THE COMPLEXITIES OF LESSON STUDY IN A EUROPEAN SITUATION

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This study combines elements of Japanese Lesson Study in mathematics education in Dutch schools. It considers the first year of a four-year project completed by three upper level high school teachers working cooperatively to introduce the derivative in the calculus. The approach is based on the distinction between instrumental and relational understanding and the transition between the conceptual embodiment of the derivative as the changing slope of a locally straight curve and the operational symbolism of the calculus. This paper reports two cycles of lesson study which took place in the first year, the first focusing on the notion of the derivative (introduced for polynomials) and the second on trigonometry (as the concepts shift from ratios in a right angled triangle to functions in the calculus).

The experience resulted in changes in the teachers' awareness of students' misconceptions, errors in calculations and their use of algorithms. However, their desire to be good teachers, their perceived need to prepare students for standard examinations and their reluctance to use computers impeded their progress in developing a lesson study approach. The introduction of Japanese Lesson Study to a European context merits further reflection in the later years of the project.

Keywords: Lesson study; long-term development of mathematical thinking; three worlds of mathematics; conceptual embodiment; operational symbolism; axiomatic formalism; sensible approach to the calculus; local straightness.

1. INTRODUCTION

The aim of the present study is to investigate teachers' professional development to achieve a sensible approach to the calculus through elements of lesson study. It is widely acknowledged that teachers' professional development leads to improvements in instructional practices and student learning (Penuel, Fishman, Yamaguchi, & Gallagher, 2007). Recent research suggests that two main factors contributing to professional teacher development are collaborative learning, and active involvement in curriculum design (Desimone, 2009). These are both foundational ideas in Lesson Study, which has its origins in Japanese elementary education for over one hundred and thirty years (Lewis, 2002).

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Japan and Asian countries score more highly in international studies than European countries, occupying numbers one, two, three and four in the ranking list of the Trends in International Mathematics and Science Study (TIMMS, 2007). This reports a remarkable percentage of students in Asian countries that reached the Advanced International Benchmark for mathematics, representing fluency on items involving the most complex topics and reasoning skills (Olson, Martin, & Mullis, 2008).

A salient difference from European countries is that these students work cooperatively to make sense of mathematics in a way that not only improves their understanding, but maintains longer-term success (Tall, 2008b). European governments establish guidelines for teaching and learning approaches that are controlled in more or less directive ways. As a consequence, teachers focus on preparing for the exams and often work in isolation in their own classrooms. The complexity of their daily work rarely allows them to converse with colleagues about what they discover about teaching and learning (Cerbin & Kopp, 2006). In this reported study, originally five teachers participated; two were unable to complete the study because they were not given enough time to do so in their very busy schedule.

If it is possible for teachers to share their ideas about teaching and learning, it is likely to be based on the form of knowledge that they develop from their experiences in the classroom. This type of knowledge seems immediately useful, but tends to be tied to concrete and specific contexts and is not always in a form that can be accessed and used by others (Hiebert, Gallimore, & Stigler, 2002). This suggests that practitioner knowledge should preferably be made public, shareable, and verifiable so that it may become professional knowledge, leading to the formation of a professional knowledge base that can be widely shared with other practitioners.

1.1 Lesson study

Lesson study is a collaborative design research framework which emphasizes teachers' leadership for learning and improving teaching, interaction between students in the classroom, and their individual needs and learning differences (Matoba, Shibata, Reza, & Arani, 2007). The starting point of this approach is the selection of a topic and goals for student learning in a module of work over a period of time, followed by an overall plan of development, including the focus on a specific research lesson as an integrated part of an in-depth study. The lesson is intended to address immediate academic learning goals (e.g., understanding specific concepts and subject matter) and broader goals for development of intellectual abilities, habits of mind and personal qualities. Work on these study lessons begins with teachers jointly drafting a detailed lesson plan within an overall module of work so that one of them can eventually teach the lesson to his or her students while others observe. The teachers then meet to discuss their observations and ideas for how to improve the lesson related to the whole module. These discussions are either followed by the group choosing to work on a new lesson or, as is often the case, by the group revising the lesson plan, re-teaching the lesson in a different classroom, and meeting again to discuss the development (Fernandez, Cannon, & Chokshi, 2003; Fernandez & Yoshida, 2004; Puchner & Taylor, 2006; Sowder, 2007). Figure 1 presents the essential features of Japanese lesson study given by Isoda (2010, pp.18, 19).

