continued fraction $[e, d, \ldots]$ and its complementary continued fraction $[2, 2, \ldots, 3, \ldots]$ has two different “scissors” embeddings into the $L, M$-plane (as the dots of Figures 2.2 and 4.2) and into the $A, B$-planes, and the Pretty Polytope $\Pi(d, e, k)$ is just the diagonal embedding into the product.

### 4.2 The quotient $Q$ and the Padded Cell

The exponents of $x_{0-k}, y_{0-k}$ in Proposition 4.1 also behave in a characteristic way modulo the integers (see Figure 4.3). To understand this, we write $M' = \mathbb{Z} \cdot (L, M, A, B) \subset M$ for the sublattice generated by $A, B, L, M$, and $Q = M/M'$ for the quotient. We think of $Q$ pictorially as a fundamental domain in $M$ for the translation lattice $M'$, as in Figure 4.3.

![Figure 4.3: The Padded Cell (with sides identified): the values of $x_i$ and $y_i$ in the torus $Q = M/M'$. The $x_i$ cycle around the 4 points $(\pm \frac{1}{d}, 0)$ and $(0, \pm \frac{1}{e})$ closest to the origin, while the $y_i$ walk around the path of Figure 4.3, performing $k - 1$ quarter-circuits around the padding of the cell, starting from $x_3 \equiv y_0$. Each quarter-circuit takes place in steps of $x_1$ and has endpoint $x_{i+1}$.](image)

**Corollary 4.7** (iv) $Q \cong \mathbb{Z}/d \oplus \mathbb{Z}/e$, based by:
if $k = 2\kappa$ is even:

$$x_0 \equiv (-\frac{1}{\kappa}, 0, \pm \frac{1}{\kappa}, 0); \quad \text{and} \quad y_0 \equiv (0, -\frac{1}{\kappa}, 0, \pm \frac{1}{\kappa});$$

(4.8)

if $k = 2\kappa + 1$ is odd:

$$x_0 \equiv (-\frac{1}{\kappa}, 0, 0, \pm \frac{1}{\kappa}) \quad \text{and} \quad y_0 \equiv (0, -\frac{1}{\kappa}, \pm \frac{1}{\kappa}, 0),$$

(4.9)

where in either case $\pm = (-1)^\kappa$.

(v) The classes in $Q$ of monomials $x_0, \ldots, y_l$ are given as follows (for even $k$):

$$x_1 \equiv -y_0 \equiv (0, \frac{1}{\kappa}, 0, \mp \frac{1}{\kappa}), \quad x_i \equiv -x_{i-2} \quad \text{for } i \geq 2$$

and $y_{j+1} = y_j + x_{i(j)}$

for $j$ in the appropriate interval. In particular, in $Q$, the $x_i$ are periodic with period 4, with $x_3 \equiv y_0$.

Note that in $Q$, the different corner tags on the two long rectangles say the same thing; thus

$$x_1y_0 = x_1^0A^\alpha B^\beta = x_0^dL \quad \text{both give} \quad x_1 \equiv y_0^{-1} \in Q$$

$$x_0y_1 = y_0^{(e-1)}A^\gamma B^\delta = y_0M \quad \text{both give} \quad y_0 \equiv x_0y_1 \in Q$$

because $x_0^d, y_0^e \in \mathbb{M}$.

5 Proof of Theorem 1.1: main case

We prove the existence of the diptych variety $V_{ABLM}$ for any pair of toric extensions of the tent $V_{AB} \supset T \subset V_{LM}$ arising from the Classification Theorem 3.3 under the assumption that $d, e \geq 2$ and $de > 4$.

5.1 Structure of the proof

The proof of Theorem 1.1 builds a staircase: first, we drop a chain of projections down from the top of $V_{AB}$ to eliminate the generator $x_{2..k}$ and $y_{2..l}$ one at a time. This chain will serve as a guiding rail in the main construction; it records the order of variables and the current state of the tags and annotations as we eliminate them (Proposition 5.2): as each $s_\nu = x_{i+1}$ or $y_{j+1}$ is eliminated from $V_{AB,\nu+1}$, it has tag 1, and appears in an equation $s_\nu h_\nu = x_i y_j$ with its neighbours, where $h_\nu = h_\nu(A, B)$ is the monomial in $A, B$ defined in 5.2.3.

We then build the 6-fold $V_{ABLM}$ up from the bottom, holding tight to our guiding rail, the chain of projections of $V_{AB}$. Each step $V_{\nu+1} \rightarrow V_{\nu}$ of the induction is a Kustin–Miller unprojection (see [PR]), and adjoins an unprojection variable $s_\nu = x_{i+1}$ or $y_{j+1}$. The
current \( V_r \) is contained in the ambient space \( \mathbb{A}_r = \mathbb{A}^{i+j+6}_{(x_{0...i}, y_{0...j}, A, B, L, M)} \). The main point is to set up the unprojection divisor \( D_r \subset V_r \); we define it by the ideal
\[
I_{D_r} = (x_{0...i-1}, y_{0...j-1}, \nu),
\]
with \( \nu(A, B) \) as in 5.2.3, so that \( D_r \) is the hypersurface
\[
D_r : (\nu(A, B) = 0) \subset \mathbb{A}^6_{(x_{i}, y_{j}, A, B, L, M)}.
\]
Thus \( D_r \) is by definition the product of affine 4-space \( \mathbb{A}^4_{(x_{i}, y_{j}, L, M)} \) with the monomial curve \( \nu(A, B) = 0 \); the elements \( L, M \) form a regular sequence for \( D_r \), and the section \( L = M = 0 \) in \( D_r \) is the unprojection divisor \( D_{AB, \nu} \) for \( V_{AB, \nu+1} \rightarrow V_{AB, \nu} \). The remaining issue is to prove that \( D_r \subset V_r \), or equivalently, that
\[
I_{V_r} \subset I_{D_r} = (x_{0...i-1}, y_{0...j-1}, \nu).
\]
For this, rather than working with the actual equations of \( V_r \) (that we cannot always calculate in closed form, and include complicated terms), we prove the stronger result: any monomial in \( x_{0...i}, y_{0...j}, A, B, L, M \) with the same \( T \)-weight as a generator of \( I_{V_r} \) is in \( I_{D_r} = (x_{0...i-1}, y_{0...j-1}, \nu) \). Thus, every \( T \)-homogeneous generator of \( I_{V_r} \) is a sum of monomials in \( I_{D_r} \).

