It is an honour to be invited to present to you the framework I have formulated for the long-term development of *How Humans Learn to Think Mathematically*. In particular I acknowledge the contribution arising from an early encounter with Masami Isoda where he spoke about the tangent in circle geometry touching the circle at one point and not passing through it in a manner that conflicts with the more general concept of tangent in the calculus. Our common interests and differences and subsequent collaboration on Lesson Study have been a constant encouragement in the development of the theory I present today.

I also declare that this framework is the distillation of many contributions from the literature and from my colleagues and research students. At a time when new technology is causing civilisation to evolve seemingly faster than ever before, it is time to reflect on how humans learn to think mathematically and build a new theory to encourage different communities to interact and plan together for the future.

This presentation celebrates the publication of the Japanese translation of *How Humans Learn to Think Mathematically*, which I will refer to as *HHLTTM*. It offers an evolutionary framework of long term development that should be read reflectively over several weeks.

Here I present an outline of the main ideas beginning with the overall framework of

- **Three worlds of mathematics**
  - the underlying Biological development of mathematics
  - its accompanying Emotional aspects, particularly anxiety
  - and the very different Social aspects in different communities
  - I will include observations about international comparisons and implications

The three worlds of mathematics is a framework for the cognitive development of mathematical thinking, where each world is based on human perception, action & reason. Actions performed with a specific purpose are called operations including constructions in geometry and symbolic operations in arithmetic and algebra.

- The embodied world explores the perceptual properties of physical and mental objects, formulating verbal definitions used at a more sophisticated level to reason about relationships.
- The symbolic world develops out of mathematical operations performed initially on real world objects, where the operations are symbolised and the symbols themselves are manipulated as mental objects. These symbols can then be operated upon at successively higher levels in arithmetic, algebra, symbolic calculus, vector algebra and so on.
- The formal world is based on verbal/logical definition with properties deduced by mathematical proof.
The Three Worlds of Mathematics

In each world there is a long-term development in sophistication.
- Objects develop sophisticated structure
- Operations are symbolised and the symbols may then be conceived as mental objects with structure.
- Properties are later formulated verbally to define formal concepts whose other properties are deduced by formal proof.

In the long-term, powerful mathematical thinking builds what I term crystalline concepts.
- These have structures that are determined as a consequence of their context.
- I will also suggest that failure to develop these flexible structures can lead to limited procedural learning.

Examples of crystalline concepts include:
- An isosceles triangle in Euclidean geometry
  - It can be a triangle with two equal sides, with two equal angles or with various other properties
  - Different properties can specify the same mathematical object
- A crystalline concept may be a number in arithmetic, for example, 6, which is also 4+2, 2+4, 5+1, 3x2, 2x3
  - Here different symbols give the same mathematical object
- It can be a formal axiomatic system, for example the set of integers, a group, a complete ordered field.
  - Here each system specifies a structure with properties that can be proved.
- The same system may also be specified by a different list of axioms.

The Three Worlds of Mathematics

Examples of crystalline concepts:
An isosceles triangle in Euclidean geometry
- Two equal sides
- Two equal angles
- Symmetry
- Perpendicular bisector
- Angle bisector

A number in arithmetic, e.g. 6, 4+2, 2+4, 5+1, 3x2, 2x3
- Different symbols, same mathematical object

A formal axiomatic system, e.g. the set of integers, a group, a complete ordered field.
- Here each system specifies a structure with properties that can be proved.
- The same system may also be specified by a different list of axioms.

For the vast majority, in school, in practice & in application, mathematical thinking is a combination of embodiment and symbolism based on experience.
- The young child is born with a brain still maturing and spends the first months building connections between perception and action.
- Using perception and construction of operations on objects to find properties in what I term the @Embodied world.
- Action on objects, one, two, three, four leads to the number concept 4 where operations can be symbolised flexibly as process and manipulable object in what I term the @Symbolic world.
- This may develop into a more sophisticated level of Definition and Deductive Proof represented by Plato in Euclidean Geometry
- with corresponding levels of definition and deduction in symbolism
- as part of a higher level of Theoretical Mathematics
- with a continuing interchange between embodiment and symbolism.
At a more sophisticated level, pure mathematicians shift to set-theoretic definition and formal proof introduced by Hilbert.
- which extends the Formal world to a new level of
- Formal definition with properties deduced only from the definition using formal proof
- as distinct from the Natural form of definition and deductive proof based on familiar properties.
This extends Formal proof from Theoretical proof to Axiomatic Formal.

From here, the research mathematician can develop new theories arising from problems by considering possibilities, suggesting conjectures and seeking proof.

