Problem posing in the long-term conceptual development of a gifted child

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This paper is dedicated to our friend András Ambrus who has shared ideas in mathematical problem solving with the first author for several decades. In his papers on teaching problem solving, András (Ambrus 2014, Ambrus & Barczi-Veres, 2016) considers the issues encountered in attempting to teach average-ability students using techniques that are successful with gifted children. In this paper, we consider the situation from the reverse angle, focusing on the development of a single gifted learner who is encouraged to ask questions about ideas that intrigue him. As a result, he is not only a problem *solver*, but a problem *poser*, seeking insights to make sense of mathematical concepts. The learner is the third author, Simon, now eleven years old, who speaks about mathematical ideas with his father, the second author, Nic. The paper is written by the first author, Simon's grandfather (known to Simon as 'Grampa').

As first author, it has been my privilege to talk with Simon and listen carefully to what he says. This has shed a different light on accepted theoretical frameworks formulated by mathematicians and educators about the learning and teaching of mathematics. It has caused me to rethink the fundamental ideas of mathematics and how they develop in increasing sophistication in the long term.

The way in which my son Nic developed over the long term had a profound effect on the theoretical framework that I developed in my book *How Humans Learn to Think Mathematically* (Tall, 2013). At the age of four and a half, he heard the weather forecast on the television mention the temperature would drop to 'minus two degrees' that night and asked what 'minus two' meant. This led to us looking at a thermometer as a vertical number line with higher temperatures being warmer, ice forming at zero degrees and negative temperatures being colder. The simple question 'If the temperature is two degrees centigrade and it goes down three degrees, what is the new temperature?' not only elicited the response 'minus one', it led to a long fascination with numbers of various kinds, including philosophising about the concept of infinity (Tall, 2001). Nic eventually went to university to read mathematics but soon found pure mathematics was not to his taste and changed to a degree in psychology, later taking a first-class external degree at Oxford in Theology.

He gave several reasons for his disillusion with university mathematics, but one was the introduction of the set-theoretic definition of a vector space which made no sense to him. This incident led to my work on advanced mathematical thinking focusing on the problematic change from 'natural' mathematics based on intuition to ‘formal’ mathematics based on set-theoretic definition and mathematical proof.

After leaving university, his wife had a responsible job as a County Archivist and, when his children were born, he chose to be the home maker during the day, caring for his daughter Emily and son Simon. They live 200 km away, so we meet in person only a few times each year, while speaking often using FaceTime on iPads, with occasional e-mails to interchange written materials.

Simon

Simon’s interest in mathematics started at an early age. At two years old, he watched a TV programme called ‘Mr Maker’ with characters dressed as shapes, dancing and singing “I am a square,” “I am a rectangle,” “I am a triangle,” “I am a circle.” When he was on holiday with us, he played with some square table mats and began making shapes. These included a 2×2 square, a 2×3 rectangle and two symmetric shapes that he called triangles. (Figure 1.)