It turns out in the end, much to our regret, that our proof does here not involve any explicit pentagrams or Pfaffians; however, they are important in the constructions of [BR2] when \( de = 4 \).

### 5.2 The projection sequence of \( V_{AB} \)

This section and the next set out facts and notation for the chains of birational projections down from \( V_{AB} \) and up from \( V_{LM} \). Either chain is provided by the blowdown of Proposition 2.1(d) applied to the conclusion \( [a_2, \ldots, b_1] = 0 \) of Corollary 2.8.

**Example 5.1** Consider the long rectangle of Figure 1.2. The concatenated continued fraction \([4, 2, 1, 3, 2, 2] = 0 \) is deconstructed as
\[
[4, 2, 1, 3, 2, 2] \mapsto [4, 1, 2, 2, 2] \mapsto [3, 1, 2, 2] \mapsto [2, 1, 2] \mapsto [1, 1] = 0
\]
This is a recipe for a chain of birational projections, each eliminating a monomial from \( \sigma_{AB} \) with tag 1:

<table>
<thead>
<tr>
<th>A2</th>
<th>1B</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>AB1</th>
<th>2B</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>AB2</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>A3B2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>A2B3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>A3B5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>AB2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>A4B7</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>-1</td>
</tr>
</tbody>
</table>

36
For example, on the second line, we read \( x_1 y_2 = x_2^2 A^2 B^3 \) and \( x_2 y_1 = y_2 AB^2 \) from the tags and annotation of the first rectangle, that we can check against (1.7). Each rectangle is the monomial cone \( \sigma_{AB,\nu} \) of a Gorenstein affine toric variety \( V_{AB,\nu} \) with the given monomials in \( A, B \) as annotations, and each step \( V_{AB,\nu+1} \to V_{AB,\nu} \) is a birational projection.

### 5.2.1 Order of monomials

Our construction inverts this type of chain, up from a codimension 2 complete intersection in \( x_0, x_1, y_0, y_1, A, B \), adding \( x_2, y_2 \) and so on one at a time, to recover \( V_{AB} \). For this, we order the \( k + l - 2 \) steps inverse to the elimination of the monomials \( x_2..k, y_2..l \); that is, we rename the \( n \)th eliminated monomial \( s_\nu \) with \( \nu = k + l - 2 - n \), so that \( s_0 = x_2 \) and \( s_1 = y_2 \). We work by induction on this \( \nu \). At the same time, we name the annotation \( h_\nu \) on the monomial \( s_\nu \) as it is eliminated; the chain starts from the top with

\[
s_{k+l-3} = y_l \quad \text{and} \quad h_{k+l-3} = B \quad \text{and} \quad b_l = 1. \tag{5.4}
\]

(This uses the main case hypothesis \( d, e \geq 2 \) so that \( b_l = 1 \).

Thus in Example 5.1, \( [s_0, s_1, s_2, s_3, s_4] = [x_2, y_2, y_3, x_3, y_4] \) and

\[
[h_0, h_1, h_2, h_3, h_4] = [A^3 B^5, AB^2, AB^2, AB, B].
\]

The scissors of Figure 4.2 strongly suggest this ordering of the monomials, although there is a choice to make at the end between \( y_1 \) and \( x_2 \), which both have tag 1; we always eliminate \( s_1 = x_2 \).

### 5.2.2 The projection \( V_{AB,\nu+1} \to V_{AB,\nu} \) and the bar \( x_i \longrightarrow y_j \)

The projection sequence gives cones \( \sigma_{AB,\nu} \) that depend on the induction parameter \( \nu \). The top corners of each \( \sigma_{AB,\nu} \) are monomials \( x_i \) and \( y_j \) with \( i = i(\nu) \) and \( j = j(\nu) \) (Table 5.1 keeps track of these functions), and we know the equations of \( V_{AB,\nu} \) including

\[
x_{i-1} y_j = x_i^{\alpha_\nu} A_\nu \quad \text{and} \quad x_i y_{j-1} = y_j^{\beta_\nu} B_\nu, \tag{5.5}
\]
given by the tags and annotations at \( x_i \) and \( y_j \) in \( V_{AB,\nu} \) as in Figure 5.1. We think of this action happening at the top of a sub-rectangle, which we refer to as the bar \( x_i \longrightarrow y_j \); the bar cascades down the long rectangle as variables are projected away (see Figure 5.2), and the tag equations (5.5) at each bar provide the key pieces of quantitative data about the convexity of \( V_{AB} \) that we use throughout the proof.

![Figure 5.1: The bar \( x_i \longrightarrow y_j \) at the top of \( \sigma_{AB,\nu} \), with tag equations (5.5).](image-url)
Proposition 5.2 The chain of projections $V_{AB,\nu+1} \to V_{AB,\nu}$ reduces $V_{AB}$ down to a codimension 2 complete intersection $V_{AB,0} \subset A^b_{(x_0, x_1, y_0, y_1, A, B)}$. The step $V_{AB,\nu+1} \to V_{AB,\nu}$ eliminates $s_\nu = x_{i+1}$ or $y_{j+1}$, with two possible cases for the top of $\sigma_{AB,\nu+1}$:

\[
\begin{array}{c|c|c|c}
& s_\nu & & s_\nu \\
\hline
x_i & y_j & or & x_i & y_j \\
\hline
x_{i-1} & y_{j-1} & & x_{i-1} & y_{j-1}
\end{array}
\]

In the left case $s_\nu = x_{i+1}$, the top of $\sigma_{AB,\nu}$ and of $\sigma_{AB,\nu+1}$ are related by

\[
A_\nu = A_{\nu+1}, \quad B_\nu = A_{\nu+1}B_{\nu+1}, \quad \alpha_\nu = 1, \quad \alpha_\nu = a_i - 1, \quad \beta_\nu = \beta_{\nu+1} - 1, \quad (5.6)
\]

and similarly in the right case by

\[
A_\nu = A_{\nu+1}B_{\nu+1}, \quad B_\nu = B_{\nu+1}, \quad \alpha_\nu = \alpha_{\nu+1} - 1, \quad \beta_{\nu+1} = 1, \quad \beta_\nu = b_j - 1. \quad (5.7)
\]

5.2.3 Choice of $h_\nu(A, B)$ and the unprojection divisor $D_{AB,\nu} \subset V_{AB,\nu}$

Proposition 5.2 described the projection $V_{AB,\nu+1} \to V_{AB,\nu}$ that eliminates the variable $s_\nu$; inverting this, we construct $V_{AB,\nu+1}$ as an unprojection from $V_{AB,\nu}$ adjoining $s_\nu$. For this, we set $h_\nu = \text{hcf}(A_\nu, B_\nu)$, equal to $A_\nu$ or $B_\nu$ by (5.6–5.7) and define $D_{AB,\nu} \subset A^b_{(x_0, \ldots, x_0, \ldots, A, B)}$ by the ideal $(x_{0, i-1}, y_{0, j-1}, h_\nu)$; thus $D_{AB,\nu}$ is the hypersurface $(h_\nu = 0) \subset A^b_{(x, y, A, B)}$.

