Applying Algebraic Geometry to 3–Manifold Topology

Vivien Easson

Calf Algebraic Geometry Seminar, Warwick

11th December 2003

Contents

1 **Some 3–manifold topology** 2

2 **Hyperbolic 3–manifolds and Kleinian groups** 4

3 **Representations of Kleinian groups in $SL(2, \mathbb{C})$** 6

4 **Groups acting on trees** 7

5 **Valuations and the tree for $SL(2, K)$** 9

6 **The character variety of a Kleinian group** 10

7 **Proving the two boundary slopes theorem** 11

8 **Major successes of Culler–Shalen theory** 13

Introduction

It's always good when you find the right way to look at a problem. It's even better when a simple idea turns out to have big consequences. Here we'll look at a few simple ideas in algebraic geometry, and explain what big consequences they've had for the study of 3–dimensional manifolds.

The example involved is a connection between 3–manifold topology and algebraic geometry, usually known as Culler–Shalen theory after the mathematicians Marc Culler and Peter Shalen. In their 1983 paper [CS83] they explained a method of finding non–trivial actions of certain groups on trees, by examining the ideal points of a complex algebraic variety. This insight led on to many new discoveries, and continues to be an important tool for topologists today.

In Sections 1 and 2 we'll explain some background from low–dimensional topology. Then in Sections 3 to 6 we explore the main ideas involved in Culler–Shalen theory.

One use of the theory is that it provides a way of showing that essential surfaces exist in certain hyperbolic 3–manifolds. Ideas discussed involve the Bass–Serre theory of group actions on trees, some elementary theory of valuations, and an analysis of the character variety of a Kleinian group considered as the fundamental group of a 3–manifold. Details of the background required can be found in Peter Shalen's recent survey [Sha02], which we have relied on heavily throughout.

In Section 7 we give a very specific application of the theory: a proof that one–cusped hyperbolic manifolds have at least two boundary slopes. This solved a conjecture of Lee Neuwirth – can every knot group be expressed as an amalgamation over a free group?

In fact, Culler–Shalen theory is capable of providing much stronger results. In Section 8 we describe a few results where the proofs have relied heavily on this theory: these range from an alternative proof of the Smith Conjecture to the proof that knots are determined by their complements. Results such as these have had profound implications for the study of 3–dimensional manifolds, and the theory remains a powerful tool.

This essay was given in the form of a talk to postgraduate students studying algebraic geometry. Thus concepts within algebraic geometry will not usually be explained, whereas terms from low–dimensional topology will be defined more explicitly. This is a brief overview intended to encourage readers to explore the references for themselves.

1 Some 3–manifold topology

We will assume throughout that M is an orientable 3–dimensional manifold with possibly empty boundary. All the results we will discuss relate in some way to the fundamental group $\pi_1(M)$ of this manifold. In particular, we will be considering two kinds of representations of $\pi_1(M)$. For these, our choice of basepoint for $\pi_1(M)$ is essentially irrelevant, so we will omit mention of it.

Currently, the major goal of 3–manifold topology is to examine the extent to which a manifold's topology determines its geometry. This hope is expressed by Thurston's geometrization conjecture stated below. Compare this to the Uniformization Theorem for 2–manifolds: if you know the topology of a surface, you know all possible Riemann structures and geometries. Moreover, the Riemann structures occur precisely as discrete quotients of $\mathbb{C} \cup \infty$ (spherical), $\mathbb{C}$ (euclidean) or the unit disc $\mathbb{H}$ (hyperbolic).

Conjecture 1.1 (Geometrization Conjecture, 1979)
We can decompose any 3–manifold along a canonical collection of essential spheres, tori and annuli such that the resulting pieces are all geometric.

This means that each piece has geometry modelled on one of the following: S^3, $\mathbb{E}^3$, $\mathbb{H}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{SL(2,\mathbb{R})}$, Nil *or* Sol. *Equivalently, the piece is a quotient of one of these spaces by a discrete subgroup of the group of isometries of the space.*

Thurston proved this for all 3–manifolds with non–empty boundary. The canonical decomposition is due to Kneser (1929) and Jaco–Shalen, Johannson (1979). It remains to show that every resulting piece does have a geometric structure, and various approaches have been tried over the past twenty years.

If this conjecture were true, other long–standing conjectures would follow. The most famous of these is the Poincaré conjecture, one of the Clay Mathematics Institute's seven \$1 million dollar problems. Poincaré asked if every closed, simply–connected 3–manifold is homeomorphic to S^3.