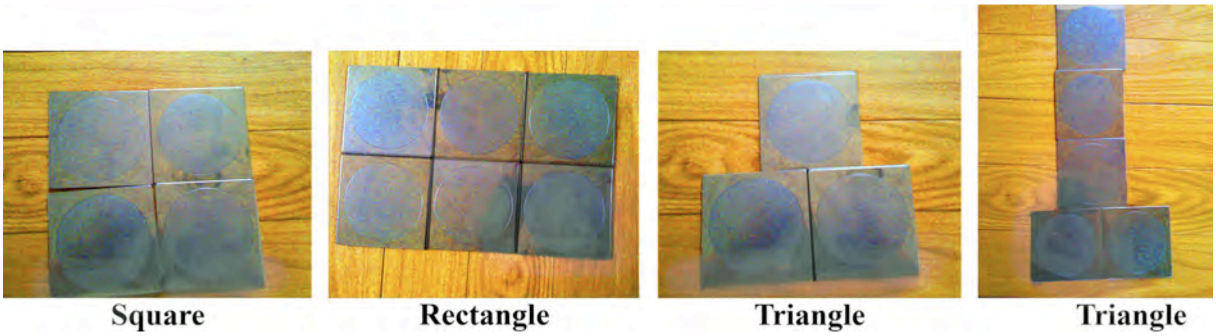

Figure 1: Simon’s shapes

As a mathematics educator, I saw that this fitted neatly with van Hiele’s theory that a child’s first stage of learning about geometric shapes involves visual appearance. Simon linked shapes to the characters in his television programme, distinguishing between a square (with equal sides) and a rectangle (with different sides). With only square table mats available, he couldn’t make a triangle as a shape with three straight sides, instead he laid out the table mats as shapes that are wide at the bottom, narrow at the top, with a vertical axis of symmetry.

As he grew older, I followed his progress at a distance as his father encouraged him to take an interest in a wide range of activities, with Simon fully in charge of what he wished to do. In mathematics, he focused on ideas that interested him and regularly asked questions to develop deeper insights.

As the mathematics became more sophisticated, Simon began to contact me using Facetime on his iPad to talk about ideas he had discovered and to ask me questions too. This offered me the possibility of participating in his long-term growth and to consider how his development fitted with various theoretical frameworks in mathematics education. What I learnt from him through his problem posing dramatically changed the ways in which I viewed long-term learning and the role of problem solving.

Developing a long-term theoretical framework

When I first met Andras Ambrus in the late 1980s, we shared interests in a problem-solving approach to learning. I had used the book *Thinking Mathematically* (Mason et al, 1982) as a basis for an undergraduate course, which is itself a rethinking of Polya's famous problem-solving book on *How to Solve It* (Polya, 1945). Fundamentally I found that the vast majority of students took delight in the course and learnt to reflect positively on their thinking processes, but this did not always relate directly to the demands of their regular mathematics courses where they had a large amount of content to learn.

Distilling the essence from many theoretical frameworks including Piaget, Dienes, Bruner, van Hiele, Skemp, etc., I saw the long-term growth for most individuals in terms of:

> ***Practical mathematics*** involving calculations and recognition of properties of mathematical concepts and operations, including a focus on *coherence*,

and

> ***Theoretical mathematics***, based on mathematical definitions and deduction with a focus on *consequence*.

The first of these is appropriate for the wider population, the second is necessary for more advance forms of mathematical thinking in Euclidean geometry and in applications requiring more sophisticated theory. A third form of mathematics, spear-headed by Hilbert at the turn of the twentieth century, has become foundational among pure mathematicians:

> ***Formal mathematics*** (based on set-theoretic and logical definitions with formal properties deduced by *mathematical proof*).

For most people, problem solving relates to theoretical mathematics – thinking about problems that need to be formulated and solved, perhaps by specialising to specific cases, then later generalising to other contexts. This involves a level of creativity that is different from the demands of regular mathematics courses following a specified curriculum. The essential conundrum is how to balance the learning of specific mathematical concepts with the more creative aspects of open-ended problem solving.

Working with Simon opened up new possibilities. He uses problem solving to *pose* his own problems to help him make sense of ideas and to develop relationships between concepts. In particular, his approach caused me to radically re-think basic ideas in arithmetic and algebra that are widely understood but may not be an explicit focus in the design of the mathematics curriculum. These will arise in greater detail as we follow his development.

Simon's long-term development

As Simon grew older, his interest in mathematics and science expanded. He loves new ideas that are unusual in some way, and working out how they fit together.