1. **Process/lesson study cycle**: Plan (Preparations), Do (Observations) and See (Discussion and Reflection) activities involving with other teachers;
2. **Various Dimensions of Open Classroom**: Personal (by master teacher), Whole School, Regional and National lesson study but Systematic;
3. **Theme of lesson study**: Study Topics and Objective are different. Study Topics such as Developing Mathematical Thinking, Learning for/by themselves in relation to development, reform or improvement. Objective is specified at each class related with curriculum. In the case of Japan, the objective is often described by the sentence 'Through A, students learn/understand/enable to do B' because Japanese curriculum asked teachers to teach learning how-to and achievement as for outcome.
4. **Lesson Plan**: A format is not fixed, usually developed/improved depending on a study topic of lesson study. Some countries recommend a set of national lesson plans as a part of curriculum, but lesson study is implemented for new challenges and pushes the new format of lesson plan and ways of teaching approach.
5. **Teachers' mind**: Lesson study is conducted by teachers for developing students in a classroom and making each student developing him/herself, not for researchers who just observe a classroom through their telescopes. Even though researchers participate as for their research, if they do not understand teacher's objectives for developing children, and if they do not work together with them, it is just the activities as a social scientist as an observer. In this sense, lesson study recommends that researchers are teachers who propose improvement of class, as well as teachers are researchers who analyze children's understanding;
6. **Results:** Lesson study usually considers achievement in relation to study topic and objective. At the same time, aims of lesson study change depending on participants and are not always the same as seen in the following: Model teaching approach, New ideas for traditional approach, Understanding objectives, What students learned before the class, What students learned and could not learn in the class, Teachers' values, Students' values, Professional development, Ideas for the curriculum reform, Theory of mathematics education, and so on;
7. **Sequential experience for sharing the heritage:** Lesson study cycle continues beyond the generation. It's usually opened for newcomers and changes experienced bearers. On this context, similar experiences usually are recognized as new experiences with challenges. That is the reason why lesson study develops the learning community.

Figure 1: The Essential Features of Japanese Lesson Study

The aim of the study was for the teachers, the university members and the researcher (the first author) to work together to design, observe and analyze a research lesson in collaboration. This was intended to culminate in at least two tangible products: (a) a detailed, usable lesson plan, and (b) an in-depth study of the lesson that investigates teaching and learning interactions, explaining how students responded to instruction, and how instruction might be further modified based on the evidence collected (Cerbin & Kopp, 2006). The outcomes of the study would then be shared with the teachers and managers in the participating schools.

The authors were confident that the introduction to the calculus using an approach that began with the dynamic visible properties of the changing slope of a graph and its computation as a numeric and symbolic derivative would prove meaningful to the students.

In the first iteration of the study, the researcher encouraged the teachers to take control of their own activities. Therefore it was not expected at this stage that the full development of the calculus or all the characteristics of lesson study would be implemented. In practice, instead of producing a widely useable lesson, each teacher was initially concerned with the specifics of their own approach in the classroom, each using different facilities and each feeling the need to follow the sequence prescribed and tested by the national examinations.

1.2 Long-term mathematical thinking in three worlds of mathematics

Our theoretical framework for the long-term development of mathematical thinking is based on human perception and action, developing in sophistication using language and symbolism to develop more sophisticated ways of thinking as formulated by Tall (2008) in terms of a blend of three mental worlds of mathematics where each develops in sophistication, as follows:

Conceptual embodiment builds on human perceptions and actions that are verbalized in increasingly sophisticated ways to become mental entities in our imagination;

Operational symbolism grows out of physical actions into mathematical procedures. While some learners may remain at a procedural level, others may conceive the symbols flexibly as operations to perform and also as entities to be operated on through calculation and symbolic manipulation;

Axiomatic formalism builds formal knowledge in axiomatic systems specified by set-theoretic definition, whose properties are deduced by mathematical proof.

In school mathematics, the main worlds encountered are those of conceptual embodiment and operational symbolism. As the learner matures, making sense of mathematics becomes more sophisticated over the long-term, leading to the development of more formal definition and deduction. In the conceptual embodied world, meaningful links are made between perceptions and actions and, over time, language is developed to explain and to deduce these relationships in more sophisticated ways (as, for example, in initial experiences of space and shape developing over the long-term into Euclidean geometry).

The operational symbolic world of arithmetic and algebra, is initially concerned with making sense of physical perception of operations such as noticing that if we change the arrangement of a set, we always find that counting the set gives the same number. Then the learner begins to make sense of the operations on numbers, such as the fact that the sum of two numbers is independent of the order. Then, as the learner becomes aware of the properties of arithmetic, these may become used as 'rules of arithmetic' that can give a structure to the manipulation of algebraic expressions.