Recall that homeomorphism means that two manifolds are topologically the same: we can create a pair of mutually–inverse continuous maps between them. Closed means the manifold is compact and has empty boundary; simply–connected means that it has trivial fundamental group.

The following definition is crucial for what follows. A surface Σ is essential in a 3–manifold M if it is not parallel to the boundary, and if it is incompressible: every curve on it bounding a disc in M also bounds a disc in Σ. It also should be ∂–incompressible: if there exist arcs $\sigma \subset \Sigma$ and $\alpha \subset \partial M$ such that σ and α cobound a disc in M, they cobound a disc in Σ. From now on, we will also assume that essential surfaces have no 2–sphere components.

Geometrization is also known for any 3–manifold which contains an essential surface; such 3–manifolds are said to be Haken. Culler–Shalen theory is useful because it provides a way of finding essential surfaces in 3–manifolds. These 3–manifolds always have boundary, so are known to be geometrizable. However it's still extremely useful to know more about which surfaces occur. For example, we can try to extend from the case of a manifold with boundary to the closed case (surgery), and observe directly whether the surface remains essential after this kind of surgery.

2 Hyperbolic 3–manifolds and Kleinian groups

We've just explained why we might like to find essential surfaces in a 3–manifold M with non–empty boundary ∂M. In fact we will usually restrict attention to the case where M is a hyperbolic 3–manifold with one boundary component, a torus.

Hyperbolic 3–manifolds are quotients $\mathbb{H}^3/\Gamma$ of the unique simply–connected 3–manifold $\mathbb{H}^3$ with constant curvature -1. The group Γ, which is isomorphic to the fundamental group $\pi_1(M)$ of M, is a discrete subgroup of the group of orientation–preserving isometries of $\mathbb{H}^3$. Since

$$\mathrm{Isom}^+(\mathbb{H}^3) \cong PSL(2,\mathbb{C}) \cong SL(2,\mathbb{C})/\{\pm I\},$$

Γ is a discrete subgroup of $PSL(2,\mathbb{C})$. Thus we may make the following definition.

Definition 2.1 (Kleinian groups)
A finitely generated group Γ is Kleinian if it acts properly discontinously on $\mathbb{H}^3$ by isometries. Equivalently, Γ is Kleinian if it is a discrete subgroup of $PSL(2,\mathbb{C})$.

It's good to have a picture in mind when thinking about hyperbolic space. One possible model is the Poincaré unit ball model: take the interior of the unit ball in $\mathbb{R}^3$, and give it the metric

$$ds^2 = \left(\frac{2}{1-|r|^2}\right)^2 |dr|^2$$

where $|r|$ is the Euclidean distance from the origin and $|dr|$ is the Euclidean metric in $\mathbb{R}^3$.

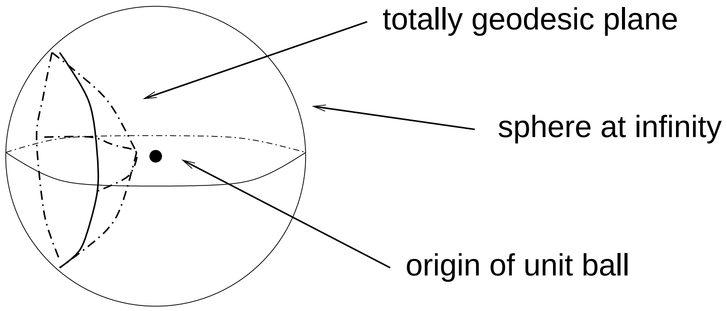

Figure 1: Poincaré unit ball model of $\mathbb{H}^3$

As in the picture, geodesics in this model are lines meeting the boundary sphere at right angles; geodesic planes are parts of spheres. Note that the boundary sphere lies at infinity in the hyperbolic metric and isn't part of $\mathbb{H}^3$.

It's also useful to have an example of a hyperbolic manifold in mind. The canonical and best–understood example of a hyperbolic manifold with one torus boundary component is the complement of the figure–eight knot in S^3. This is shown schematically below. You can either imagine

S^3 as the unit sphere in $\mathbb{R}^4$, or as $\mathbb{R}^3$ with a point attached at ∞, or as two solid balls with their boundaries glued together. Low–dimensional topologists would probably use the last of these three pictures, since this makes it easy to see what happens locally, near to a particular point.