In axiomatic formal mathematics it is possible to prove Structure Theorems: that a formal structure has visual and symbolic meanings. This gives a spiral development of more sophisticated levels of embodiment, symbolism and formalism.

The biological development of the mathematical mind begins with
- a single fertilised cell.
- (video)......
- to produce a whole living person by cell division
- with a fundamentally symmetric brain.
Here is a view of the (mainly symmetric) brain seen from above.

- The Left
- and the Right are essentially symmetric.
- Perception occurs as perceptual input passes signals to the back half of the brain.
- Physical action occurs at the rear of the forebrain.
- Reason occurs through links with the front of the brain.

Where does language input and output occur?

- For 99% of right handers and 81% of left handers
  - Hearing is in Broca’s area in the left rear brain.
  - Speech is in Wernicke’s area in the front left brain.
- However, if injury occurs in the child’s brain, speech can develop in the right brain
  - So the brain is essentially symmetric and speech resides on one side (usually the left)
  - Sequential operations, such as counting occur mainly in the left.
- Global estimation occurs mainly in the right.

The brain is essentially symmetric
Where does language input and output occur?

Sequential e.g. counting
Global e.g. estimating

Sequential
Global

13 Taking a cross-section down the middle of the brain

14 We now see a cross-section between the two halves. The interconnections between the two halves of the brain are connected through the corpus callosum.

15 Here we see the long interconnections between different parts of the brain displayed by functional magnetic resonance imaging. Thinking occurs through multiple connections across the whole brain.
16 @ In the centre of the brain is the **limbic system**.
@ The **limbic system** has several functions, including making links to long term memories and emotional response to perceptual input. 
@ perceptual data received in the back brain passes through the **limbic system** to the **prefrontal cortex**.
@ Unconscious emotional 'fight or flight' reaction occurs before logical reason.
@ See, for example, Kahnemann's book on 'Thinking Fast, Thinking Slow'.

**Emotional aspects of long-term learning**

Unconscious emotional ‘fight or flight’ reaction occurs before logical reason

Mathematical thinking is affected subconsciously by ideas that fit together or that cause conflict.

Philosophers speak of thinking in 'metaphors'.

To relate new ideas to previous experience, I introduced the term 'met-before'.

A **supportive met-before** refers to previous experience that is now consistent with new learning.

A **problematic met-before** is in conflict with new learning.

What is supportive in one context may become problematic in another.

17 @ Mathematical thinking is affected subconsciously by ideas that fit together or that cause conflict.
@ Philosophers speak of thinking in ‘metaphors’.
@ To relate new ideas to previous experience, I introduced the term ‘met-before’.
@ A supportive met-before refers to previous experience that is now consistent with new learning.
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@ What is supportive in one context may become problematic in another.

Examples of problematic changes:

counting is supportive for simple arithmetic but problematic as the only strategy for larger numbers.

Procedures occur in time and flexible thinking involves seeing different symbols can represent the same thing.

4+2, 2+4, 5+1, 12/2, 3x2, 2x3 all represent the number 6.

**Procept**: different processes, same concept.

As mathematics becomes more sophisticated, equivalent concepts are later considered as the same:

\[ \frac{3}{6} = \frac{1}{2} \]

are equivalent as fractions,

the same as rational numbers.

Different procedures but they represent a single flexible crystalline concept.
Most current curriculum frameworks specify positive goals to be attained and assessed, e.g. SIMSS, PISA.

Negative aspects, e.g. mathematical anxiety are usually researched separately.

The three-world framework integrates both supportive and problematic aspects of mathematics.

Historically, mathematics advanced from (unsigned) quantities to signed numbers.

Cognitively, our understanding of long-term mathematical thinking may be advanced by shifting from focusing only on positive aspects to including both positive and negative effects of mathematical thinking.

Skemp’s theory (1986, 1989) focuses on both goals (to be achieved) and anti-goals (to be avoided).

Skemp’s theory (1986, 1989) focuses on both goals (to be achieved) and anti-goals (to be avoided). This is the full Picture ...

This has profound implications for making sense of mathematics. In this presentation there is only time to focus on one main aspect:

- the teacher and learners’ goals in learning mathematics: is it long-term sense-making appropriate to the individual or short-term success passing the test.

Is the goal of teacher or learner:
(1) long-term sense-making appropriate to the individual or (2) short-term success to pass the test?