Making sense in the axiomatic formal world focuses on developing the methods of proof from formal definitions, building theorems that formulate the structure of the defined concepts. Over the long term, a learner may develop perceptual, operational then formal ways of making sense of mathematics, at each stage blending together experiences in increasing levels of sophistication. Indeed, at the highest level, theorems may often be proved that specify the structure of axiomatic systems in embodied and symbolic terms that fully integrate the interdependence of embodiment, symbolism and formalism.

1.3 A sensible approach to the calculus

At a formal level, pure mathematicians see the limit concept as the logical basis for the calculus. However, research shows that students have subtle cognitive difficulties with the limit concept in which they imagine a limit as a *process* of *getting close* involving arbitrarily small quantities (Cornu, 1991). This leads mathematicians to teach students that the expression dy/dx is a limit concept that should not be considered as a quotient of tiny quantities dy by dx.

However, a sensible approach can begin first with the perceptual idea of continuity as the drawing of a curve by a dynamic movement of the hand (Tall, 2010). Some curves have the property that, when tiny portions are viewed under increasingly higher magnification, they look more and more straight (a property termed 'locally straight'). One can then *see* the changing slope of a (locally straight) curve by running one's hand along the curve and visualizing the changing value of the slope as a new graph. This graph is the changing slope of the original curve and if the slope is calculated symbolically as

$(f(x+h)-f(x))/h$, for reasonably small *h*, then the graph stabilizes to give what mathematicians call the 'derived function' or the 'derivative'. Such an approach leads to an embodied idea of the derivative as the changing slope of the tangent line, which has components *dx* and *dy* and has slope *dy/dx* which is the quotient of the components of the tangent vector. Once this idea is established, it can be used to visually determine the derivative of standard functions such as x^2, x^3, sin*x*, cos*x*, and even to suggest that there is a value *e* between 2 and 3 such that the derivative of e^x is e^x (Tall, 2010). Such experiences provide a context where the learner can begin to link the dynamic visual pictures to the corresponding symbolic calculation of the derivative.

To develop a sensible lesson-study approach to the derivative using local straightness requires the teacher to grasp how learners make sense of the ideas at their current level of expertise, linking the visual ideas to operational symbolism. This requires the teacher to shift beyond the traditional approach to the calculus to a new sensible approach building on students' interpretation of dynamic visualization and its transition to operational symbolism to develop the methods for calculating symbolic derivatives for combinations of known functions.

After an initial stage based on embodiment of locally straight graphs and symbolic calculations of the slope function, we anticipated the translation from embodiment to symbolism to be verbalized in terms of the notion of limit, to describe how the numerical and symbolic approximations stabilize to give the resulting slope function of a locally straight graph. It is at this point that the limit concept can be used to develop the symbolic rules for computing the slope of the sum, difference, product, quotient and composite of functions.

This relates to the sequence of modes of operation formulated by Bruner (1966) in terms of enactive, iconic and symbolic to be translated into the three-world model. The embodied world involves enactive gestures and iconic pictures while the symbolism of Bruner includes both the operational symbolism of arithmetic and algebra and the later formal language of set-theoretic definition and deduction.

This distinction between the world of operational symbolism and axiomatic formalism highlights the schism between school calculus and mathematical analysis. When students are presented with an informal introduction to a formal verbal definition, then a few may make sense of it and use it meaningfully. However, many find the potential infinity of the limit concept to be problematic and this causes a switch from attempting to make relational sense of the calculus to the instrumental learning of the rules of operation (Juter, 2007).

In our initial analysis, we believed that a ‘locally straight approach’ would focus mainly on the conceptual embodiment of the changing slope of the graph of a function and relating this to the operational symbolism of calculating the derivative. We did not at first consider the role of formalism as being central in a first course. However, this proved to be short-sighted because the teachers and those observing the teaching were all conscious of the need to introduce the limit concept to fit in with the examination syllabus. This introduced subtle formal aspects that profoundly affected the interpretation of the data.

2. THE STUDY

The lesson study team completed two lesson study cycles in the school year 2009 – 2010, within a half-year module. The first focused on the introduction of the derivative, and the second on the transition from the trigonometric relationships involving ratios in right-angled triangles to functional representations of trigonometric functions appropriate for the calculus.

Our research question is:

> ‘What changes in teachers’ educational goals, preferred strategies and teaching sequences occur when using lesson study in a European situation?’