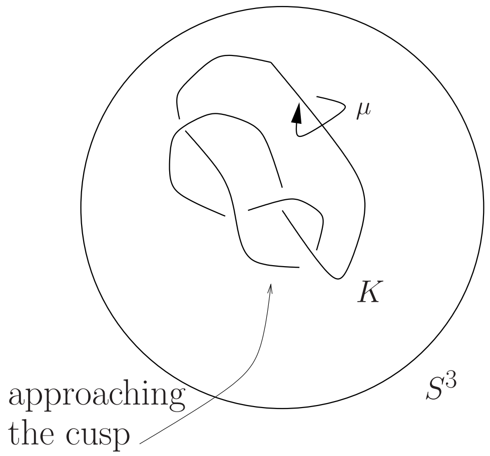

Figure 2: The figure-of-eight knot complement

The hyperbolic 3–manifold is $S^3 - K$, where K is the knotted embedding of a circle in the 3–sphere S^3. Not all knot complements have a hyperbolic structure, but the only knots which don't are the small number which have an essential torus (not boundary–parallel!) in their complement. Of prime knots with up to sixteen crossings, only 32 out of 1.7 million fail to have hyperbolic complement.

With the figure–of–eight knot, we can show directly that it has a hyperbolic structure. Draw a regular tetrahedron in the unit ball model of $\mathbb{H}^3$, such that all its vertices lie on the sphere at infinity. We can tessellate $\mathbb{H}^3$ with such tetrahedra such that the repeating unit (fundamental domain) consists of two tetrahedra glued by a face. Gluing the remaining faces of these tetrahedra in the correct way (see [Thu97], 1.4.8) we obtain the complement of the figure–of–eight knot. Indeed, we can check that, after gluing, the link of the ideal vertex (at infinity) is a collection of eight triangles which glue to form a torus knotted in the required way.

For the algebraists reading this, we can also write down the fundamental group of $S^3 - K$ as

$$\Gamma_8 = \langle x, y : (x^{-1}yxy^{-1}x^{-1})x(x^{-1}yxy^{-1}x^{-1})^{-1}y^{-1}\rangle$$

giving a one–relator presentation for the Kleinian group $\Gamma_8 = \pi_1(S^3 - K)$.

Note that the knot itself doesn't lie in the hyperbolic manifold: it lies at infinity. We call such a feature a cusp. More generally, whenever some boundary component of a hyperbolic manifold is a torus, it forms a cusp. Near the boundary component, the geometry looks like the product of a torus and an interval. This means we could describe the figure–of–eight knot complement as a one–cusped hyperbolic 3–manifold.

3 Representations of Kleinian groups in $SL(2,\mathbb{C})$

As we said above, a Kleinian group Γ is isomorphic to a discrete subgroup of $PSL(2,\mathbb{C})$. Thurston showed that we can always lift this representation to a representation of Γ into $SL(2,\mathbb{C})$. Some versions of Culler–Shalen theory work directly with the $PSL(2,\mathbb{C})$–representations to obtain more subtle results. Nevertheless we will work throughout with $SL(2,\mathbb{C})$, and hence write Γ directly in terms of 2×2 complex matrices.

For example, the fundamental group Γ_8 of the figure–of–eight knot complement has a representation where the generators x, y are sent to non–commuting parabolic generators

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}.$$

Elements of Kleinian groups are said to be parabolic, loxodromic or elliptic depending on whether they fix one, two or infinitely many points in the closure $\mathbb{H}^3 \cup S^\infty$ of $\mathbb{H}^3$. This corresponds to an appropriate matrix in $SL(2,\mathbb{C})$ having trace ± 2, trace with magnitude greater than 2 or non–real, or real trace with magnitude less than 2 respectively.

Let's summarize what's known about uniqueness and existence of these representations. For M a hyperbolic 3–manifold , $\pi_1(M)$ is Kleinian and there is a faithful discrete representation

$$\rho_0 : \pi_1(M) \to SL(2,\mathbb{C})$$

If M is closed, ρ_0 is unique. When M has cusps, we can deform ρ_0 through other representations.

One way of looking at this is by creating a algebraic variety known as the representation variety $R(\Gamma)$. Suppose Γ is generated by elements $\gamma_1, \ldots, \gamma_n$, and let ρ be a representation $\Gamma \to SL(2,\mathbb{C})$. Then ρ is completely determined by the matrices $\rho(\gamma_1), \ldots, \rho(\gamma_n)$, which have $4n$ complex variables. Thus we can think of all possible representations of Γ as those points of $\mathbb{C}^{4n}$ satisfying certain polynomial equations. These equations arise from the relations in the group Γ.