Claim The ideal of $V_{AB,\nu}$ is contained in the ideal $(x_{0, i-1}, y_{0, j-1}, h_\nu)$ of $D_{AB,\nu}$, or in other words, $D_{AB,\nu} \subset V_{AB,\nu}$.

Proof We know that $I_{V_{AB,\nu}}$ is generated by equations for $x_i y_j'$, $x_i' x_i''$ and $y_j y_j''$ for suitable values of the indexes $i', i'', j', j''$. Consider for example an equation $x_i y_j' = x_i' y_j'' A^\alpha B^\beta$ for some $j' < j$. First $\xi = 0$, for otherwise dividing by $x_i$ gives a monomial expression for $y_j'$ that contradicts Figure 2.3, where $(y_{0, i})$ is a 2-face of $\sigma_{AB}$. Substitute for $x_i$ from the tag equation $x_i y_{j-1} = y_j^{\beta_i} B_\nu$ to give

\[
y_j y_j^{\beta_j} y_{j-1} = A^\alpha B^\beta B_\nu^{-1}. \quad (5.8)
\]

Now both sides of (5.8) are 1, since the 4-dimensional vector space $\mathbb{M}_Q$ is the direct sum of the 2-dimensional subspace spanned by $y_{0, i}$ and that spanned by $A, B$ (compare Figure 2.3). Therefore $A^\alpha B^\beta = B_\nu$, and both sides of our equation are in the ideal. The other equations are similar. \text{Q.E.D.}

The initial case $n = 0$ or $\nu = k + l - 2$ is $V_{AB,\nu} = V_{AB}$; in our construction of $V_{ABLM}$, it is the final goal: if we reach it, there is nothing more to check. Then $A = A_\nu$, $B = B_\nu$, $h_\nu = 1$, and divisibility by $h_\nu$ is trivial.
5.3 Crosses, pitchforks and pentagrams

5.3.1 The spreadsheet for $V_{AB}$

Our construction of $V_{\nu+1}$ from $V_{\nu}$ reverses the projection sequence down from the top of $V_{AB}$. Our proof also needs information derived from the projection sequence up from the bottom of $V_{LM}$. Thus in Extended Example 1.2, we deconstructed $V_{LM}$ by eliminating $y_0, y_1, y_2, x_0, x_1$ from the bottom of Figure 1.3. Here we establish how the two projection sequences interleave, as an exercise in patient bookkeeping.

Table 5.1 gives the function $i = i(\nu), j = j(\nu)$ of 5.2.2 describing the top of $V_{AB,\nu}$ as in Figure 5.1. The table repeats periodically with period $d + e - 2$, or alternate half periods of $d - 1, e - 1$. We set $v = \nu \mod d + e - 2$ and write $\nu = C(d + e - 2) + v$.

<table>
<thead>
<tr>
<th>$v$</th>
<th>$i$</th>
<th>$j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2C + 1$</td>
<td>$(d + e - 4)C + 1$</td>
</tr>
<tr>
<td>$a$</td>
<td>$2C + 2$</td>
<td>$(d + e - 4)C + a$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>for $1 \leq a \leq e - 1$</td>
</tr>
<tr>
<td>$e + b - 1$</td>
<td>$2C + 3$</td>
<td>$(d + e - 4)C + e - 2 + b$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>for $1 \leq b \leq d - 1$</td>
</tr>
<tr>
<td>Final</td>
<td>$k = 2\kappa + 1$</td>
<td>$l = (d + e - 4)\kappa + 2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$v$</th>
<th>$i$</th>
<th>$j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2C + 1$</td>
<td>$(d + e - 4)C + 1$</td>
</tr>
<tr>
<td>$a$</td>
<td>$2C + 2$</td>
<td>$(d + e - 4)C + a$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>for $1 \leq a \leq d - 1$</td>
</tr>
<tr>
<td>$d + b - 1$</td>
<td>$2C + 3$</td>
<td>$(d + e - 4)C + d - 2 + b$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>for $1 \leq b \leq e - 1$</td>
</tr>
<tr>
<td>$a$</td>
<td>$2C + 2$</td>
<td>$(d + e - 4)C + a$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>for $1 \leq a \leq d - 1$</td>
</tr>
<tr>
<td>Final</td>
<td>$k = 2\kappa$</td>
<td>$l$</td>
</tr>
</tbody>
</table>

Table 5.1: Numbering the unprojection sequence for $V_{AB}$. The even case $k = 2\kappa$ has one fewer half round. The final line is irregular: it adds a final $y_i$ instead of $x_{k+1}$ with $l = (d + e - 4)\kappa + 2$ or $l = (d - 2)\kappa + (e - 2)(\kappa - 1) + 2$.

The starting point $\nu = 0$ is $V_{AB,\nu}$ with $x_1, y_1$ at its top bar. The table is split into two, the $k$ odd and $k$ even cases; we describe the odd case $k = 2\kappa + 1$. Set $\kappa = 0$ and enter the first round: the line $v = a = 1$ adds an $x_i$, then $a = 2, \ldots, e - 1$ is a half round that adds $e - 2$ terms $y_j$; similarly, the line $v = e$ (so $b = 1$) adds an $x_i$ and then $b = 2, \ldots, d - 1$ is a half round that adds $d - 2$ terms $y_j$. We then increment $\kappa \Rightarrow \kappa + 1$ and loop. Each half round adds one $x_i$ and $d - 2$ or $e - 2$ terms $y_j$. There are $k - 1$ half rounds, ending with $\nu = (d + e - 2)\kappa$ if $k = 2\kappa + 1$ or $\nu = (d - 1)\kappa + (e - 1)(\kappa - 1) + 1$ if $k = 2\kappa$.

