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THE COMPLEXITIES OF A LESSON STUDY IN A DUTCH SITUATION: MATHEMATICS TEACHER LEARNING

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ABSTRACT. This study combines the Japanese lesson study approach and mathematics teachers' professional development. The first year of a 4-year project in which 3 Dutch secondary school teachers worked cooperatively on introducing making sense of the calculus is reported. The analysis focusses on instrumental and relational student understanding of mathematical concepts and the transition between the conceptual embodiment and the operational symbolism of the calculus. This paper reports on 2 cycles of lesson studies that took place in the first project year, the first cycle focussing on the notion of the derivative (introduced for polynomials) and the second on trigonometry (as the concepts shift from ratios in a right-angled triangle to functions in the calculus). The lesson study cycles resulted in changes in the teachers' educational goals and instructional strategies in relation to student understanding. However, the teachers' desire to be good teachers, their perceived need to prepare students for standard examinations and their reluctance to use computers impeded their progress in developing a lesson study approach. The introduction of a Japanese lesson study approach into a Dutch context merits further reflection in the later years of the project.

KEY WORDS: lesson study, making sense of the calculus, student understanding, teachers' professional development

INTRODUCTION

In 2008, the Dutch government recognized a stagnated progress in numeracy in scientific studies as a consequence of a lack of students' mastering of mathematical skills. This resulted in renewed attention on secondary school algorithms and correct calculations, requiring new curricular materials and new teaching approaches. For class implementation, two aspects were crucial: (a) the focus on students' understanding of mathematical concepts as well as procedures to solve problems and (b) professional development. We decided to use a lesson study approach incorporating two main factors for effective professional development: collaborative learning and active involvement in curriculum design (Desimone, 2009, 2011; Levine & Marcus, 2010).

The origin of lesson studies is deeply integrated into Japanese culture and can be traced back to the early 1900s (Nakatome, 1984). Lesson studies, based on the terms 'in school' (*konai*) and 'training' (*kenshu*),

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have always remained voluntary (Fernandez & Yoshida, 2004). An important reason for the popularity of a lesson study might have to do with the fact that Japanese teachers find participation in *konaikenshu* (in school training), and in particular, a lesson study, very helpful (Inagaki, Terasako & Matsudaira, 1988).

The aim of the present study is to investigate changes in mathematics teachers' educational goals and instruction strategies through lesson studies. In the next sections, we will first look at characteristics of a lesson study and the way we used it. Then we focus on mathematical knowledge for teaching and student understanding of mathematical concepts. Finally, we present a theoretical basis for teaching mathematics based on concept development and problem solving procedures.

Theoretical Framework

A Lesson Study

A lesson study is a collaborative design research framework focussing on (1) teachers' leadership for learning and improving teaching, (2) interaction between students in the classroom and (3) students' individual needs and learning differences (Matoba, Shibata, Reza & Arani, 2007). It is a cyclic process of collaboratively designing a *research* lesson, implementing this in class and reflecting on student learning. The starting point of this approach is the selection of a topic and corresponding goals for student learning. A lesson study can be extended with various additions, like hosting an open house or using scientific literature, but collaborative lesson planning, live classroom observation and post-lesson discussion constitute the core of a lesson study (Lewis, 2002; Lewis & Hurd, 2011; Stepanek, Appel, Leong, Mangan & Mitchell, 2007). The research lesson is intended to address immediate academic learning goals (e.g. understanding specific concepts and subject matter) and broader goals for the development of intellectual abilities, habits of mind and personal qualities.

A research lesson begins with teachers jointly designing a detailed lesson plan enabling one of the teachers to eventually teach the lesson to students while the others observe (Saito, Hawe, Hadiprawiroc & Empedhe, 2008). After the class, the teachers meet to share their observations and discuss ideas for how to improve the research lesson. These discussions are either followed by the group choosing to work on a new lesson or, as is often the case, by the group revising the lesson plan, re-teaching the lesson in a different classroom and meeting again to discuss learning gains (Fernandez & Yoshida, 2004; Puchner & Taylor, 2006; Sowder, 2007). Through a lesson study, teachers develop a (shared) understanding of education goals, curricula, instructional materials, teaching and student thinking and learning. A lesson study generates new ideas for teaching and student learning. It is the basis for considerations to improve instruction emphasizing critical components of a single research lesson (Sims & Walsh, 2009). It is noteworthy that a lesson study does not focus on the development of a perfect lesson. Instead, it focusses on examining teaching and learning in one's personal classroom, aimed at achieving 'both action and research: action for change and research for understanding' (Pérez, Soto & Serván, 2010, p. 77). The core of a lesson study is the cycle: planning the research lesson collaboratively, observing it live in the classroom and discussing and reflecting on it with a focus on teaching and learning (Lewis, Perry & Murata, 2006).