Thus the representation variety $R(\Gamma)$ is the solution set of a collection of polynomial equations, which makes it an algebraic variety in $\mathbb{C}^{4n}$.

We also need to consider some related representations of Γ. If R_0 is an irreducible subvariety of $R(\Gamma)$, we can construct the tautological (or universal) representation

$$\mathcal{P} : \Gamma \to SL(2, \mathbb{C}[R_0]); \qquad \gamma \mapsto \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}.$$

Here we define $a_\gamma, \ldots, d_\gamma$ by the property that $\rho(\gamma)$ has entries $a_\gamma(\rho), \ldots, d_\gamma(\rho)$ for any representation ρ. These functions $a_\gamma, \ldots, d_\gamma$ lie in the coordinate ring $\mathbb{C}[R]$ for $R = R(\Gamma)$ and hence in the coordinate ring of any subvariety.

In Section 6 we'll return to these concepts and define a variety based on the character of representations into $SL(2, \mathbb{C})$. The variety $R(\Gamma)$ is too fine for our purposes, since we are interested in representations of a group which is only defined up to conjugacy (change of basepoint in the fundamental group). Characters are constant on conjugacy classes, so we will consider them instead. But first we'll look at some more background material.

4 Groups acting on trees

Suppose a finitely generated group Γ acts on a tree T, a contractible graph. We always assume that Γ acts without inversions, so that any element γ preserving an edge of T also fixes each of its endpoints rather than swapping them.

Definition 4.1 (Non–trivial action)
An action of a group Γ on a tree T is non–trivial if it has no global fixed point. That is, no point of T is fixed by every element $\gamma \in \Gamma$.

For example, the free group $\mathbb{Z} * \mathbb{Z}$ acts non–trivially on the infinite tree shown. This is an example of a group acting on its Cayley graph. Vertices correspond to elements of $\mathbb{Z} * \mathbb{Z}$ and are joined by an edge if they differ by a generator or its inverse; the action corresponds to the action of the group on itself by left multiplication. We note that $\mathbb{Z} * \mathbb{Z}$ is the fundamental group of a 3–manifold, the genus two handlebody.

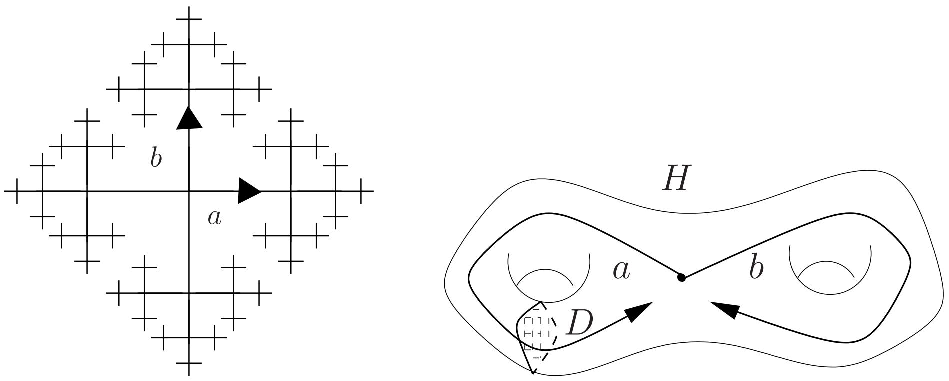

Figure 3: Cayley tree for $\mathbb{Z} * \mathbb{Z}$ and genus two handlebody

Another example of a 3–manifold fundamental group acting on a tree non–trivially arises from the Seifert surface construction for a knot complement. Every knot complement $S^3 - K$ has an essential, non–separating surface known as the Seifert surface. This follows by examining the first homology group. Any knot has homology $\pi_1(S^3 - K)^{\mathrm{ab}} = H_1(S^3 - K) \cong \mathbb{Z}$ with generator a cycle around the meridian of the knot (μ in the figure–of–eight knot diagram).

We can use this to obtain an action of $\pi_1(M)$ via abelianization on $\mathbb{R}$, with the generating cycle $[\mu]$ a unit translation. By placing vertices at integer points, we get a (simplicial) action on a tree.