He reads books that combine ideas with humour, with titles like *Murderous Maths* or *Horrible Science*. When he walked to and from school, his father talked with him and his sister about a wide variety of topics, telling stories, speaking about current events and anything else that interested them. When his sister moved to another school, Simon's interests turned more and more to mathematics. These involved a wide range of puzzles and problems. Here we focus on his development in arithmetic and algebra. As Nic explained,

> When Simon was in Year 3 or 4 [aged 8 or 9] I would occasionally talk maths but by year 5 [when he was 10], we would talk algebra and I would ask him something like "if $2x+3 = 7$, what is x?". Because we were walking to school, he didn't have a piece of paper, so he would work them out in his head. Sometimes he would challenge me and I would say, "Do you want something really tricky?" and he'd say, "Yes." So I would invent something like "If $2x$ squared $+ x + 1 = \ldots$" and make a problem up on the spot. So he went through a period doing mental arithmetic using algebra.

This interest in mental arithmetic included talking about negative numbers, square numbers, square roots of positive numbers, properties of powers and other topics. On one occasion when he and I met in person, I wrote down the product $2^2{\times}2^3$ and asked him to say what the answer might be. His immediate guess was to multiply the powers to get 2^6. He later reflected on his experience and told me that his method would not work because it would suggest that $2^1{\times}2^1$ would give $2^{1\times1}$ which is 2 but 2×2 is 4, not 2. He now agreed that multiplying powers of the same number is calculated by *adding* the powers. Using this new rule, he found it fascinating to develop meanings for negative and fractional powers such as 3^{-1}, $2^{1/2}$, x^0.

This proved to be a general strategy that he used when faced with a new situation. He attempted to use his previous knowledge to predict how to make sense of the new context and if it didn't work, he wanted to know what changes were necessary to get a working solution.

Support from his father Nic often happened without forward planning. For example, when the two of them were walking around a country park, Nic began a conversation asking Simon what different sorts of numbers he knew about. Simon replied, "Whole numbers, decimals, fractions. I know about negative numbers." As they talked, Nic asked, "Did you know there are things called imaginary numbers?" As soon as Simon realises there is something interesting that he doesn't know about, he wants to know more. "What's an imaginary number?", he asked. Nic responded to this question with a question: "What is the square root of 1?". Simon replied, "I know 1 times itself makes 1" and Nic followed this up with "Are there any other numbers?". They both agreed that minus one was also a square root of 1. Then Nic asked the $64,000 question: "Right, so what's the square root of minus one?" When Simon realised that he could not think of such a number, Nic said, "In that case let's say we'll call it something called 'i' and 'i' is an imaginary number because it is not a real number.” Simon says “OK.” Nic asks, “Now what is i times i?” Simon thinks about it and says, “Ah well, if i is the square root of minus 1, i times i is minus 1.”

The conversation continued, with Simon enjoying adding and multiplying simple expressions involving i while replacing i^2 by –1, including questions like “What is the square root of –9?”. All this conversation took place verbally. It was only when they returned home that Nic took a piece of paper and drew the Argand diagram for Simon, with real numbers on the horizontal axis and i on the vertical axis, one unit above the origin.

Nic enjoyed broadening Simon’s horizons, but his main objective was always to respond to Simon’s interests. After his own experience enjoying maths as a child, but giving it up at university to turn to other interests, he encouraged Simon to follow his own path with maths and science as part of a range of activities including mixed martial arts, playing the clarinet, taking part in live action role play and ‘chilling out’, reading, and playing ‘Angry Birds’ on his iPad.

Over the years, Simon continued to take an interest in anything he found intriguing.

What is 2+2×2?

In November 2015, a week before his tenth birthday, Simon saw a Japanese video on Facebook that tickled his sense of humour. A student lay on the ground with a wall moving towards her. On the wall was the statement ‘2+2×2’ and there were two doors, one labelled ‘6’, the other ‘8’. The student had to decide which door to choose before being pushed into a swimming pool. The door marked ‘6’ would open and let the student through, but she chose the door marked ‘8’ and was pushed into the pool. (Figure 2.)

Figure 2: What is 2+2×2?