Mathematical Knowledge for Teaching

Mathematical knowledge for teaching is strongly related to Pedagogical Content Knowledge (PCK) which 'identifies the distinctive bodies of knowledge for teaching. It represents the blending of content and pedagogy into an understanding of how particular topics, problems or issues are organized, represented and adapted to the diverse interests and abilities of learners and presented for instruction' (Shulman, 1987, p. 4). Several scholars have emphasized that not only content knowledge conditions teachers' effectiveness, but also knowledge of how to teach such content (Borko, Eisenhart, Brown, Underhill, Jones & Agard, 1992; Wilson, Shulman & Richert, 1987).

For mathematics, Ball (1990, 1991) conceptualized PCK as the mathematical knowledge for teaching (MKT). This type of knowledge allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures and examine and understand unusual solution methods to problems. Research findings showed that teachers' MKT positively predicted student gains in mathematics achievement in the first and third grade (Hill, Rowan & Ball, 2005; Hill, Ball & Schilling, 2008). The focus on MKT has resulted in studies designed to answer questions about how teachers' mathematical behaviour—in particular their classroom explanations, representations and interactions with students' mathematical thinking—affects student outcomes. The COACTIV project in Germany, for example, that dealt with MKT in secondary schools (Krauss, Baumert & Blum, 2008), showed that expert teachers view mathematics as a process and believe that mathematics should be learned by means of self-determined active discovery including reflecting on one's errors (Baumert & Kunter, 2006). As we wanted to improve mathematics education by professionalising teachers, and changes in MKT are to be expected in the context of a lesson study (e.g. Lewis et al., 2006; Lewis, 2009), our research question is:

What changes in mathematics teachers' educational goals and instruction strategies occur when using a lesson study in a Dutch situation?

Student Understanding of Mathematical Concepts

Our theoretical framework for the long-term development of mathematical thinking is based on student understanding of the mathematical concepts (Skemp, 1976). Skemp distinguished instrumental understanding which pertains to learning how to perform mathematical operations and relational understanding in which relationships are constructed between concepts. Instrumental understanding implies that students master operations to calculate algorithmically without reasoning. Relational understanding means that students know what to do and why, focussed on reasoning. These different types of understanding imply subtle processes occurring in learning in which operations that take place over time become thinkable concepts that exist outside a particular time. We used Tall's (2008a) theoretical framework of long-term development in mathematical thinking to relate the teachers' preferred instruction strategies to student thinking and understanding in the case of introducing a new mathematical concept. Tall (2008a) introduced long-term development of mathematical thinking as being a growth from child to adult, based on fundamental foundations of human perception, operation and reason. He described this long-term development in mathematical thinking as a journey through three mental worlds:

(a) The conceptual embodied world, which is based on perception of, and reflection on properties of objects
(b) The proceptual-symbolic world that grows out of the embodied world, through actions (such as counting) and symbolization, into thinkable concepts such as numbers; the development of symbols that function both as processes to perform operations and concepts to think about (called procepts)

(c) The axiomatic-formal world (based on formal definitions and proofs) which reverses the sequence of construction of meaning from definitions based on known concepts to formal concepts based on set-theoretical definitions.

Calculus in schools is a blend of the worlds of embodiment (e.g. drawing graphs) and symbolism (e.g. manipulating formulae). Two examples illustrate this blend. Firstly, the concept of the derivative focuses in detail on the property of *local straightness* referring to the fact that, if one looks closely at a magnified portion of the curve where the function is differentiable, then the curve looks like a straight line. Secondly, the transition from a geometric to an analytic representation of the (co)sine focusses in detail on the property that a (co)sine is a chord in a circle, represented as a $y(x)$-coordinate in the unit circle, and seeing the *dynamics*. This making sense of the calculus proposes that a more natural approach to the calculus blends together the dynamic embodied visualisation and the corresponding symbolic calculation (Tall, 2010).

The framework of the three worlds offers new insight into making sense of the calculus. It builds on the dynamic embodiment of the changing slope of a locally straight graph in our first lesson study cycle and the chord in a circle as a coordinate in the unit circle in our second lesson study cycle, blended with the operational symbolism using good-enough arithmetic and precise symbolism. The concept of local straightness (conceptual embodiment) develops, through instrumental as well as relational understanding, to the limit definition (operational proceptual-symbolism). The students' learning process develops from instrumental understanding of the derivative as a ratio to a relational understanding of the derivative as a rate of change. The concept of co(sine) as a ratio in a triangle (conceptual embodiment) develops, through instrumental as well as relational understanding, to a function (operational proceptual-symbolism). The students' learning process develops from an instrumental understanding of the co(sine) as a ratio to a relational understanding of a periodic movement.

Theoretical Basis for Teaching Mathematics

Teachers' MKT depends on their personal orientation towards teaching and learning mathematics. This orientation can globally be divided in two main categories, both with their own educational goals: (a) to apply procedures to solve mathematical problems as well as practical problems in different fields of application and (b) to understand mathematical concepts as well as mutual relationships and to think logically in order to construct mathematical proofs (Bransford, Brown & Cocking, 2000; Kilpatrick, Swafford & Findell, 2001). The first goal, with its focus on procedures to solve problems, has been related to problem solving skills and the application of mathematical techniques to achieve a practical purpose (Davis & Vinner, 1986; Jaffe & Quinn, 1993). The second goal, with its emphasis on understanding, has been related to (a) mathematical concepts represented in units of instruction, (b) connections of these units in a structure and (c) an emphasis on mathematical proof with logical arguments (Inglis, Mejia-Ramos & Simpson, 2007; Thurston, 1990).