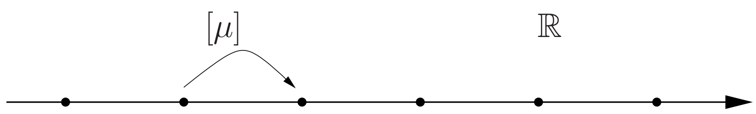

Figure 4: Action on a tree homeomorphic to $\mathbb{R}$

To such an action of $\pi_1(M)$ on a tree T we can associate a surface in M as in the following sketch. Full details are given in [Sha02], §2; non–topologists should probably just note that there is a sensible way of doing this. Roughly, if we map a 3–manifold onto a 1–dimensional tree, then the preimage of a point in the tree (say the midpoint of an edge) is 2–dimensional, and we can ensure that it is a surface with good properties.

- Construct a continuous $\pi_1(M)$–equivariant map $\tilde{f} : \widetilde{M} \to T$ from the universal cover of M, by taking a triangulation of $\widetilde{M}$ induced from a fixed triangulation of M and construct $\tilde{f}$ on the i–skeleta for $i = 0, 1, 2, 3$ in turn. Also ensure that $\tilde{f}$ is simplicial with respect to triangulations of $\widetilde{M}$ and T.
- For $E = \{\text{all midpoints of edges in } T\}$, $\tilde{f}^{-1}(E)$ is a properly embedded 2–manifold in $\widetilde{M}$. Since the set $\tilde{f}^{-1}(E)$ is $\pi_1(M)$–invariant, it covers a properly embedded 2–manifold in M.
- Compressing this surface (and discarding sphere components) gives us an essential surface. Such a surface is far from unique, but we say that it is dual to the action of $\pi_1(M)$.

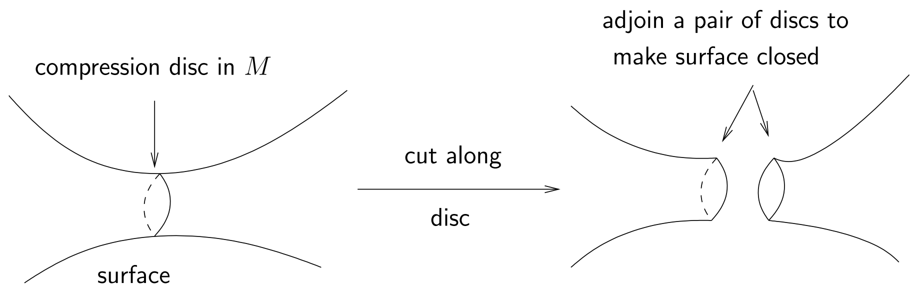

Figure 5: Compressing a surface

This construction provides the link from a non–trivial action of $\pi_1(M)$ on a tree to an essential surface in M. In the next section we describe the tree upon which actions of $\pi_1(M)$ will be constructed.

5 Valuations and the tree for $SL(2,K)$

Recall that a valuation of a field K is a surjection $v : K - \{0\} \to \mathbb{Z}$ satisfying

$$v(ab) = v(a) + v(b) \qquad \text{for all } a, b \neq 0 \tag{1}$$

$$v(a+b) \geq \min(v(a), v(b)) \qquad \text{for all } a, b \text{ such that } a, b, a+b \neq 0 \tag{2}$$

The valuation ring $\mathcal{O} = \mathcal{O}_w$ of K consists of all elements x of K which have valuation $w(x) \geq 0$. This includes the units of K which all have valuation zero.

For K a field and $w : K - \{0\} \to \mathbb{Z}$ a valuation, Serre constructed a tree T_w upon which $SL(2,K)$ acts by isometries. Full details are given in [Ser80], or in Section 3.6 of [Sha02]. We'll give only a brief picture of this tree, which is an example of a Bruhat–Tits building.

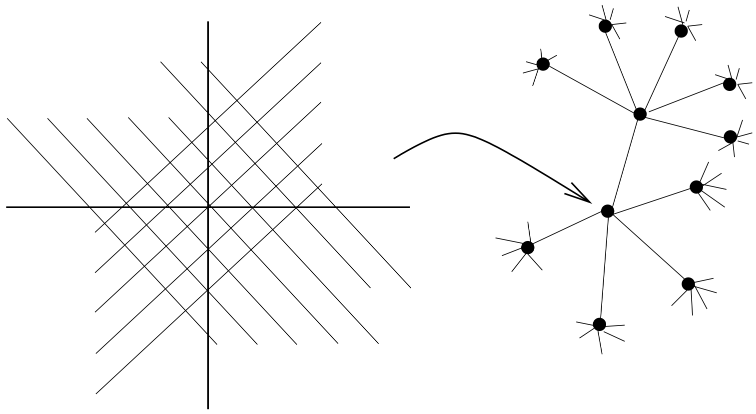

Figure 6: Sketch of tree for $SL(2,\mathbb{F}_p)$

As indicated in the sketch, the vertices of the tree T_w are homothety classes of lattices Λ in K^2, where two lattices Λ and Λ' belong to the same class if they satisfy $\Lambda = \alpha\Lambda'$ for some non–zero element α of the field K.