Simon found it very funny and called me using his iPad to talk about it. As a mathematician, I was naturally familiar with the convention, ‘multiplication takes precedence over addition’. However, for a child, and evidently for the Japanese student in the video, seeing the symbols in sequence can lead to‘2+2×2’ being evaluated sequentially, first ‘2+2 is 4’, then ‘×2’ gives ‘8’. As we spoke about this, we realised that the two different interpretations of the meaning of the sentence could be communicated by the way it is spoken. Saying ‘two plus two (pause) times two' gives 'four times two', which is eight. Saying 'two plus (pause) two times two', gives 'two plus four', which is six. Using language flexibly makes it possible to give different meanings to expressions which can then be addressed in writing by using brackets.

Simon called me on several occasions to develop the meanings of similar expressions in other contexts. A few days later he asked, "What is the square root of nine times nine?". As he knew about negative numbers, I thought he was trying to test me and I replied, "plus or minus nine." "No," he commented, "it's 27 … because 'the square root of nine (pause) times nine' is '3×9' which is 27."

Another question was, "What is 7 take away 9?". I said "–2" but he gave the answer as 11. He reasoned that if it was 7 o'clock in the evening, 9 hours earlier it would be 11 o'clock in the morning. This opened up the possibility of talking about clock arithmetic and arithmetic modulo a whole number, which has very interesting consequences when the number is prime, but there was too much going on at the time to follow this line of reasoning.

These interactions led me to a period of deep reflection on the fundamental properties of whole number arithmetic. Expressions involving more than one operation depend on how the expression is said, not only with '2+2×2' but also with '√9×9', '4–3+2' and so on. For instance, in the last case, the expression could mean 'four minus three … plus two' or 'four minus … three plus two.'

I also saw that the pure mathematician's way of defining binary operations using brackets and separate associative, commutative and distributive laws obscured a much simpler general property that is satisfied by the two fundamental operations of addition and multiplication. If we add any (finite) set of numbers, say '3+2+5+4', then the result is always the same, no matter in what order the operations are performed. This is also true multiplying any finite set of numbers, say '3×4×2×5'. What is not immediately obvious is that *this general property is also true for other forms of numbers that learners meet in school*, including whole numbers, negative numbers, fractions, decimals, real numbers, and even complex numbers. This offers a major principle that would be helpful in the long term for a broad range of learners: to keep in mind the flexible general properties of addition and multiplication and to contrast these meaningfully with the different ways that related operations behave.

Late night maths moments

Simon had quiet times when he was alone and could think deeply about his ideas. He regularly went to bed at 9.00 every evening and fell asleep in his own time. Sometimes he had a 'late night maths moment'. As Nic explains, "When this happens, he has been thinking about maths and he can't sleep; he calls excitedly down to me because he wants to explain something before he is able to rest. … He gets excited with his theories. If he has an idea that he has worked out some big overarching theory that means he can do all one mathematical branch, and he's unified it all with his insight, that's what excites him."

Dividing a whole number by a fraction

In January 2016, after 10 o'clock at night Simon called down to his father. Nic went up to find Simon with bits of scrap paper he'd been scribbling on. Simon announced, "I've got a theory so that I can do all fractions." As Nic explained, "He had been lying in bed, working out that if you divide a number by half it doubles in size, and then if you divide it by a third it triples in size." Simon then declared

"To calculate *n* divided by *x*/*y* you work out *n* times *y* over *x*."

In school, he had followed the National Curriculum introducing simple fractions such as a half and a quarter and this had not progressed beyond adding fractions with the same denominator. At home, he had developed a flexible approach to whole number arithmetic with a simple insight into equivalence of fractions and his father had introduced him to operations such as '5 times two thirds' being the same as '5 times 2 divided by 3', written as

$$5\times\frac{2}{3}=\frac{5\times 2}{3}$$

Simon used his knowledge to work out how to divide a whole number by a fraction before he ever met the general rules for adding and multiplying fractions.

What is Calculus?