2.1 Participants

The lesson study team consisted of three upper level high school teachers from different regional Dutch schools and five staff members of the University of Twente (two educational teacher trainers, a mathematician, a researcher and a PhD candidate). The schools, about 2000 students each, were broadly oriented. The upper level high school student groups of about 30 students (age 15-16) each, worked into pairs.

The first author, as researcher and member of the lesson study team, had formed good relationships with the teachers, based on good experiences in collaboration in teacher trainee supervision. The male school teachers, A, B and C, indicated their interest in professional development. Teacher A used a computer in his classroom. His students used graphic calculators. Teacher B did not use any computer facility and teacher C used a digital white board.

Teacher A (age 56) attained a Bachelor’s degree in Mathematics and a Master’s degree in Mathematics Education. He had worked as a mathematics teacher for 17 years with lower level to upper level high school students.

Teacher B (age 48) attained a Bachelor’s degree in Mathematics and a Master’s degree in Mathematics Education. He had worked as a mathematics teacher from 1988, mostly with upper level high school students.

Teacher C (age 48) attained a Bachelor's degree in Engineering and a Master's degree in Mathematics Education. He was a contract member of the University of Twente for seven years. Since 2009 he has worked as a mathematics teacher with mostly upper level high school students.

2.2 Preparation for lesson study

Each participant was given a research paper with the intention to introduce scientific research outcomes and to develop a collective technical language. To attain these goals, teachers were asked to present and to discuss their papers in a lesson study team seminar at the university. Teacher A was given a paper about students' perspectives on a mathematical concept from Godfrey and Thomas (2008), teacher B had a paper about the role of metonymy in mathematical understanding and reasoning from Zandieh and Knapp (2006) and teacher C had a paper about long-term mathematical thinking from Tall and Mejia-Ramos (2009) in the first lesson study cycle. The researcher taught additional mathematics educational theory based on Skemp's (1976) instrumental or relational understanding and Tall's (2008a) embodied and symbolic worlds. The reading of these papers was not directly focused on the lesson preparation. Between the first and second lesson study cycle, the teachers were given a paper on teachers as mentors to encourage power and simplicity in active mathematical learning (Tall 2007).

The teachers decided to start by uncovering the students' initial learning and thinking processes, through designing an observation list as well as criteria to analyze the students' responses. They chose criteria based on Skemp's (1976) types of understanding (instrumental or relational) and Tall's (2008a) journeys through the two mental worlds of conceptual embodiment and operational symbolism (embodiment or symbolism).

They chose not to refer to formalism, which seemed appropriate at the time. However, with only embodiment and symbolism, they categorized the limit as symbolic (consistent with Bruner's use of the term including both operational symbolism and verbal language). In analysing the choices of the assessors, the authors used Bruner's framework to distinguish between operational symbolism and verbal symbolism, which could then be related to the shift from operational symbolism towards the more formal verbal definitions and general arguments involving the limit concept.

Reliability. Two assessors (the university teacher trainers in the lesson study team) worked separately to categorise the responses and in cases of disagreement (21%), the third assessor (the university mathematician) was called upon to arbitrate to seek a resolution that led to an agreed categorisation between all three assessors.

2.3 Data gathering instruments

Evolution of teachers' general goals and preferred starting strategies. The team discussed and agreed a list of twelve possible goals for the lessons and eight possible ways of starting the first lesson. The three teachers were then asked to place these in order of preference in written tasks administered before and after the first lesson study cycle and repeated after the second cycle. Tables (1) and (2) present their most and least favoured preferences in each case (four items each out of twelve in table (1) and three items each out of eight in table (2)).

Evolution of teachers' conceptions of the subject-specific curriculum. Before and after each cycle, the three teachers were asked to write down what they considered to be important in teaching the given topic (the derivative in the first cycle and the transition from a geometric to an analytic representation of trigonometric functions in the second). The purpose of these questions was to reveal changes and differences in their perceptions concerning teaching methods (table 3).

2.4 Procedure and data analysis

The teachers implemented the collaborative designed lesson successively, first teacher A, then teacher B, followed by teacher C. The observers, consisting of the lesson study team and additional school colleagues and school managers (as many as possible), made field notes related to students' oral remarks. The observers analysed the development of the lessons using these field notes.

The evolution of the teachers' goals was analysed by the assessors using Skemp's (1976) instrumental understanding [i] and relational understanding [r].