More generally, if an element A of the general linear group $GL(2,K)$ maps Λ onto Λ', we can define a distance between Λ and Λ' by $\delta(\Lambda,\Lambda') = w(\det(A))$. The distance d between classes can be defined from this if we're careful about the representatives we choose. Specifically we require representatives such that $\Lambda' \subset \Lambda$ and Λ/Λ' is a cyclic $\mathcal{O}$–module.

We then connect vertices by an edge if they are at distance 1 from each other. The 1–complex obtained is a connected tree.

The action of $SL(2,K)$ on this tree T_w arises by restricting the natural action of $GL(2,K)$ on lattices in K^2, and is an action without inversions. We also need to know that the $SL(2,K)$–stabilizer of a vertex is a conjugate of $SL(2,\mathcal{O})$ in $GL(2,K)$.

6 The character variety of a Kleinian group

Recall the definition of the representation variety of a Kleinian group.

Definition 6.1 (Representation variety $R(\Gamma)$)
For Γ a Kleinian group on n generators, the representation variety $R(\Gamma)$ of Γ is the algebraic variety of points in $\mathbb{C}^{4n}$ which satisfy polynomial equations obtained from the relations in Γ.

Suppose that $\pi_1(M)$ is the fundamental group of a hyperbolic 3–manifold, and that we choose two different basepoints. The resulting Kleinian groups Γ_x and Γ_y will differ by a conjugation corresponding to some path running from x to y in M. We wish to consider representations of Γ_x and Γ_y as being equivalent, but in $R(\Gamma)$ we can distinguish them. Thus we consider only the characters of representations of Kleinian groups to $SL(2,\mathbb{C})$.

The character variety $X(\Gamma)$ can be described as the quotient of $R(\Gamma)$ by $SL(2,\mathbb{C})$ in the category of algebraic sets. However we will give a more direct and elementary definition.

Definition 6.2 (Trace map)
For each element γ in a Kleinian group Γ, define an element I_γ of the coordinate ring $\mathbb{C}[R(\Gamma)]$ by $I_\gamma =$ trace $\mathcal{P}(\gamma) = a_\gamma + d_\gamma$ where $\mathcal{P}$ is the tautological representation described above. The ring $\langle I_\gamma \rangle_{\gamma\in\Gamma}$ is finitely generated by $I_{\delta_1}, \ldots, I_{\delta_N}$ for some integer $N \leq 2^n - 1$. Thus we define a map

$$t : R(\Gamma) \to \mathbb{C}^N; \qquad \rho \mapsto (I_{\delta_1}(\rho), \ldots, I_{\delta_N}(\rho))$$

known as the trace map. This map encapsulates all the character information in $R(\Gamma)$.

Indeed, $t(\rho) = t(\rho')$ if and only if ρ and ρ' are representations with the same character. Thus we can define the character variety of Γ via this map.

Definition 6.3 (Character variety $X(\Gamma)$)
For a Kleinian group Γ, define the character variety $X(\Gamma)$ to be the image $t(R(\Gamma))$ of the trace map in $\mathbb{C}^N$.

Proofs that this does define an algebraic variety can be found in [CS83] (original) and [GAMA93] (more elementary). We will simply assume this fact.

As before, consider a hyperbolic 3–manifold M. It has a faithful, discrete representation $\rho_0 : \pi_1(M) \to SL(2,\mathbb{C})$ corresponding to the hyperbolic structure. The character χ_0 of ρ_0 may be thought of as a point in the character variety $X(\pi_1(M)) \subset \mathbb{C}^N$.

If M is closed, this point χ_0 forms a 0–dimensional irreducible component of $X(\pi_1(M))$. This follows from Weil's local rigidity theorem [Wei60], [Wei62].

If M has n cusps, any irreducible component containing χ_0 has dimension n. It is not known if such a component is unique, and a lot of research is being carried out into potential "bumping" behaviour where two components touch at a point.