The above treatment assumes that we are in the main case $d, e \geq 2$; everything remains true when $d$ or $e$ or both are 2. Then the intervals $2 \leq a \leq d - 1$ or $2 \leq b \leq e - 1$ are
empty, so the corresponding half periods add one \( x_i \) and no \( y_j \).

### 5.3.2 Comparing the projection sequences for \( V_{AB} \) and \( V_{LM} \)

We want to compare the bars \( x_i, y_j \) at the top of \( V_{AB,\nu} \) with the corresponding thing at the bottom of \( V_{LM} \) after a number of projections. To see this, we divide the monomials \( y_j \) up into intervals according to the lines of Table 5.1, writing \( Y_{i-1} \) for the \( i \)th half period.

![Diagram](image)

Figure 5.2: Projecting \( V_{AB} \) from the top and \( V_{LM} \) from the bottom

In more detail, for \( k \) even, the line for even \( i = 2C + 2 \) gives the interval

\[
Y_{i-1} = \left\{ y_j \mid j \in [n_i + 1, \ldots, n_i + d - 1] \right\}
\]

(5.9)

where \( n_i = (d + e - 4)\frac{i - 2}{2} \); similarly, the line \( i' = 2C + 3 \) gives

\[
Y_{i'-1} = \left\{ y_j \mid j \in [n_{i'} + 1, \ldots, n_{i'} + e - 1] \right\}
\]

(5.10)

where \( n_{i'} = (d + e - 4)\frac{i' - 3}{2} + d - 2 \).

Notice the adjacency between the intervals: the last entry \( n_i + d - 1 \) of \( Y_{i-1} \) equals the first entry \( n_{i'} + 1 \) of the following interval \( Y_{i'} \) with \( i' = i + 1 \), and vice versa. For \( d \) or \( e = 2 \), the interval \( Y_i \) reduces to one element (which, in Figure 5.2, is tagged with 4, rather than 3).

**Lemma 5.3** The bars at the top of \( V_{AB,\nu} \) are precisely \( x_{i+1}, y_j \) with \( j \in Y_i \).

The bars at the bottom of \( V_{LM,\nu'} \) (after projecting out \( \nu' \) monomials from \( V_{LM} \), starting with \( y_0 \)) are precisely \( x_{i-1}, y_j \) with \( j \in Y_i \). See Figure 5.2.

The first clause is a more digestible rephrasing of the information contained in Table 5.1 about the order of projection. The projection sequence of \( V_{LM} \) from the bottom is enumerated by a symmetric spreadsheet, which proves the second clause.

The following simple consequence is a key point of our proof in 5.4.

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Corollary 5.4 Suppose that we project out $n_1$ monomials from the top of $V_{AB}$ down to the top bar $x_i, y_j$ and $n_2$ monomials from the bottom of $V_{LM}$ up to the bottom bar $x_i', y_j'$, where $n_1 + n_2 = k + l - 2$, so that just 4 monomials remain. Then $i' < i$ and $j' < j$.

Equivalently, either $i' = i - 1$ and $j' = j - 1$ or $i' = i - 2$ and $j' = j$, so that any such projection leads to a "cross" or "pitchfork" of the shape

$$\begin{align*}
&x_i \quad y_j \\
&x_{i-1} \quad y_{j-1} & \quad \text{or} & \quad x_{i-1} \quad y_j
\end{align*}$$

(5.11)

The same phenomenon was already implicit in the cascade of pentagrams of Example 1.2; we include this, although it is not essential for our proof.

Corollary 5.5 Projecting out $n_1$ monomials from the top of $V_{AB}$ and $n_2$ from the bottom of $V_{LM}$ with $n_1 + n_2 = k + l - 3$ gives a pentagram of one of the two shapes

$$\begin{align*}
&\text{or} & \quad \text{or}
\end{align*}$$

(5.12)

5.4 Proof by induction

We construct $V = V_{ABLM}$ by serial unprojection. The induction starts from the codimension 2 complete intersection

$$V_0 \subset A^{8}_{(x_0,x_1,y_0,y_1,A,B,L,M)}$$

defined by

$$x_1y_0 = T_{x_0}(V_{AB}) + T_{x_0}(V_{LM}) \quad \text{and} \quad x_0y_1 = T_{y_0}(V_{AB}) + T_{y_0}(V_{LM})$$

where $T_{x_0}(V_{AB})$ is the righthand side of the tag equation at $x_0$ in $V_{AB}$, and similarly for the other three terms. Clearly $V_0$ is Gorenstein and $A, B, L, M$ is a regular sequence, with the regular section $L = M = 0$ in $V_0$ the variety $V_{AB,0}$. We use the following elementary fact about unprojection.

Lemma 5.6 Unprojection commutes with regular sequences: let $X, D$ be as in [PR], Theorem 1.1 and $Y \to X$ the unprojection of $D$ in $X$. Suppose that $z_1, \ldots, z_r \in \mathcal{O}_X$ is a regular sequence for $X$ and for $D$. Then $z_1, \ldots, z_r$ is also a regular sequence for $\mathcal{O}_Y$, and $Y_z$ is the unprojection of $D_z$ in $X_z$, where $Y_z : (z_1 = \cdots = z_r = 0) \subset Y$ and similarly for $D_z$ and $X_z$. \hfill \square

Inductive assumption 5.7 We own a variety $V_{\nu} = V_{ABLM,\nu}$ having a $T$-action, together with a regular sequence $L, M$ made up of $T$-eigenfunctions such that $V_{\nu} \cap (L = M = 0) = V_{AB,\nu}$.

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We start with $\nu = 0$, and $V_{ABLM,\nu} = V_0$ as above. The induction has $k + l - 2$ steps, adjoining $x_{2..k}$ and $y_{2..l}$ in the order determined in 5.2.1. When $\nu$ reaches $k + l - 2$ then $V_{ABLM} = V_\nu$ and we are finished. Otherwise, if $\nu < k + l - 2$, the induction step consists of proving that $V_\nu$ has a divisor $D_\nu$ on which $L, M$ is a regular sequence, and the section $D_\nu \cap (L = M = 0)$ is the divisor $D_{AB,\nu} \subset V_{AB,\nu}$.