(1) is the more desirable goal, but if (1) fails at any stage, the goal may change to (2). Problematic transitions can cause a change to learn to pass the test. The new goal of passing the test also can have a pleasurable outcome. So many of us gain pleasure through repeating (2) which may then fail to attain (1). My belief is that this is happening in many, perhaps most, communities. in particular where standards are specified to be attained.
Social Aspects in Differing Communities

There are differing communities in (and within)
- Mathematics
- Mathematics Education
- Teaching
- Government
- Philosophy
- Science
- Other Applications
in addition to:
- differing international communities
- differing communities within a given country

Communities of advanced mathematicians have different beliefs and goals
- For example, Pure mathematics – formal axiomatic proof with various foundations
- Engineering – natural proof based on modelling problems visually and symbolically
- Differing communities of math educators, teachers, researchers, curriculum designers, administrators etc. have very different perspectives
- The three world framework offers an overall view to compare, contrast and evolve new understandings to improve long-term mathematical thinking.

International Comparisons & Implications

UK: continuing difficulties in ‘raising standards’
USA: ongoing debate over Core standards
Netherlands: a disconnect between realistic math and later development of skills
Japan: earlier concern that children had good skills but hated maths is now focused on performance in PISA
- many other instances
- with some exceptions (Shanghai, Singapore, Finland?)
- So what does the Three World Framework suggest?

Communities of advanced mathematicians have different beliefs and goals
- The UK has continuing difficulties in ‘raising standards’.
- In the USA: There is an ongoing debate over Core standards.
- In the Netherlands: There is a disconnect between realistic math and later development of skills
- In Japan: There was an earlier concern that children had good skills but hated maths. The concern is now focused on performance in PISA.
- There are many other instances
- with some exceptions (Shanghai, Singapore, Finland?)
- So what does the Three World Framework suggest?
### Implications of the Three World Framework

#### Structurally
- Each world of mathematics increases in sophistication.

#### Operationally
- There is a distinction between learning procedures and having a flexible (proceptual) meaning for symbols.
- There is a difference in school between practical mathematics and theoretical mathematics and in more advanced mathematics in the transition to axiomatic formal.
- ‘Making sense’ changes subtly as mathematics becomes more sophisticated, with powerful supportive links and also problematic transitions that impede sense making.

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### Implications of the Three World Framework

#### Structurally
- Making sense involves successive levels, particularly from practical to theoretical and from theoretical to axiomatic formal.
- Practical properties are simultaneous.
- Theoretical properties can be proved one from another.
- Theoretical properties may be based on imagery.
- Formal properties are based only on formal definition.
- It is important for the teacher and learner to be aware of subconscious problematic aspects that impede learning, to make sense of the new context.

#### Operationally
- Long-term sophistication requires the recognition of crystalline structures and the construction of crystalline concepts.
- For instance, as procedures $2 \times 3$ and $3 \times 2$ are different but as crystalline concepts they are the same object.
- The same set may be subdivided in two different ways, or the same set may be reorganised dynamically in different ways.
- Math Educators know that students sense the difference but in the long-term would it not be better to focus on the crystalline structure?
Implications of the Three World Framework

Conjecture: The best long-term solution in the symbolic world involves building flexible crystalline concepts. This involves making sense of the ideas in context, to link ideas together coherently and to perform lengthy procedures efficiently (as in the Hakase principle).

Most curricula focus on the positive and fail to consider the implications of aspects that become problematic. The consequence is that many seek the pleasure of being able to perform the required skill in context but not be aware of later problematic met-befores that require a new kind of sense-making.

I conjecture that this is the reason why so many skill-based curricula round the world fail to improve.

Implications of the Three World Framework

In advanced mathematics, there is a major difference between
• natural proof based on thought experiment and imagery
• axiomatic formal proof based on only set-theoretic definition.

Axiomatic formal proof includes structure theorems that prove an axiomatic structure has embodied & symbolic structures.

For example the Peano Postulates give a unique crystalline structure: the natural numbers visually as successive points on a number line with the familiar symbolic properties of arithmetic.

A complete ordered field has a unique crystalline structure visually as points on a line and numerically as decimals.

A group has a unique structure as permutations of a set and finite groups can be classified symbolically in terms of generators and relations.

A finite dimensional vector space over F has structure as $F^n$ with both visual and symbolic interpretations.

Implications of the Three World Framework

The full framework from birth to maturity is given in How Humans Learn to Think Mathematically.

The framework at university including structure theorems, embodiment & symbolism is in the 2nd edition of Foundations of Mathematics (Stewart & Tall 2014) planned to be translated into Japanese by Kyoritsu Shuppan. Due 2018.
Implications of the Three World Framework

The three world framework offers a coherent framework for long-term mathematical thinking with successive higher levels of embodied, symbolic and formal sophistication. It takes account of the biological evolution of human thought cognitively and emotionally, and the natural and formal development of mathematics in both the development of the individual and the historical evolution of mathematics.

The Three Worlds of Mathematics

Thank you for listening.

http://homepages.warwick.ac.uk/staff/David.Tall/