7 Proving the two boundary slopes theorem

Why study the character variety? We've created a algebraic geometry framework relating to concepts from 3–manifold theory. It would be good to see an example of this framework being used to prove something. In this section we will sketch a proof the following result as an example of the kind of arguments used in Culler–Shalen theory.

Theorem 7.1 (Two boundary slopes theorem)
Let M be a one–cusped hyperbolic 3–manifold. Then M has at least two boundary slopes.

We have already encountered one boundary slope: the curve on the torus boundary of a knot complement which occurs as the boundary of its Seifert surface. Precise definitions are as follows. An isotopy of a curve is a homotopy such that at every point in time, the image is a homeomorphic image of the original curve (knots can't be pulled so tight that they disappear).

Definition 7.2 (Slopes and boundary slopes)
A slope is an isotopy class of non–trivial simple closed curves on a torus component of ∂M. A boundary slope is a slope for which there exists an essential surface F in M with all components of ∂F isotopic to the slope.

Proof sketch. Let M be a one–cusped hyperbolic manifold. It has a Seifert slope s, and we let γ be an element of $\Gamma = \pi_1(M)$ representing a curve which travels once round s. Since γ represents a curve in the cusp of M, it is a parabolic element of $\Gamma = \pi_1(M)$. Thus the coordinate function I_γ takes the value ± 2 (the trace of a parabolic element) at the discrete faithful representation ρ_0 arising from the hyperbolic structure of M.

As mentioned at the end of the last section, the character χ_0 of ρ_0 lies in an irreducible 1–dimensional component of $X(\Gamma)$, a curve X_0. We can deform the representation along this curve. By Weil's local rigidity theorem, ρ_0 is isolated with the property that $I_\gamma(\rho_0) = \pm 2$. Thus I_γ is not constant on the curve X_0.

Suppose we compactify $X(\Gamma)$ by adding ideal points, and assume that these are smooth. Then, since I_γ is non–constant on X_0, it has a pole at some ideal point $x \in \bar{X}_0 - X_0$. The existence of this pole will lead to a non–trivial action on a tree.

Since every ideal point x is a smooth point, the function field $\mathbb{C}(\bar{X}(\Gamma))$ of the compactification of $X(\Gamma)$ is the same as $\mathbb{C}(X(\Gamma))$, the function field from the coordinate ring $\mathbb{C}[X(\Gamma)]$. For each point

$x \in \bar{X}(\Gamma)$ we can create a valuation v_x of $\mathbb{C}[\bar{X}(\Gamma)]$ in a natural way by assigning to each rational function f/g the order of the zero of f/g at x.

By valuation theory we can extend this valuation to a valuation w_x of the field $K = \mathbb{C}(R_{X_0})$ where R_{X_0} is the irreducible subvariety of $R(\Gamma)$ which maps onto the curve X_0 by the trace map. It follows from Section 5 that there exists a tree $T = T_{w_x}$ on which $SL(2, K)$ acts without inversions. Moreover, we can use the tautological representation $\mathcal{P} : \Gamma \to SL(2, K)$ to pull back this action and obtain an action of $\Gamma = \pi_1(M)$ on T.

The final step is to show that this action is non–trivial: there are no global fixed points. It is enough to show the following lemma.

Lemma 7.3 (Non–trivial action lemma)
The element $\gamma \in \Gamma$ fixes a vertex of T_{w_x} if and only if $I_\gamma(x) \in \mathbb{C}$.

Proof. If $I_\gamma \in \mathbb{C}$, we have $v_x(I_\gamma) \geq 0$ and hence $w_x(\text{trace } \mathcal{P}(\gamma)) \geq 0$. Thus the trace of $\mathcal{P}(\gamma)$ lies in the valuation ring $\mathcal{O}$. This implies that either $\mathcal{P}(\gamma) = \pm I$ or that $\mathcal{P}(\gamma)$ is a conjugate of an element of $SL(2, \mathcal{O})$ and hence stabilizes a vertex of T. Under the pullback, γ must also stabilize a vertex. Conversely, if γ and hence $\mathcal{P}(\gamma)$ stabilize a vertex of T, $\mathcal{P}(\gamma)$ lies in a conjugate of $SL(2, \mathcal{O})$ and has trace in $\mathcal{O}$. This implies that $v_x(I_\gamma) \geq 0$, so that I_γ cannot have a pole at x and must lie in $\mathbb{C}$ instead. □

Since we know that our chosen element γ gives us a pole in I_γ at x, it cannot fix any vertex of T_{w_x}. Therefore there will be no global fixed points and the action will be non–trivial. This in turn provides a non–empty essential surface F in M.