If $\nu < k + l - 2$, by 5.2.3, the step $V_{AB,\nu+1} \to V_{AB,\nu}$ of the chain down from $V_{AB}$ is the unprojection adjoining the element $s_\nu$ with unprojection ideal $(x_{0..i-1}, y_{0..j-1}, h_\nu)$, where $h_\nu$ is the monomial in $A, B$ defined in 5.2.3. We seek to imitate this for the 6-fold $V_\nu$; for this, define $D_\nu$ by

$$D_\nu \subset \mathbb{A}^8_{(x_{0..i}, y_{0..j}, A, B, L, M)} \text{ with ideal } I_{D_\nu} = (x_{0..i-1}, y_{0..j-1}, h_\nu).$$

Clearly, it is the hypersurface $D_\nu : (h_\nu = 0) \subset \mathbb{A}^6_{(x, y, A, B, L, M)}$, and is the product of $\mathbb{A}^4_{(x, y, A, B, L, M)}$ with the plane curve $h_\nu(A, B) = 0$. The issue is to prove that $D_\nu \subset V_\nu$.

**Proposition 5.8 (Key point)** $I_{V_\nu} \subset I_{D_\nu} = (x_{0..i-1}, y_{0..j-1}, h_\nu)$ for every $\nu < k + l - 2$.

We prove this by a general argument on $T$-weights of monomials that may appear in a relation, without any need to analyse the actual equations of $V_\nu$. We introduce the notation $R(\nu)$ for the $T$-weights of homogeneous generators of $I_{V_{AB,\nu}}$ or equivalently, of $I_{V_\nu}$ (by $T$-equivariance); we write $f \in R(\nu)$ to indicate that $f$ is a homogeneous polynomial with $T$-weight in $R(\nu)$. The precise statement we prove is the following:

**Claim 5.9** Any monomial $x_i^a y_j^b A^\alpha B^\beta L^\lambda M^\mu \in R(\nu)$ is divisible by $h_\nu$. (We emphasise the prevailing hypotheses: $d, e \geq 2$ and $de > 4$.)

Recall that $h_\nu = \text{hcf}(A_\nu, B_\nu)$; we usually prove divisibility by $A_\nu$ or $B_\nu$. By definition, any alleged monomial in $R(\nu)$ is $T$-equivalent to a relation in $I_{V_{AB,\nu}}$ for $x_i y_j^\nu$ or $x_i x_j^\nu$ or $y_j y_j^\nu$. The main mechanism of the proof is to compare it with one of the two equations (5.5), or more precisely, with one of the model monomials

$$x_{i-1} y_j \overset{\nu}{\sim} x_i^{\alpha_\nu} A_\nu \; \text{and} \; x_i y_{j-1} \overset{\nu}{\sim} y_j^{\beta_\nu} B_\nu,$$

coming from the top corners of $V_{AB,\nu}$ as in Figure 5.1.

**Step 1** Claim 5.9 holds for every monomial in $R(\nu - 1)$. Indeed, it is divisible by $h_{\nu-1}$ by induction, and by (5.6–5.7) the $h_\nu$ increase as $\nu$ decreases.

**Step 2** The first actual calculation in the proof: Claim 5.9 holds for all the monomials $x_i y_j$ with $i' = 0, \ldots, i - 1$ and $x_i y_{j'}$ with $j' = 0, \ldots, j - 1$ appearing in cross-over relations.
Proof} We write out the proof for \( x_i y_j \) in detail as a model case. The method is to compare an alleged monomial
\[
x_i^\xi y_j^n m \sim x_i y_j \in R(\nu), \quad \text{where } m \text{ is a monomial in } A, B, L, M
\]
with the known monomial \( x_i^\alpha A_\nu \sim x_{i-1} y_j \) from (5.13). (Of course, we never need consider monomials which involve \( x \) or \( y \) variables from \( I_{D_\nu} \).) We have \( \eta = 0 \): otherwise dividing both sides by \( y_j \) contradicts Corollary 4.2(iii). Consider
\[
\frac{x_i}{x_{i-1}} \xrightarrow{\sim} \frac{x_i^\xi - \alpha_\nu}{A_\nu} m
\]
(5.14)
By Corollary 4.2(ii) and the fact that \( i' \leq i - 1 \), the lefthand side has \( L, M \) exponents \( \pi_{LM}(\frac{x_i}{x_{i-1}}) \leq 0 \) (see (4.3) for the notation \( \pi_{LM} \) and \( \pi_{AB} \)); thus \( \alpha_\nu \geq \xi \), and the equivalence takes the form
\[
x_i^\alpha \xrightarrow{\sim} \frac{x_i}{x_{i-1}} \xrightarrow{\sim} \frac{m}{A_\nu} \quad \text{with } \alpha_\nu \geq \xi.
\]
(5.15)
Now for the same reason, \( \pi_{AB}(\frac{x_i}{x_{i-1}}) \) \( \geq 0 \). The same goes for \( x_i^{\alpha_\nu - \xi} \), except for the case \( x_i = x_k \), at the top left of the rectangle for \( V_{AB} \). In the former case, we are done: \( \pi_{AB}(m/A_\nu) \geq 0 \) so \( m \) is divisible by \( A_\nu \), as required.

The initial case \( x_i = x_k \) is important: \( \pi_{AB}(x_k) = (-\frac{1}{d}, 0) \) by Proposition 4.1; because of the negative exponent, we cannot get our conclusion by convexity alone. Instead we use a congruence argument on the Padded Cell Figure 4.3: in fact, the negative exponent is the smallest possible value \( -\frac{1}{d} \), and we claim that \( \alpha_\nu \) is one of \( d - 1, d - 2, \ldots, 1 \). Indeed, if \( \nu = k + l - 2 \) we are at the end of the induction, and there is nothing to prove. Otherwise, the tag at \( x_k \) has decreased by at least one from its pristine value \( d \). It follows that the lefthand side of (5.15) has \( A \) exponent \( > -1 \) and \( B \) exponent \( \geq 0 \). On the other hand, the righthand side of (5.15) is a Laurent monomial. Therefore also in the initial case \( m \) is divisible by \( A_\nu \), as required.