In order to examine whether the participants had changed during their lesson study experience with regard to the educational goals they had aimed for, general goal statements were compressively formulated by two mathematicians (members of different universities). Six of these goal statements are related to the first goal of mathematics education and can be typified as instrumental understanding. Six other goal statements related to the second goal of mathematics education can be typified as relational understanding (see Table 1). We use Table 1 to categorize teachers' educational goal statements in our method section.

Classifying the phrase 'Definitions as a starting point' as instrumental rather than relational is noteworthy. Before this study, the Dutch teachers taught calculus based on an informal definition of 'limit', and so it was natural for them to consider the students' difficulties with the limit concept to involve only instrumental understanding.

Many studies with the focus on school mathematics instruction have emphasized the importance of sense making related to students' learning activities (e.g. Ambrose, Clement, Philipp & Chauvot, 2004; Hiebert & Grouws, 2007). These studies demonstrated that memorization of facts or

TABLE 1

Classification of educational goal statements

Educational goal statements	
Instrumental [i]	*Relational [r]*
Be able to execute correctly	Structures as basis for thinking
Acquire skills to solve problems	Learn to understand concepts
Use the graphic calculator adequately	Be able to relate concepts
Use computer applications	Learn to argue (generally)
Apply techniques in practice	Learn to argue (sequentially)
Definitions as a starting point	Use realistic practical situations

procedures without understanding often resulted in fragile learning. Relational understanding combined with factual knowledge and procedural facility would make an alliance in powerful ways (e.g. Carpenter & Lehrer, 1999; Hambleton & Pitoniak, 2006). These findings are in line with the assumption that mathematics teachers will integrate their teaching methods with their goals of mathematics education. For the purpose of exploring this assumption, possible teaching methods were grouped into two categories: (a) starting with an abstract mathematical concept (in symbols) and solving abstract problems followed by practical worked examples and solving practical problems or (b) starting with situated examples followed by solving practical problems in that situation and later generalizing to abstract mathematical concepts. Both approaches were theoretically based and supported (Borovik & Gardiner, 2007; Schoenfeld, 2006).

The two university mathematicians related the first category to four statements with regard to Tall's (2008a) embodied world. They continued with the second category in relation to Tall's symbolic world (see Table 2). We have used Table 2 to categorize teachers' preferred instruction strategies in our method section.

The second statement in the first column 'Realistic examples' was related to the Dutch educational approach of Realistic Mathematics Education (RME) (Gravemeijer, 1994; Van den Heuvel-Panhuizen, 2000). Dutch primary school education is based on RME, which implies strong attention on relational understanding. Some of the secondary school mathematics teachers try to integrate this approach in their instruction. The last statement in the first column 'Representations' focussed on different representations, including graphs. The symbolic classification may be seen as related more to Bruner's (1966) notion of

TABLE 2

Classification of preferred instruction strategies

Preferred instruction strategies	
Embodied [e]	*Symbolic [s]*
Different examples (focus on differences)	General concepts
Realistic examples	Worked examples
Practical situations	Definitions
Representations	Thinking models

‘symbolic’, including both operational symbolism and verbal language. The university mathematicians subdivided the symbolic category [s] into operational symbolic [os] and verbal symbolic [vs] which revealed verbal symbolism as an initial step into the world of formalism. Thus ‘Worked examples’ was classified as [os] while the others—‘General concepts’, ‘Definitions’ and ‘Thinking models’—were classified as [vs] representing the shift to verbal representations, placing the informal definition of the limit concept as an informal introduction to formal thinking.

The Study

Some teachers, affiliated to the university in the context of in-service training programs, intended to improve their teaching methods. The researcher (first author) invited three of them to start a lesson study team. The choice was based on good experiences in collaboration in teacher trainee supervision. The teachers aimed to revise their instructional strategies in the context of the stagnated progress in numeracy in scientific studies. The teachers differed with regard to the intended solution of this revision. The lesson study team completed two lesson study cycles in the school year 2009–2010. The first cycle focussed on the introduction of the derivative and the second on the transition from the trigonometric relationships involving ratios in right-angled triangles to functional representations of trigonometric functions appropriate for the calculus. The choice of these topics depended on the teachers’ standard curriculum. They had to attain examination objectives in a limited time. In the first half year, they taught the derivative, and in the second half year, they taught the co(sine) as a function.

Participants

The lesson study team consisted of three male mathematics secondary school teachers (with fictive names) from different regional Dutch schools of about 2,000 students each. The classes had about 30 students (aged 15–16). The students worked in pairs.