In June 2016, when Simon was aged 10, his family visited Bletchley Park where the British decoded German signals during the war. His father Nic explained:

> When we went to Bletchley Park, Simon was very interested in how codes are cracked. Back home the family watched *The Imitation Game* DVD about Bletchley and Alan Turing, which involved a mention of the term 'calculus'. Simon asked what it was. I replied, "It's a mathematical technique". Simon was not satisfied. He wanted to know what 'calculus' is and how to do it.

Nic hadn't thought about it since he studied maths some twenty years earlier, so he called me to refresh his ideas. Simon had already experienced drawing graphs for simple algebraic expressions such as x^2 and x^3 using signed numbers. Nic explained:

> I sat down with Simon at bedtime that evening and just went through the basics of differentiation, drawing a graph of *x* squared and then taking two points on the graph and showing how you can work out the slope and then moving the two points together and then having a *dx* which gets infinitely small. We did one or two graphs including *x* squared and *x* cubed and the corresponding algebraic calculations and also looked at the different signs that occurred when the graph is increasing or decreasing.

Simon became very absorbed with the ideas and wanted to know more about dealing with higher powers. Together with his father he explored simple cases of the Binomial Theorem and Pascal's Triangle, which included talk about factorials.

Two weeks later Simon had his most amazing late night maths moment.

Finite Differences

On June 19th, 2016, an insight occurred which Nic described in the following terms:

> This was a complete bolt out of the blue. I put him to bed. Clearly, he must have been lying in bed thinking about square numbers, 1, 4, 9, 16 and so forth and, because he likes patterns, he saw that the difference between the square numbers increased by two each time to get 3, 7, 9, and so on. He then did the same for *x* cubed and noticed that the difference between the cubes was increasing each time but then the difference between the difference of the cubes increased by six every time. Then he thought, "I wonder what would happens for *x* to the power of 4?". Working out successive differences led to the difference between the difference between the differences giving the constant number 24. Now he had 2, 6 and 24. Knowing about factorials, he suddenly realises these are factorial numbers and says to himself, "Well, if I did the numbers for *x* to the fifth, eventually the increase in the difference will settle down at 5 factorial, which is 120." At this point, half past ten at night, he calls down, "Daddy, I need to talk to you." He says, "I've got a maths theory," and I knew that he wouldn't be able to sleep until he has had enough time to talk about it and explain it to let his mind reach a point of satisfaction. He's got school next day, it's half past ten at night, I need to let him explain what he is doing. I am just blown away with what he has come up with. I go downstairs, get a calculator and explain to my wife that it could be a late one. I work out the sequence of powers of *x* to the fifth and look at the difference between those numbers, following it through until I find it is indeed 120. Simon's face is just delighted. We talk to Grampa on FaceTime and explain it all. Simon is happy and can now go to sleep. Next day we get him to write it all up so that he can show it to his teacher who is very impressed but finds it difficult to understand. (Figure 3.)