The argument for \( x_i y_j' \) is similar but slightly easier. Suppose that
\[
x_i y_j' \xrightarrow{\sim} x_i^\xi y_j^n m \quad \text{with } m = A^\alpha B^\beta L^\lambda M^\mu.
\]
(5.16)
First \( \xi = 0 \), because otherwise dividing through by \( x_i \) would contradict Corollary 4.2(iii). Next, dividing through by the monomials in the second expression of (5.13) gives
\[
\frac{y_j'}{y_{j-1}} = y_j^{\eta-\beta_\nu} \times \frac{m}{B_\nu}.
\]
(5.17)
As before, since \( j' \leq j - 1 \), Corollary 4.2(ii) gives that \( \pi_{LM}(\frac{y_{j'}}{y_{j-1}}) \leq 0 \). Therefore \( \eta - \beta_\nu \leq 0 \). Taking that term to the lefthand side gives
\[
y_j^{\beta_\nu - \eta} \frac{y_j'}{y_{j-1}} = \frac{m}{B_\nu} \quad \text{with } \beta_\nu \geq \eta.
\]
(5.18)
Now $j' \leq j - 1$, so $\pi_{AB}(\frac{y_j}{y_{j-1}}) \geq 0$; the same goes for $y_j$ except if $j = l$ and $y_j$ is at the top of the long rectangle, and we are finished, with $V_\nu = V_{ABLM}$. Therefore the exponents of $A, B$ on the left hand side are $\geq 0$, and hence $m$ is divisible by $B_\nu$.

This proves Step 2. Q.E.D.

The proof of Step 2 used Corollary 4.2(ii) to compare the exponents of $x_i/x_{i'}$ and $y_j/y_{j'}$, with typical implication $i > i' \Rightarrow \pi_{LM}(x_i) > \pi_{LM}(x_{i'})$. For Step 3 we need a similar comparison for monomials $x_i/y_{j'}$ and $y_j/x_{i'}$. Care is needed here to distinguish the order of monomials in the projection sequences from the top of $V_{AB}$ and from the bottom of $V_{LM}$: the $L, M$ exponents behave monotonically in the projection sequences of $V_{AB}$, and vice versa.

**Lemma 5.10** Given two monomials $m_1, m_2 \in \{x_0 \ldots k, y_{0 \ldots l}\}$, suppose that the projection sequence for $V_{AB}$ eliminates $m_1$ before $m_2$; then

$$\pi_{LM}(m_1) \geq \pi_{LM}(m_2). \quad (5.19)$$

Similarly, if the projection sequence for $V_{LM}$ eliminates $m_1$ before $m_2$ then

$$\pi_{AB}(m_1) \geq \pi_{AB}(m_2). \quad (5.20)$$

See Scissors, Figure 4.2 for a picture. The proof is simply to observe that when a variable is introduced in an unprojection sequence, it appears linearly in the new tag equation as its corner. Example 4.5 provides a numerical sanity check, with the respective orders of elimination

$$V_{AB} : y_0, y_1, y_2, x_0, y_3, y_4, x_1, y_5, y_6, x_2, y_7, y_8, x_3, y_9, y_{10}, x_4, y_{11}, x_5, y_{12}, x_6, y_{13}, x_7, y_{14}, x_{15}, y_{16};$$

$$V_{LM} : y_0, y_1, y_2, x_0, y_3, y_4, x_1, y_5, y_6, x_2, y_7, y_8, x_3, y_9, y_{10}, x_4, y_{11}, x_5, y_{12}, x_6, y_{13}, x_7, y_{14}, x_{15}. $$

**Step 3** Claim 5.9 holds for all monomials $y_j y_a$ with $a = 0, \ldots, j - 2$.

First, Corollary 5.4 implies that the $V_{LM}$ projection sequence eliminates $y_a$ before $x_{i-1}$. Indeed, $x_{i-1}$ is joined to $y_j$ in a cross or pitchfork involving at most $y_j$ and $y_{j-1}$, and these are parties to which no $y_a$ with $a \leq j - 2$ is invited. Therefore Lemma 5.10 gives

$$\pi_{AB}(\frac{y_a}{x_{i-1}}) \geq 0. \quad (5.21)$$

As before, comparing the alleged monomial $x_i^{\xi} y_j^{\eta} m \sim y_j y_a \in R(\nu)$ with the first of (5.13) gives

$$\frac{y_a}{x_{i-1}} \pi_{LM}(x_i^{\xi-\alpha_\nu} m \sim y_j y_a \in R(\nu) \text{ with the first of (5.13) gives} \quad (5.22)$$

The proof divides into two cases.
CASE 1  The projection sequence for \(V_{AB}\) eliminates \(x_i\) before \(y_a\).

Lemma 5.10 says that \(\xi - \alpha_\nu \geq 1\) is impossible in (5.22) (the lefthand side would have \(\pi_{LM}\) strictly smaller than the right). Thus

\[
x_i^{\alpha_\nu - \xi} \frac{y_a}{x_i - 1} \sim \frac{m}{A_\nu} \quad \text{with} \quad \alpha_\nu \geq \xi,
\]

and (5.21) implies that \(A_\nu\) divides \(m\).

CASE 2  The projection sequence for \(V_{AB}\) eliminates \(y_a\) before \(x_i\). This means that \(y_a, y_j\) are both contained in the interval \(Y_{i-1}\) of Lemma 5.3, and that \(y_a\) is not at the bottom:

\[
\begin{array}{c}
x_i \\
y_j \\
y_a \\
y_b
\end{array}
\]

Suppose that \(x_i\) is tagged with \(d\) (or simply replace \(d \leftrightarrow e\) in what follows), and write \(Y_{i-1} = [b, b + d - 2]\) for the interval of Lemma 5.3. Our conclusion in this case is that \(\xi - \alpha_\nu < a - b + 1\) and \(\equiv a - b + 1 \mod d\), and so \(\xi \leq \alpha_\nu\), and the argument of Case 1 works as before.

The proof goes as follows:

(a) For \(y_a \in Y_{i-1}\)

\[
\pi_{LM}(y_a) = (a - b)\pi_{LM}(x_i) + \pi_{LM}(y_b) < (a - b + 1)\pi_{LM}(x_i). \quad (5.24)
\]

(b) On the other hand, taking \(\pi_{LM}\) in (5.22) gives

\[
(\xi - \alpha_\nu)\pi_{LM}(x_i) \leq \pi_{LM}(y_a) - \pi_{LM}(x_{i-1}) < \pi_{LM}(y_a) \quad (5.25)
\]

Therefore \(\xi - \alpha_\nu < a - b + 1 \leq d - 2\).