Alfred (aged 56) attained a bachelor’s degree in mathematics and a master’s degree in mathematics education. He was a mathematics teacher for 17 years, working with both lower- and upper-level high school students. Before this, he was a primary school teacher.
Bobby (aged 48) has a bachelor’s degree in mathematics and a master’s degree in mathematics education. He was a mathematics teacher starting in 1988, mostly with upper-level high school students.

Carlo (aged 48) holds a bachelor's degree in electrical engineering and a master's degree in mathematics education. Since 2009, he has worked as a mathematics teacher with mostly upper-level high school students.

Context of the Study

The researcher requested that the teachers read a research paper with the intention of them becoming familiar with scientific research and realizing a collective jargon. The teachers presented aspects of the paper relevant for them and then discussed the relevant topics in a lesson study team seminar at the university. The teachers received one other paper in the second lesson study cycle. The papers focussed on student understanding and the development of long-term mathematical thinking. The researcher familiarized the teachers with Skemp's (1976) instrumental and relational understanding and with Tall's (2008a) embodied and symbolic world.

The teachers' collaborative goal was to stimulate sense making of mathematics. They jointly planned the first research lesson. They observed the lesson and made field notes.

Data Gathering Instruments and Procedure

Three instruments were used: one listing possible educational goals, one listing instructional strategies and one where teachers were asked to freely associate on the question of what aspects regarding the topic they would like to specifically bring to the attention of their students.

All instruments were used for both lesson study cycles: the introduction of the derivative in the first lesson study cycle and the transition from the trigonometric relationships involving ratios in right-angled triangles to functional representations of trigonometric functions in the second lesson study cycle. Table 3 shows the instruments in relation to their use in the two lesson study cycles.

The list of educational goals consisted of 12 possible goals (six related to instrumental understanding and six related to relational understanding; see Table 1), arbitrarily ordered. The list of preferred instruction strategies consisted of eight possible starts (four related to embodiment and four related to symbolism; see Table 2), arbitrarily ordered. The teachers were asked to rank these lists and to justify their priorities. The free associations consisted of teachers' personal reports on what they considered important aspects to bring to their students' attention with regard to the topic in both cycles.

TABLE 3

Data gathering instruments

	Data gathering instruments	
	List of priorities regarding …	*Free associations related to …*
At the start of the first cycle	…educational goals …instruction strategies	…the derivative
Between the two cycles	…educational goals …instruction strategies	…the derivative …the (co)sine
At the end of the second cycle	…educational goals …instruction strategies	…the (co)sine

Data Analysis

The results of the rankings were combined in word tables (Tables 4 and 5). The free associations were gathered and ordered in another word table (Table 6).

TABLE 4

Teachers' educational goal statements with highest and lowest priority

Goal statements			
	Alfred	*Bobby*	*Carlo*
First preference			
Pre-test	Structures as a basis for thinking [r]	Acquire skills to solve problems [i]	Learn to understand concepts [r]
Intermediate	To learn to understand concepts [r]	Apply techniques in practice [i]	Structures as a basis for thinking [r]
Post-test	Structures as a basis for thinking [r]	Structures as basis for thinking [r]	Learn to argue (generally) [r]
Last preference			
Pre-test	Acquire skills to solve problems [i]	Learn to argue (generally) [r]	Definitions as a starting point [i]
Intermediate	Definitions as a starting point [i]	Use the GC adequately [i]	Use the GC adequately [i]
Post-test	Use computer applications [i]	Learn to argue (sequentially) [r]	Learn to argue (sequentially) [r]

GC graphic calculator, *i* instrumental understanding, *r* relational understanding

Results

The 'The Development of the Lessons' section reports what happened in the successive research lessons, first Alfred, followed by Bobby and Carlo. The 'Changes in Teachers' Educational Goals' section reports on teachers' ranked goal statements at the start of the first lesson study cycle, between the two cycles and at the end of the second lesson study cycle. The 'Changes in Teachers' Preferred Instruction Strategies' section continues with teachers' ranked preferred instruction strategies at the start of the first lesson study cycle, between the two cycles, and at the end of the second lesson study cycle. The 'Teachers' Free Associations" section reports teachers' free associations at the start and at the end of the first lesson study cycle and at the start and at the end of the second lesson study cycle.

The Development of the Lessons

Alfred usually used graphic calculators in his class. He now introduced an applet to demonstrate local straightness as the most meaningful way to understand the derivative. His goal was relational understanding. After a short introduction, he concentrated on the ratio $\Delta y/\Delta x$ to connect to the textbook approach. His lesson was crammed full with ideas as his explanation was very detailed. As a consequence, his students did not have any questions, and he was unable to initiate a lesson in which the students shared their developing ideas or to begin to uncover his students'

TABLE 5

Teacher's choice of statements with highest or lowest priority at the lesson start