If ∂F is a slope distinct from the Seifert slope s, we're done: M has two boundary slopes. Otherwise, ∂F is empty or equal to s. Take a simple closed curve c with slope s, and isotope c so that it is disjoint from F. The loop represented by γ is therefore conjugate to an element of $\pi_1(C)$ where C is a component of $M - F$. But elements of C stabilize a vertex ([Sha02], Section 2.3), so γ must also stabilize a vertex of T, contradicting the lemma. Therefore this cannot happen.

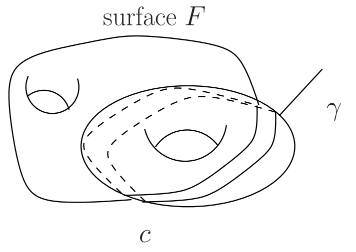

Figure 7: Examining the surface F

This shows that M must have at least two distinct boundary slopes: one corresponding to the Seifert surface arising from homology, and one arising from this Culler–Shalen construction. □

8 Major successes of Culler–Shalen theory

There have been many theorems concerning 3–manifolds proved using these kind of algebro–geometric techniques. Here are a few to tempt the appetite.

Weak Neuwirth Conjecture. Adapting the argument used to prove the Two Boundary Slopes theorem shows that every knot complement has a separating essential surface. Thus every knot group $\pi_1(S^3 - K)$ can be expressed as an amalgamated free product over a surface subgroup. [CS84]

Smith Conjecture (strong version). Suppose that $\widetilde{N}$ is a finite cover of degree > 1 of a closed manifold N branched over a knot K. If $\widetilde{N}$ is simply–connected, the knot K is trivial. [CS83]

Cyclic Surgery Theorem. Let M be a compact irreducible, non Seifert–fibred 3–manifold whose boundary is a torus. If two Dehn fillings $M(\alpha)$ and $M(\beta)$ of M are cyclic, then the primitive slopes α and β on ∂M have at most one geometric intersection. [CGLS87]. See also [Dun99].

Finite Surgery Theorem. Similar to its cyclic counterpart; see work of Boyer–Zhang. [BZ01]

Knots are determined by their complements. As a non–trivial consequence of the Cyclic Surgery Theorem, two knots with homeomorphic complements must be isotopic. [GL89]

References

[BZ01] Steven Boyer and Xingru Zhang. A proof of the finite filling conjecture. *J. Diff. Geom.*, 59:87–176, 2001.

[CGLS87] Marc Culler, Cameron McA. Gordon, John Luecke, and Peter B. Shalen. Dehn surgery on knots. *Ann. Math.*, 125:237–300, 1987.

[CS83] Marc Culler and Peter B. Shalen. Varieties of group representations and splittings of 3–manifolds. *Ann. Math.*, 117:109–146, 1983.

[CS84] Marc Culler and Peter B. Shalen. Bounded separating, incompressible surfaces in knot manifolds. *Invent. Math.*, 75:537–545, 1984.

[Dun99] Nathan M. Dunfield. Cyclic surgery, degrees of maps of character curves, and volume rigidity for hyperbolic manifolds. *Invent. Math.*, 136:623–657, 1999.

[GAMA93] F. Gonzalez-Acuna and J.M. Montesinos-Amilibia. On the character variety of group representations in $SL(2, \mathbb{C})$ and $PSL(2, \mathbb{C})$. *Math. Z.*, 214:627–652, 1993.

[GL89] Cameron McA. Gordon and John Luecke. Knots are determined by their complements. *J. Amer. Math. Soc.*, 2:371–415, 1989.

[Ser80] Jean-Pierre Serre. *Trees (translated by John Stillwell).* Springer–Verlag, 1980.

[Sha02] Peter B. Shalen. Representations of 3–manifold groups. In R.J. Daverman and R.B. Sher, editors, *Handbook of Geometric Topology*, pages 955–1044. North Holland, Amsterdam, 2002.

[Thu97] William P. Thurston. *Three-Dimensional Geometry and Topology, Volume I.* Princeton University Press, edited by Silvio Levy, 1997.

[Wei60] André Weil. On discrete subgroups of Lie groups. *Ann. Math.*, 72:369–384, 1960.

[Wei62] André Weil. On discrete subgroups of Lie groups, II. *Ann. Math.*, 75:578–602, 1962.