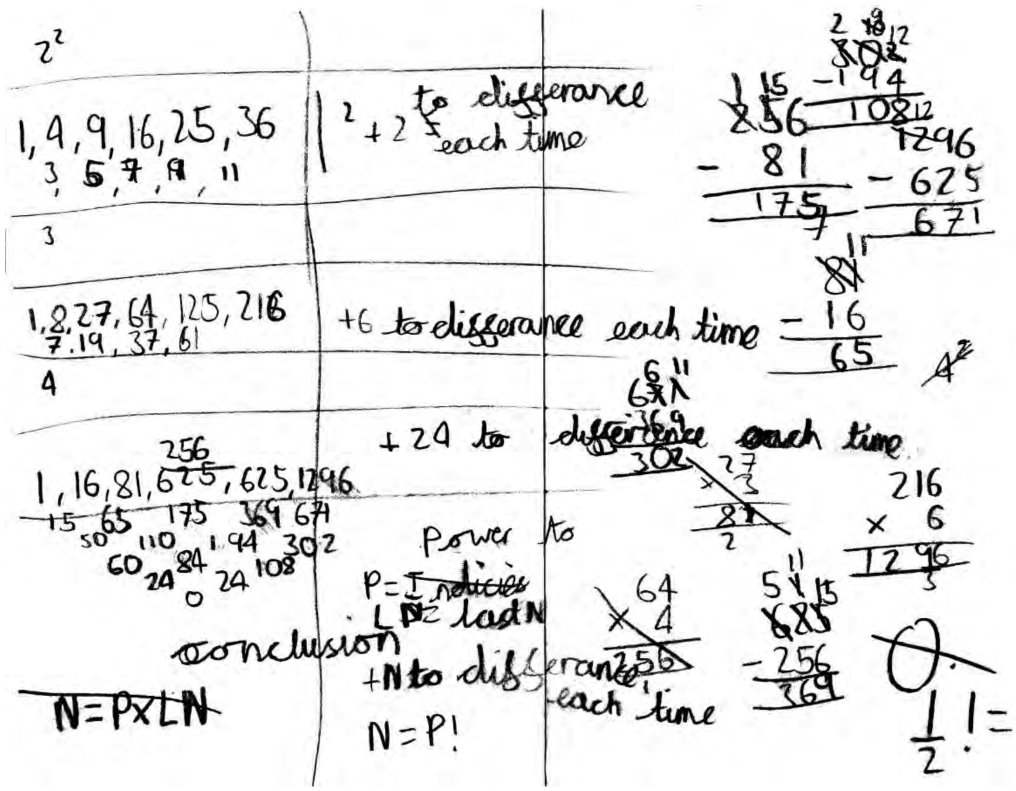

Figure 3: Simon's finite differences

The page shows him working out the differences between sequences of squares, cubes and fourth powers on the left with some of his calculations on the right. Lower down he chooses to denote the power by P and the last number by LN to get the recurrence relation

$$N = P \times LN$$

which he writes as

$$N = P!$$

In the lower right corner he writes

$$\frac{1}{2}! =$$

as he wonders what might be the value of "half factorial". Having earlier generalised whole number powers to fractions and negative numbers, it is only reasonable to attempt to do the same thing for factorials. His grandfather knows that this is related to the theory of analytic continuation in complex analysis and even has an answer ($\sqrt{\pi}/2$) but this is a step too far.

Fortunately, Simon turns to what seems to be a simpler question: zero factorial. Given his recent interest in the binomial theorem with coefficients using factorials, it is natural to define 0! = 1, so this does not seem to be too big a problem. But for Simon, his ideas are genuinely more complex.

He returns to his ideas for nth powers where $n \geq 2$ and attempts to work back with $n = 1$, then $n = 0$.

For $n = 1$, he finds the sequence of powers is

$$1^1, 2^1, 3^1, 4^1, \ldots$$

and the first finite difference is

1, 1, 1, …

so

1! = 1.

However, he then finds that, for $n = 0$, the sequence is

$1^0, 2^0, 3^0, 4^0, \ldots$

which is

1, 1, 1, 1, …

and, calculating the finite differences, he gets the sequence

0, 0, 0, …

and so declares

0! = 0.

He explains it to me and I attempt to explain why most mathematicians define 0! to be 1. Simon is not convinced. His method assumes that it is necessary to perform at least one finite difference operation. For the power n, the nth final difference is n! and the (n+1)th is 0. For $n = 0$, the nth finite difference is actually the original sequence 1, 1, 1, … and the next finite difference is 0. So the mathematical definition allows the 0^{th} difference to be the original sequence and 0! = 1. Meanwhile Simon's method gives a sensible, but different, answer.