Preferred starting strategy			
	Alfred	*Bobby*	*Carlo*
First preferred instruction strategies			
Pre-test	Different examples [e]	Worked examples [os]	Different examples [e]
Intermediate	Realistic examples [e]	Different examples [e]	Different examples [e]
Post-test	Definitions [vs]	Different examples [e]	Thinking models [vs]
Last preferred instruction strategy			
Pre-test	General concepts [vs]	General concepts [vs]	Definitions [vs]
Intermediate	General concepts [vs]	Definitions [vs]	Definitions [vs]
Post-test	Realistic examples [e]	General concepts [vs]	Definitions [vs]

e embodied, *s* symbolic, *vs* verbal symbolic, *os* operational symbolic

TABLE 6

Teachers' associations related to student understanding

Alfred	*Bobby*	*Carlo*
Lesson study cycle 1: the derivative		
Pre-test		
Change Growth Average rate of change Difference quotient Differential quotient Velocity Slope Tangent line Equation of tangent line	Rate of change Help with more information on functions Difference in growth over an interval and at a point (concept of limit) Different notations (not at the start) Remarks: (1) I teach this abstractly because it develops into a skill. (2) I look for more practical aspects to use at the start.	Average rate of change Instantaneous rate of change General formula for the derivative (to take the limit?) Arithmetic rules for the derivative Equation: tangent line To calculate extreme values *Change*, a scheme with components: table, function, rule, symbol, physics symbolised in mathematics
Post-test		
Slope of a graph Difference quotient Limit Differential quotient Change Slope Tangent line (equation) Interval Maximum/minimum	Tangent line: what is it? Slope Difference quotient→ differential quotient (what is the difference?) Rate of change–average change The concept of the limit Can we define the derivative as a function? What to do with the derivative? To calculate slope and extreme values	Rate of change Change at one moment (speed camera) Tangent line on the graph→ understand this concept (visual) Be aware of the accompanying formula: see the problem (one point), work around (understanding $\Delta x \rightarrow 0$) Understand relation between drawing the graph and accompanying formulas Limit at the end
Lesson study cycle 2: the developmental shift of the conception of sine and cosine		
Pre-test		
Able to draw the unit circle, able to explain its role Ratio of sides: indicate the sides of a right-angled triangle Convert between degrees and radians The concepts: period, amplitude, phase-shift and zero crossing, as well as	Coordinates of a point on the unit circle Standard graphs of sin x and cos x followed by transformations of these to $y=a+b \sin c(x-d)$, using concepts such as amplitude, phase Ratios of sides in right-angled triangles (soh/cah/toa)	Memorize sin/cos/tan in right-angled triangles, formula soh/cah/toa Unit circle, height h, width b, angle x; Deduce $h=\sin x$, $b=\cos x$ Walk around a circle. The distance is x. Circumference$=2\pi r$ (2π in unit circle)

frequency, wavelength Reproduce the standard values of sine, cosine and tangent Calculate sides and angles in a triangle, eventually using sine and cosine rules Able to solve equations using sine, cosine and tangent Determine a function rule in relation to a sine/cosine graph Draw a graph given by a function rule	Applications: (1) Resonances in physics; therefore, knowledge is needed to calculate rules related to sine and cos, $\sin 2x=\ldots$, $\cos 2x=\ldots$, $\sin(x+y)=\ldots$, etc. (2) Sometimes polar coordinates or (3) Parametric curves, Lissajous figures	Relation angle α and distance x. Function: $h=\sin x$ graph, with x on the x-axis and h on the y-axis. The function $y=d+a\sin b(x-c)$, $p=\pi/b$ (period) (1) Concept of parameters (2) From parameter to graph (and back) (eventually derivative of $\sin x/\cos x$) Tan $x\rightarrow$unit circle Tan $x=h/b$, function relation in unit circle, $\tan x=\sin x/\cos x$, $\sin^2 x+\cos^2 x=1$ (Pythagoras) Sum/difference, double angle formulas
Post-test		
Standard triangles Ratios of side Unit circle Degrees, Radians Graph of $f(x)=\sin x$, $f(x)=\cos x$, zero points, domain, range Derivative$\rightarrow$calculate max–min Formulas: $\sin 2x=\ldots$, $\sin^2 x+\cos^2 x=1$ Parametric functions, sine and cosine Calculate a combined period of two sine functions	Ratios of sides within right-angled triangles (soh/cah/toa) Drawing a unit circle and developing the graph of sine Radians Concepts: amplitude, period, translation ($y=a+b\sin c(x-d)$) In a later phase trigonometric formulas like $\sin 2a=\ldots$, $\sin(a+b)=\ldots$ Using 1–1–$\sqrt{2}$ and 1–2–$\sqrt{3}$ triangles or 'the unit circle' or better *both*	Geometry: soh/cah/toa in a triangle Need for the unit circle: look! $\sin x$=height, $\cos x$=width, x is distance walking around unit circle From the graph of $\sin x$ and/or $\cos x$ to sin and cos function Relationship between radians and degrees Afterwards the derivative of sin and cos functions and area under them Then relations between sin, cos, tan; formulas for these relationships

learning processes. Alfred's first experiences with the introduction of local straightness impeded him in focussing on student learning. His enthusiasm involved his own relational understanding of the derivative.