The evolution of the teachers' conceptions of the curriculum development was analysed in terms of Tall's (2008a) embodied [e] and symbolic [s] modes of operation.

The third table of the three teachers' free-associated topic specific aspects was analysed using a combination of instrumental/relational and embodied/ symbolic categorizations.

3. RESULTS

3.1 The development of the lessons

The goal of the first lesson study cycle was the integration of a sensible approach to the calculus with the focus on the introduction of the derivative. However, none of the teachers chose to use software designed to illustrate the idea of local straightness (such as van Blokland & Giessen, 2000). Teacher A introduced local straightness as an applet on the student's graphic calculators. Teachers B and C did not use software. All three teachers decided that their introduction of the derivative should be followed by the use of their regular textbooks based on the conception of the generic tangent as a line that touches the graph at one point only and does not cross it.

Teacher A usually used graphic calculators in his class and introduced an applet to demonstrate local straightness as the most meaningful way to understand the derivative. His goal was conceptual understanding. After a short introduction, he concentrated on the ratio $\Delta y/\Delta x$ to connect to the textbook approach. His lesson was crowded with ideas and he avoided possible student difficulties by giving a too detailed presentation. As a consequence his students did not have any questions and he was unable to initiate a lesson in which the students shared their developing ideas or to begin to uncover his students' learning processes.

Teachers B and C concentrated on teacher A's difficulties in uncovering students' thinking processes. Teacher B did not have any computer facility. He intended to uncover student's thinking processes by introducing practical activities followed by a plenary discussion, hoping to build on student's interactions. He began the activity by giving each student a graph of $y = x^2$ on squared paper and asked them to draw a tangent at a point that was not placed on a crossing of the grid lines. As a consequence, the tangent lines they drew were slightly different and gave small differences in the numerical slope of the tangent. He focused on the concept of the tangent line, as a basic conception in line with the textbook approach, before introducing the derivative. Teacher B's plenary discussion focused on the concept of the tangent, but also ended in the ratio dy/dx, following the strict textbook guidelines. Once again, the students did not ask any questions and the plenary discussion did not lead to sharing ideas in a manner appropriate for lesson study.

The observers (the lesson study team and other colleagues) noted that the students were not amazed at all, when their practical approach to the tangent produced different tangent lines with different slopes as compared with the graphic calculator that produced a single formula.

Teacher C had begun the project working with two colleagues in a subgroup but both his colleagues withdrew when they were unable get time from their school management. He then joined teachers A and B who had already begun their own preparation. Although he often used a digital white board, in the lesson study teaching, he began with written worksheet questions to try to uncover his students' thinking processes on paper. He focused on the formula of the tangent line and what happened to the difference quotient as

$\Delta x \to 0$. He aimed at relational understanding, basing his approach on the limit concept without naming it explicitly.

Teacher C realized that his written worksheet questions were unclear: he had intended to encourage relational understanding, yet his questions essentially focused on standard techniques that could be performed instrumentally. The resulting responses to these questions proved to be difficult to analyze.

All three teachers were amazed by their students' unexpected responses and were determined to find out more. They decided to continue the worksheet approach to uncover the students' thinking processes in a second lesson study cycle focusing on the study of trigonometric functions.

3.2 Changes in teachers' goals

The goal statements, agreed by the team, were classified by the assessors in terms of instrumental and relational understanding.

Six goal statements were classified as instrumental [i]:

'To be able to execute correctly'; 'to acquire skills to solve problems', 'to use the graphic calculator adequately', 'to use computer applications', 'to apply techniques in practice, and 'definitions as a starting point'.

Classifying 'definitions as a starting point' as instrumental rather than relational is noteworthy. Before this study, the teachers and the observers taking part had all taught calculus based on an informal definition of limit and so it was natural for them to consider the students' difficulties with the limit concept to involve only instrumental understanding. They had yet to encounter the effect of a 'locally straight' approach' designed to offer a relational embodied foundation to the limit concept.

The other six goal statements were classified as relational [r]: 'structures as basis for thinking', 'to learn to understand concepts', 'to be able to relate concepts', 'to learn to argue (generally)', 'to learn to argue (stepwise)', and 'to use realistic practical situations'.

The three teachers' higher and lower prioritized goal statements are listed in Table 1. The first column lists the three applications of the questionnaire, the pre-test, the intermediate test between cycles, and the post-test after the second cycle. The remaining columns list the teachers' choices of goals for the three successive questionnaires in order, with the first four preferences, then the last four.