Now Simon is in a quandary. He believes 0! is 0 and Grampa, the authority, says it is 1. He also finds that 0 behaves peculiarly compared with other numbers. For instance, $0 \times 3 = 0$ and if you cancel 0 from both sides, you get $3 = {}^0/_0$. The same happens with any other number, say $0 \times 2 = 0$, giving $2 = {}^0/_0$ and 2 is certainly not equal to 3. I explain that numbers other than zero have an inverse, but again, Simon wants to make sense for himself.

Speaking to me about 0! using his iPad, he explains his problem:

> If zero factorial equals one, it breaks some patterns but it keeps others. The definition of n factorial uses n minus one, all the way down to one. Zero can't do that. So, like minus one, I don't think you can actually *have* zero factorial, or you may not be able to *do* zero factorial.

I intended to focus his attention on the need to get a more general definition, but replied first in terms of his reference to minus one, suggesting, "It's like saying you can't have a square root of minus one. You can't have a *real* number that's a square root of minus one, but you can have a *complex* number." I seem to have touched a nerve. He retorted, "You can't make a complex number out of zero because it's neither imaginary or real." "It's both," I replied, and he recognised this by gesturing with his arms to represent the horizontal and vertical axes, saying, "That means it is a complex number because it is on both lines." (Figure 4.)

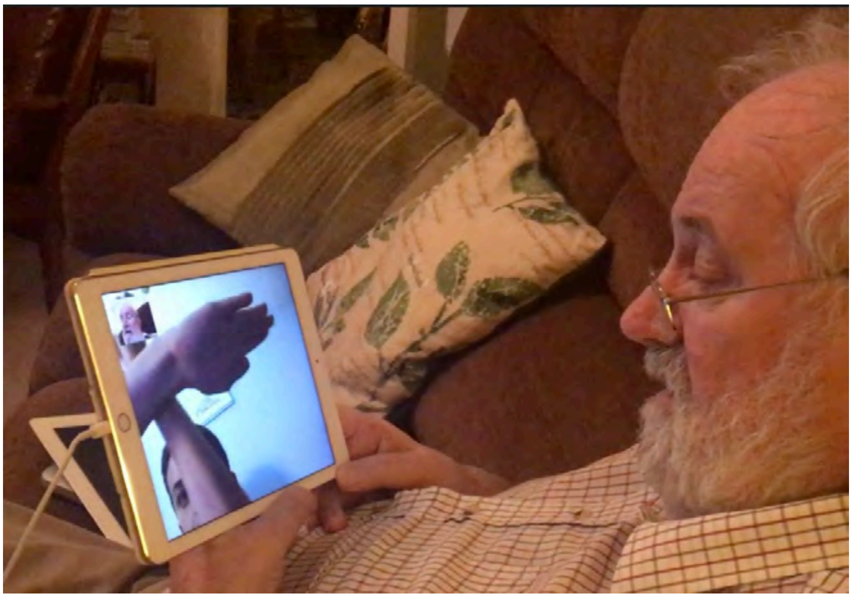

Figure 4: Simon gesturing to show that zero is a complex number

This was to have longer-term repercussions for Simon. The complicated number zero is now complex, and real, and imaginary, and is also affected by Simon's calculus experience calculating with very small numbers which are then set equal to zero. He now has an interpretation of zero as a complex number surrounded by nearby points to give what he says is "zero star, which is just zero plus the teeniest, weeniest, weeniest bit."

What is the square root of zero?

Simon now thinks that zero is surrounded by tiny complex numbers and offers his own explanation of its square root by considering what happens on the two axes:

> You have 'i zero star and minus i zero star' and 'zero star and minus zero star' and you look at the way that minus numbers and normal numbers behave with square roots. The square root of an imaginary number makes a complex number, the square root of a negative number makes an imaginary number, the square root of a positive number equals a positive or negative number. Just to tell you why the square root of i and imaginary numbers have two answers, it's just like normal numbers, it has a positive one and a negative one.

I have a video tape of him as he puzzles with these ideas, speculating with possibilities that may or may not prove to be correct. In the end, he uses his common sense to say that 0 times 0 is 0, so 0 is its own square root.