(c) Moreover modulo \(M'\), we have

\[
y_a \equiv \frac{x_i^{a - b + 1}}{x_i - 1} \in Q. \quad (5.26)
\]

(d) Therefore in (5.22), \(\xi - \alpha_\nu \equiv a - b + 1 \mod d\).

Proof  (a) follows from the tag equations for the toric variety \(V_{AB}\) at the successive \(y_{a}\); as \(y_{a+1}\) is eliminated it has tag 1 and tag equation

\[
x_{i}y_{a} = y_{a+1}A_{\alpha}B_{\alpha}. \quad (5.27)
\]

Applying \(\pi_{LM}\) gives the equality in (5.24), and the inequality comes from Lemma 5.10. (b) explains itself.
When we reach the bottom of this interval, we eliminate $x_i$, with tag 1 and tag equation
\[ x_{i-1}y_b = x_i A^\alpha B^\beta. \tag{5.28} \]
Viewing this equation modulo $M'$ gives $y_b \equiv x_i/x_{i-1}$, and together with (5.27) this gives the value of $y_a$ in $Q$ as
\[ y_a \equiv y_b x_i^{(a-b)} \equiv \frac{x_i^{a-b+1}}{x_{i-1}}, \tag{5.29} \]
which proves (c).

In the coordinates of the Padded Cell $Q$, we know that $x_{i-1}$ is $(0, \pm \frac{1}{2})$ and $x_i$ is $(\pm \frac{1}{2}, 0)$. The alleged monomial tells us that $y_a \equiv x_i^{\xi-a}/x_{i-1}$ modulo $M'$, and (d) follows. Q.E.D.

**STEP 4** Claim 5.9 holds for all monomials $x_i x_a$ with $a = 0, \ldots, i-2$.

The prevailing assumption that $d, e \geq 2$ and $de > 4$ is necessary here; when $d = e = 2$, Claim 5.9 fails on equations of this type.

**Proof** We compare an alleged monomial $x_i x_a \sim y_j^y m$ with the second expression of (5.13) as usual; move the $y_j$ term across, this time regardless of sign, obtaining
\[ y_j^{\beta_v - \eta} \frac{x_a}{y_{j-1}} \sim \frac{m}{B_\nu}. \tag{5.30} \]
Our conclusion in this case is that $\beta_v - \eta - 1 > -2$ and $d$ divides $\beta_v - \eta - 1$; this implies that $\pi_{AB}(m/B_\nu) \geq \pi_{AB}(x_m y_j/y_{j-1}) \geq 0$, so that $B_\nu$ divides $m$ as required.

The proof breaks up into cases as follows; we suppose that the pristine tag on $x_i$ is $d$:

**The case $d > 2, e > 2$** The alleged monomial is $x_i x_a \sim y_j^y m$, and we divide by the second of (5.13) to give
\[ y_j^{\beta_v - \eta} \frac{x_a}{y_{j-1}} \sim \frac{m}{B_\nu}. \tag{5.31} \]
This equation is the key at the end of the argument, but first we rewrite it trivially as
\[ y_j^{\beta_v - \eta - 1} \frac{x_a y_j}{y_{j-1}} \sim \frac{m}{B_\nu}. \tag{5.32} \]
Since $e > 2$, then as long as $i > 2$ the tag equation in $V_{AB}$ at $y_j$ when $y_j$ is about to be eliminated is
\[ x_{i-1} y_j \sim y_j A^* B^* \]
for nonnegative powers of $A, B$ that will not concern us, which we rewrite as
\[ \frac{y_j}{y_{j-1}} \sim \frac{x_{i-1}}{A^* B^*}. \tag{5.33} \]
Now (5.32) and (5.33) together give
\[ y_j^{\beta_\nu - \eta - 1} x_a^{x_{i-1}} \frac{x_{i-1}}{A^* B^*} \equiv \frac{m}{B_\nu}. \] (5.34)
Since \( \pi_{LM}(m/B_\nu) \geq 0 \), we have
\[ (\beta_\nu - \eta - 1) \pi_{LM}(y_j) \geq -\pi_{LM}(x_a) - \pi_{LM}(x_{i-1}), \]
and since \( y_j \) is eliminated in the projection sequence of \( V_{AB} \) before \( x_a \) and \( x_{i-1} \), we have
\[ \beta_\nu - \eta - 1 > -2, \]
or in other words that
\[ \beta_\nu - \eta \geq 0. \]
(The case \( i = 2 \) is simpler: it must have \( a = 0 \), so \( \pi_{LM}(x_i x_a) = (0, *) \) by Proposition 4.1 and thus \( \eta = 0 \); in particular \( \beta_\nu - \eta \geq 0 \).)

Since \( d > 2 \), the variable \( x_a \) is eliminated before \( y_{j-1} \) in the projection sequence for \( V_{LM} \) if and only if \( a < i - 2 \). When \( a < i - 2 \), this projection implies that \( \pi_{AB}(x_a) > \pi_{AB}(y_{j-1}) \), so that \( \beta_\nu - \eta \geq 0 \) already gives
\[ 0 \leq \pi_{AB}(\text{LHS}(5.31)) \leq \pi_{AB}(m) - \pi_{AB}(B_\nu). \]
In other words, \( B_\nu \) divides \( m \), and we are done. On the other hand, if \( a = i - 2 \), then we can use the tag equation in \( V_{LM} \) at the point that \( y_{j-1} \) is eliminated (again using that \( d > 2 \)): namely
\[ x_{i-2} y_j \equiv y_{j-1} L^* M^* \] (5.35)
(for nonnegative powers of \( L, M \) that will not concern us). Writing this as
\[ \frac{x_a}{y_{j-1}} \equiv \frac{L^* M^*}{y_j} \]
and substituting into (5.31) gives
\[ y_j^{\beta_\nu - \eta - 1} L^* M^* \equiv \frac{m}{B_\nu}. \]
Since \( y_j \) lies in a corner of the Padded Cell, this implies that \( d \) divides \( \beta_\nu - \eta - 1 \) (which is known from above to be \( \geq -1 \)), so \( \beta_\nu - \eta \geq 1 \). With this, the tag equation (5.35) implies again that \( \pi_{AB} \) of the left-hand side of (5.31) is \( \geq 0 \), and we conclude as before.