Bobby and Carlo, in their preparation of the research lesson, focussed on Alfred's lack of student interaction. Bobby did not have any computer facilities. He intended to activate student interaction by introducing practical activities followed by a plenary discussion, hoping to build on students' interactions. He began the activity by giving each student a paper with the graph of $y=x^2$ and asked them to draw a tangent at a point that was not placed on a crossing of the grid lines. As a consequence, the tangent lines students drew were slightly different and showed small differences in the numerical slope of the tangent. Bobby focussed on the concept of the tangent line as a basic concept, in line with the textbook approach, before introducing the derivative. Bobby's plenary discussion focussed on the concept of the tangent, but also ended in the ratio dy/dx, following the strict textbook guidelines. Once again, the students did not ask any questions, and the plenary discussion did not lead to students' sharing of ideas in a manner appropriate for lesson study. The observers (the lesson study team and other colleagues) noted that the students were not amazed at all when their practical approach to the tangent produced different tangent lines with different slopes as compared with the graphic calculator that produced a single formula.

Carlo joined Alfred and Bobby when they had already started their own preparation. Although Carlo often used a digital white board, in the lesson study teaching, he began with written worksheet questions to reveal the students' thinking processes. The questions were based on Carlo's experiences. He focussed on the formula of the tangent line and

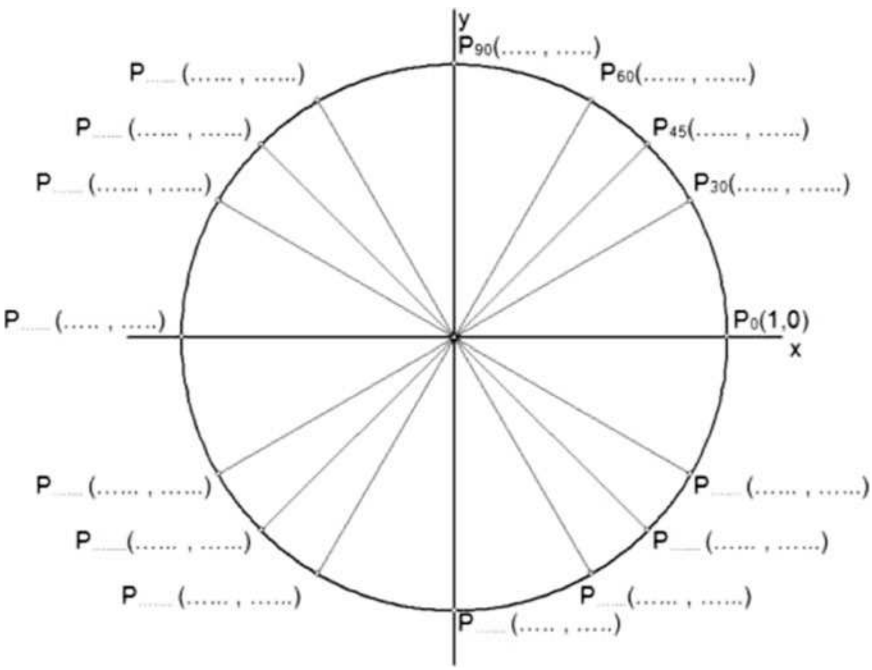

Figure 1. The unit circle

what happened to the difference quotient when $\Delta \rightarrow 0$. He aimed at relational understanding, basing his approach on the limit concept without naming it explicitly. Carlo realized that his written worksheet questions were unclear, as he had intended to encourage relational understanding, yet his questions essentially focussed on standard techniques that could be performed instrumentally.

All three teachers were amazed by their students' unexpected responses and were determined to unravel more of their students' thinking processes. They decided to continue the worksheet approach to reveal the students' thinking processes in a second lesson study cycle focussing on the mathematical concept of trigonometric functions.

The goal of the second lesson study cycle was making sense of the transition from the trigonometric relationships involving ratios in right-angled triangles to functional representations of trigonometric functions appropriate for the calculus. The teachers decided not to use graphic calculators, as they wanted to focus on collaboration and discussion in pairs. Alfred started to ask students to write down all they knew about the sine concept. He continued with the question of how to calculate the sine of 120° (obtuse angle), knowing the value of the sine of 30° (acute angle). He introduced the unit circle by asking its purpose (in relation with the sine as a ratio) and presented a unit circle (see Fig. 1).