What is the square root of i?

Simon now has an extended knowledge of complex numbers in Cartesian form, but little experience of trigonometry and polar coordinates, other than an understanding of Pythagoras based on a visual argument. He knows that the distance of the point (x,y) from the origin is $\sqrt{x^2 + y^2}$ and that a point on the unit circle satisfies $x^2+y^2 = 1$.

He is fascinated by the pattern followed by the powers i, i^2, i^3 as they move round the unit circle to $i^4 = 1$. His next question is to ask

"What is the square root of i?"

Given his focus on algebraic manipulation of complex numbers, in an iPad conversation, I suggest that he might try to square a complex number x+iy to see if he could get it to equal 1. This gives

$(x+iy)^2 = x^2+2ixy+i^2y^2 = (x^2-y^2) + i(2xy)$.

Simon can see that $x = y = \sqrt{2}/2$ is a solution. Using Pythagoras, he can see that this point lies on the unit circle making a 45° right-angled triangle with the horizontal axis.

Simon, Nic and I share a conversation about polar coordinates and how multiplying by i and then by i again turns the angle each time through 90°, and in general, multiplication in polar coordinates *adds* the angles, so squaring *doubles* the angle. Simon attempts to calculate the square of the number on the unit circle with angle 30°. He squares 30 to get 900, but on reflection, self-corrects to get 60°. As Kahneman (2011) has observed, an initial reaction often responds with immediate intuitions before thinking more deeply and logically deducing the correct answer.

Simon's journey continues into the future as he builds on his previous experience and finds it necessary to change his method in a new context. He continues to be stimulated by new ideas in mathematics and science, always seeking comprehensive new frameworks while being stimulated by new information. This may relate to the comparative height of the mountains on Mars and on the earth, the theory of space-time formulated by Einstein, or the distribution of prime numbers that he observes to be 'thinning out' and seeks to calculate huge prime numbers for himself. Mathematics for him is a journey of exploration.

Reflections

How can the progress of a single gifted child inform the long-term mathematical development of others? Individuals are so very different in their interests and abilities and what Simon does is clearly very different from others. Yet this study tells a story relating the continuing development of a child and the continuing evolution of the coherence and consequence of mathematics. While mathematics educators research the growth of ideas in the classroom and plan the curriculum to take account of the wide array of data collected in the field, this has led to sequences of learning that work for some but not for all. On the other hand, professional mathematicians, with their expertise in formal mathematics, are able to build up mathematical theories where properties are deduced exclusively from carefully chosen axiomatic definitions.

This study reveals the interplay between the growing knowledge structures of the child and the evolving meaning of mathematics as it becomes more sophisticated. There is a need to integrate both aspects. It is not just a matter of learning techniques or of developing flexible problem-solving strategies, the study reveals the value of problem posing to encourage the development of mathematics in a more meaningful way.

Even for the broader spectrum of children in school, there is a need to attend to the sophistication of the evolution of mathematical ideas. We can now see the value of being aware of the fundamental general properties of the operations of addition and multiplication and how the spoken inflection of mathematical expressions can give subtle insight into the fundamental meanings. This offers an overall framework for the development of the general properties of addition and multiplication that continue to work in the same way in successive number systems, from whole numbers through fractions, signed numbers, finite decimals, infinite decimals, real and complex numbers. Meanwhile the properties of other operations, such as subtraction, division, powers, roots and so on can be seen to relate their meanings in the spoken inflections of mathematical expressions.

While many children may benefit from a focus on practical mathematics of value in their daily lives, it is essential that others study technical mathematics to support modern society, and of vital importance for some to move on to formal mathematics with its deeper frameworks of mathematical ideas.

Problem posing – as a means for asking questions to make sense of mathematics as it becomes more sophisticated and changes in meaning – plays an essential role in understanding the long-term evolution of mathematical thinking.

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