The case \( d > 2, e = 2 \) The argument proceeds almost identically, except that the tag equation in \( V_{AB} \) when \( y_j \) is eliminated is now
\[ x_{i-2} y_{j-1} \equiv y_j A^* B^* \]
which we rewrite, to replace (5.33) in the argument, as
\[ \frac{y_j}{y_{j-1}} \equiv \frac{x_{i-2}}{A^* B^*}. \]
The only effect is to replace occurrences \( x_{i-1} \) by \( x_{i-2} \), and the conclusion \( \beta_\nu - \eta \geq 0 \) still holds.
The case \( d = 2, \ e > 2 \) This case differs from the others by having a cross \( x_i, x_{i-1}, y_j, y_{j-1} \) at this projection bar, rather than the usual pitchfork. Nevertheless, the proof follows without change to show that \( \beta_r - \eta \geq 0 \).

But now it is easier: the cross (rather than pitchfork) implies that \( x_a \) is eliminated before \( y_{j-1} \) for any \( a \leq i - 2 \), and the proof follows as before. Q.E.D.

6 Final remarks

This paper arose out of a study of Mori’s remarkable “continued division” Euclidean algorithm [M] in the divisor class group of an extremal 3-fold neighbourhood \( C \subset X \) (see also [R]). As Mori has explained to us over a couple of decades, the main result of [M] corresponds to a 2-step recurrent continued fraction \( [d, e, d, \ldots] \) as in our Classification Theorem 3.3. To paraphrase his argument: an extremal neighbourhood of type A has an exceptional curve \( C \cong \mathbb{P}^1(r_1, r_2)_{(x_0, y_0)} \) that is cut transversally by two divisors \( \text{div} x_k, \text{div} y_l \) through the terminal points of type A. Mori’s algorithm replaces these two variables successively by \( x_k, x_{k-1} = Ax_k^2/y_l, \text{then } x_{k-1}, x_{k-2} = x_{k-1}/x_k, \) continuing down the \( d, e, d, \ldots \) side of a long rectangle until it reaches \( x_0, y_0 \), that are detected by a sign reversal in their degrees. At this point, the two divisors \( \text{div} x_0, \text{div} y_0 \) in \( X \) intersect set-theoretically only in the curve \( C \subset X \). It follows that they define a pencil \( X \rightarrow \mathbb{P}^1 \), and, in the flipping case, the flip \( C^+ \subset X^+ \) as its normalised graph. Our take on this is that the canonical cover of a Mori flip of Type A arises as a regular pullback from a diptych variety. We return to this in [BR4].

We comment here on the equations of \( V_{ABLM} \), since the proof by unprojection in Section 5 deduces them by unprojection, and does not write them all out explicitly. The equations lift the toric equations of \( T \) or of \( V_{AB} \), that are of the form \( u_iy_j = \cdots \) for any pair of nonadjacent monomials on the boundary of \( \sigma_{AB} \); there are thus \( (k+l+1)/2 - 1 \) of them. Our favourites among them are the Pfaffian equations coming from magic pentagrams, which are trinomials, comparable to the binomial equations of toric geometry. These determine everything, and, as in the extended example, the full set of equations can be obtained if required. In practice, this amounts to taking a colon ideal against powers of the top monomial \( x_0 y_i AB \), or of the bottom monomial \( x_0 y_0 LM \).

In toric geometry, we assume out of habit that Jung–Hirzebruch continued fractions \([a_1, \ldots, a_k]\) have entries \( a_i \geq 2 \). In fact, if a tag equation \( x_{i-1} x_{i+1} = x_i^{a_i} \) has \( a_i = 1 \) then \( x_i \) is a redundant generator. Our Classification Theorem 3.3, already implicit in Mori [M], Lemma 3.3, has output including \( d \) or \( e = 1 \) as regular cases. It turns out that \( de = 4 \) should be treated separately (see [BR2]). We used the main case assumption \( d, e \geq 2 \) and \( de \geq 4 \) in an essential way at several point in the proof of Main Theorem 1.1 in Sections 4–5. In particular, the order of unprojecting variables from top and bottom was determined by the Scissors of Figure 4.2. In the cases \( d \) or \( e = 1 \) with \( de > 4 \), we prove Theorem 1.1 in [BR3] by an argument that keeps the variable \( x_i \) tagged with 1. Allowing \( d = 1 \) (say) changes the shape of Scissors, and the nature of the Pretty Polytope and the Padded Cell of Section 4. In fact, the \( x_i \) marked with 1 are redundant generators, and
should be projected out before their neighbours marked with $e > 4$. This obliges us to start the proof again from scratch adopting a new order of projection; the proof then goes through in parallel with the main case.

We treat the extension $T \subseteq V_{AB}$ in closed form, rather than via the infinitesimal deformations of Altmann [A]. Paul Hacking observes that our diptych varieties $V_{ABLM}$ can be seen as a variant of the construction of Gross, Hacking and Keel [GHK] in a special case. Their starting point is the vertex of degree $n$, the tent in $\mathbb{A}^n$ that is the $n$-cycle formed by the coordinate planes $\mathbb{A}^2_{(x_i, x_{i+1})}$. They construct a formal scheme that deforms this tent; their basic idea is to smooth $T$, replacing the local equation $x_{i-1}x_{i+1} = 0$ of $T$ in a neighbourhood of each punctured $x_i$-axis by the tag equation

$$x_{i-1}x_{i+1} = A_i x_i^a_i,$$  \hspace{1cm} (6.1)

where the tags $a_i$ arise from a cycle of rational curves $D = D_1 + \cdots + D_n$ on a mirror log Calabi–Yau surface $Y, D$ and the deformation parameters $A_i$ play a similar role to our annotations $A, B, L, M$. The main difference between the two constructions is summarised by the slogan “perturbative versus nonperturbative”. Whereas we work in closed form with varieties and birational unprojections, [GHK] proceed by successive infinitesimal steps: their affine pieces (6.1) are glued by formal power series expansions in affine linear transformations, specified by Gromov–Witten theory of $Y, D$ and the Kontsevich–Soibelman and Gross–Siebert scattering diagram.

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