He ended with asking the values of sin(40°) and sin(220°). Alfred and Bobby's lessons were on the same day. Bobby's lesson, prepared in a similar way to Carlo's, was a short lesson of half an hour. Both teachers were satisfied. In both teachers' lessons, there was a minimum of plenary student interaction and discussion. They forgot to introduce negative coordinates which resulted in students misunderstanding the value of sin (120°). Carlo intended to prevent this by suggesting negative coordinates. Carlo added a last question: given sin (20°)=0,342 'write down as many points (on the unit circle) as possible for which you know something about their coordinates', anticipating the connection between sine and cosine. Carlo's students wrote down rules about $\sin(2\alpha)$, $\sin^2(\alpha)+\cos^2(\alpha)=1$ and so on in order to try to answer this question. Only one student drew a geometric triangle. The only student who drew a triangle remarked 'that helps!' Carlo used an applet to show changing coordinates related to the angle in the unit circle which helped students to understand the sine (Blackett & Tall, 1991; Johnson & Walker, 2011; Ross, Bruce & Sibbald, 2011). The introduction of the unit circle was too complicated for the students. Carlo lost the students' attention. Subsequently, students mixed up *x*- and *y*-coordinates (Fig. 1). The students solved the question related to 'sin (20°)=0,342' by measuring line segments in the unit circle, followed by an angle in a triangle. The unit circle worked as a thinking model. Some students solved this problem correctly, beginning in the first quadrant. The teachers strongly focussed on the use of a worksheet as an attempt to discover how students think, a consequence of the teachers' experiences during the first lesson study cycle.

Changes in Teachers' Educational Goals

The goal statements were classified in terms of instrumental [i] and relational understanding [r]; see Table 1. The three teachers' highest and lowest ranked goal statements are listed in Table 4. The upper part of the table lists the first goal of the pre-test priorities, the intermediate test between cycles and the post-test after the second cycle. The lower table section shows the teachers' least important goals from the three successive priority lists.

Alfred remained consistent throughout, with all preferred goals being relational and all least-preferred goals being instrumental. He emphasized logical thinking. He intended to attain manoeuvrable knowledge as much as possible. Alfred switched in his lowest priorities, ending up in not using computer applications. He considered the computer as not being relevant.

Bobby initially preferred instrumental understanding, but changed at the end. He realized the importance of mathematics for physics and economics. Bobby's lowest preferences twice included 'Learning to argue'. He indicated that he did not understand these goal statements.

Carlo also began by preferring 'Relational understanding of concepts'. His final position was again mainly relational, preferring 'General arguments'. He always aimed at attaining the highest conceptual thinking level. Carlo's lowest priorities switched over time. At the beginning, he did not prefer 'Definitions'. At the end, he intertwined his priority with his experiences with students' measurements without using a graphic calculator. That was a reason for him to switch to relational understanding at the end.

Changes in Teachers' Preferred Instruction Strategies

The eight statements for possible lesson start strategies (see Table 2) were classified as embodied [e] or symbolic [s], subdivided into verbal symbolic [vs] and operational symbolic [os]. The three teachers' highest and lowest ranked statements of preferred lesson instruction strategies are listed in Table 5. The upper part of the table lists the first preference of the pre-test list, the intermediate test between cycles and the post-test after the second cycle. The lower table section shows the teachers' least important preferences from the three successive priority lists.

Alfred's preferred lesson start strategy changed to 'Definitions'. He became aware of the role of definitions as being powerful and necessary. This is related to the development of his own relational understanding. The use of realistic examples is common in Dutch education as being representative for RME. Alfred, with his long experiences with elementary mathematics at primary school, became aware of negative consequences of the RME approach (his own primitive translation of RME) during the lesson study cycles. Therefore, he indicated that realistic examples were the least-preferred strategy at the end.

Bobby remained using examples in his preferred strategy. He emphasized during lesson study the usability and the applicability of mathematics. Bobby was convinced that the use of a general concept was not advisable as a starting point for student learning. He supposed that such a start would impede student interaction.

Carlo switched to verbal symbolic 'Thinking models' as the lesson study progressed. He realized more and more the importance of a thinking model in aiming to attain the highest level of conceptual thinking.

Teachers' Free Associations

Teachers' free associations related to student understanding are listed in Table 6. In the columns, the notes by Alfred, Bobby and Carlo are displayed arranged according to the pre-test and post-test data for both lesson study cycles.

None of the teachers mentioned local straightness in their associations. After the initial research lesson, they focussed on the objectives required in the Dutch curriculum. Alfred limited his associations to pure mathematics concepts in both lesson study cycles. During the first lesson study, he became aware of the limit concept as being essential to understand the derivative. Alfred structured embodiment and symbolism in the second lesson study cycle. This is in line with his consistent preference for relational understanding, as a consequence of the influence of RME, over instrumental understanding (Table 4) and his awareness of the role of definitions (Table 5).

Bobby began with the rate of change introducing the limit concept. He realized the focus on student understanding and his role to stimulate students' reasoning by posing questions. In the second lesson study cycle, he emphasized, by underlining, his comment 'Drawing a unit circle and developing the graph of sine'. He moved on via various concepts to more general formulae. This is in line with his subtle changes in preference in Table 4, from a focus on instrumental understanding to a relational blending with symbolism and his shift in Table 5 to more focus on embodiment.

Carlo began with the traditional approach of the derivative and referred to the ‘General formula for the derivative’, questioning whether to take on the limit. He emphasized the idea of ‘Understanding [the] relation between the graph and accompanying formulas’ with ‘The limit at the end’. This related to his blending of relational understanding of concepts and instrumental acquisition of skills in Table 4 and his subtle shift from combining embodiment and symbolism to a more integrated desire to blend thinking models, representations and worked examples as seen in Table 5. In the second cycle, Carlo emphasized memorizing trigonometric formulae in right-angled triangles and the dynamic embodied idea to ‘Walk round the circle’ to relate the distance travelled to the angle turned. This led to the idea of a trigonometric function (underlined by Carlo) and to more generalised ideas including parameters, and relational ideas involving the visual and symbolic expressions of the theorem of Pythagoras and related trigonometric properties. His overall development relates to his comments in Table 4 where his initial preferences for understanding are extended to include learning to argue generally and, in Table 5, where his preference for the blending of embodied and symbolic examples is expanded to include thinking models.

Conclusions and Discussion

Answering our research question, we can see that teachers do change with respect to their educational goals and their instructional strategies, but it is a slow and idiosyncratic process. Changing educational practices involves unlearning old habits and replacing them by a new repertoire, an uncertain process as teachers do not know how students are going to respond to new instruction. Our research outcomes confirm that the teachers tended to use teaching methods with which they and their colleagues were familiar. In their old practices, they focussed on teaching methods instead of on student understanding. However, Alfred became aware of the importance of the use of definitions building upon his RME approach as a consequence of his elementary school experiences. Bobby became aware of the important role of embodiment. He learned to blend his intuitive preference for problem solving skills and applications of mathematical techniques solving practical problems with the use of thinking models. Carlo became aware of the importance of blending embodiment and symbolism. He learned to realize his intuitive preference for a traditional approach based on his engineering background.

Another observation from the data is that successive teachers used their research lesson in a number of ways not in line with the Japanese form of a lesson study. The teachers had problems introducing mathematics topics based on new mathematical insights. Instead, they followed the Dutch curriculum objectives in which the textbooks, in combination with the pressure of common study guidelines, impeded their desire to design the research lessons as part of a longer-term lesson study module (Verhoef & Tall, 2011). However, when the designed lessons did not result in active student participation in the lessons, the teachers realized that they needed to focus more on the students' thinking processes and not on the textbook. They decided to start to unravel students' thinking processes through the introduction of a different teaching approach. In this approach, teacher–student interactions were limited to short plenary discussions at the start of the lessons. Then the students would be asked to work on the issue introduced, preferably in pairs. The teachers focussed more and more on the design of student worksheets that would elicit student thinking and foster cooperation. By analysing the written student worksheets after class, the teachers started to unravel the students' thinking processes.

The question arises as to where the source of the difficulties with the implementation of the goals of the lesson study can be found. In this study, the participating researchers encouraged the teachers to read the literature and to form shared opinions to design their lessons. Although the teachers empathized with a 'locally straight' approach to the calculus, they all gave a low priority to the use of computer activities. This conscious choice was based on the intention to reveal students' thinking processes. The teachers expected less student interaction when using computers. They therefore missed the opportunity to take enough time to encourage the students to focus on the essential idea that a differentiable function gets less curved as one zooms in until it is visually straight. With this essential idea, the students might have been encouraged to 'look along the graph' to *see* its changing slope and to conceptualize the graph of the changing slope as the derivative of the function. The teachers' perceived a need to follow the objectives of the existing curriculum which was at odds with the alternative approach, and this alternative was therefore not enacted.

A Japanese lesson study requires more than the design of a single lesson: it entails the coherent approach of a series of lessons. This study reveals the significance of the complex reality of Dutch school practice. Mathematics education is driven by examination objectives, study guides based on textbooks, and the desire to realize high exam results. The response of the teachers in this study suggests that the dual attempt to introduce both a new mathematical approach (local straightness) and a new teaching approach puts great demands on teachers, already constrained to follow a tightly defined schedule preparing students for specific examinations.

There exists a typical difference with Asian cultures with regard to student learning. A salient difference from European countries is that these students work cooperatively to make sense of mathematics in a way that not only improves their understanding, but maintains longer-term success (Tall, 2008b). In lots of Western countries (the Netherlands is no exception), teachers focus primarily on preparing for the exams and often work in isolation in their own classrooms. The complexity of their daily work rarely allows them to converse with colleagues about what they discover about teaching and learning (Cerbin & Kopp, 2006). The experience of this first year of a 4-year project to introduce a lesson study approach into Dutch schools has revealed how underlying cultural forces can privilege a familiar approach to the curriculum and impede the development of a new approach that builds from the insight of conceptual embodiment to the power of operational symbolism. The teachers in the study sensed the need to help students perform well in assessments while having little time to reflect and think through new ways of teaching and learning.

The second year of the study is now in progress. The original team has been expanded with new members including those with experience of working with GeoGebra which has led to the development of a GeoGebra environment that enables the students to explore local straightness and the process of tracing along the curve to see its changing slope giving rise to the graph of the slope function that will stabilize on the derivative (Hohenwarter, Hohenwarter, Kreis & Lavicza, 2008). The existence of this tool to support dynamic visualisation still needs to be used to encourage the sense-making link between dynamic embodied meaning and operational symbolism. We hope to report the results in a follow up paper.

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