	Teacher A	Teacher B	Teacher C
First four preferences			
Pre-test 1	Structures as a basis for thinking [r]	Acquire skills to solve problems [i]	Learn to understand concepts [r]
2	Be able to relate concepts [r]	Apply techniques in practice [i]	Be able to relate concepts [r]
3	Learn to argue (sequentially) [r]	Learn to argue (sequentially) [r]	Structures as a basis for thinking [r]
4	Learn to argue (generally) [r]	Be able to execute correctly [i]	Acquire skills to solve problems [i]
Intermediate 1	To learn to understand concepts [r]	Apply techniques in practice [i]	Structures as a basis for thinking [r]
2	Structures as basis for thinking [r]	Learn to argue (sequentially) [r]	Learn to argue (sequentially) [r]
3	Learn to argue (generally) [r]	Acquire skills to solve problems [i]	Acquire skills to solve problems [i]
4	Learn to argue (sequentially) [r]	Use realistic practical situations [r]	Be able to execute correctly [i]
Post-test 1	Structures as a basis for thinking [r]	Structures as basis for thinking [r]	Learn to argue (generally) [r]
2	To learn to understand concepts [r]	Acquire skills to solve problems [i]	Acquire skills to solve problems [i]
3	To be able to relate concepts [r]	Be able to execute correctly [i]	Learn to understand concepts [r]
4	To learn to argue (generally) [r]	Definitions as a starting point [i]	Be able to relate concepts [r]
Last four preferences			
Pre-test 9	Be able to execute correctly [i]	Understand abstract concepts [r]	Use GC adequately [i]
10	Use the GC adequately [i]	Definitions as a starting point [i]	Use computer applications [i]
11	Use computer applications [i]	Use computer applications [i]	Use realistic practical situations [r]
12	Acquire skills to solve problems [i]	Learn to argue (generally) [r]	Definitions as a starting point [i]
Intermediate 9	Acquire skills to solve problems [i]	Learn to argue (generally) [r]	Definitions as a starting point [i]
10	Use the GC adequately [i]	Definitions as a starting point [i]	Use realistic practical situations [r]
11	Use computer applications [i]	Use computer applications [i]	Use computer applications [i]
12	Definitions as a starting point [i]	Use the GC adequately [i]	Use the GC adequately [i]
Post-test 9	Be able to execute correctly [i]	Learn to argue (generally) [r]	Use computer applications [i]
10	Acquire skills to solve problems [i]	Use the GC adequately [i]	Definitions as a starting point [i]
11	Use GC adequately [i]	Use computer applications [i]	Use the GC adequately [i]
12	Use computer applications [i]	Learn to argue (sequentially) [r]	Learn to argue (sequentially) [r]

Note: GC stands for Graphic Calculator

Table 1: Teachers' goal statements with high and low priorities

Teacher A remained consistent throughout with all four preferred goals being relational and all four least preferred goals being instrumental. At each stage, his four lowest preferences always included computer applications and graphic calculators, even though he used graphic calculators in his own classroom.

Teacher B began by preferring instrumental understanding through acquiring skills, applying techniques and being able to execute operations correctly, while acknowledging the need to learn to argue sequentially. Although the latter is assessed as being relational, sequential arguments may also be learnt instrumentally to reproduce in tests, so all four initial preferred choices may be instrumental in practice.

Over time, his first four preferences switch to a preference of a start with relational understanding directly followed by instrumental aspects, while the profile of definitions changes positively from a low priority to the fourth preferred goal. (This change in perception of the use of a definition may involve it switching from being seen as instrumental [i] to having a relational role [r], linking to the various structures as a basis for thinking.) At every stage, his four lowest preferences always include learning to argue generally, with his attitude to learning to argue sequentially changing dramatically to become his least preferred goal. (We will return to this change after considering his responses in Table 3.)

The use of computer applications remained his second lowest preference throughout and, after the first cycle, the adequate use of a graphic calculator became his least preferred goal, and remained a low priority at the end.

After each cycle his top two preferences are also relational and the next two end instrumental, which may be interpreted as a priority for first establishing relational understanding of structures, directly followed by the instrumental learning of essential skills.

Teacher C also began by preferring relational understanding of concepts, their relationships and using structures as a basis for thinking, followed by the instrumental goal of acquiring skills to solve problems. After the first cycle, his preferences change marginally to a balance between relational skills using structures and sequential arguments followed by the instrumental acquisition of skills and correct execution. His final position is again mainly relational preferring general arguments, followed by instrumental acquisition of skills and relational understand of concepts and relationships.

Meanwhile, his lowest preferences always included the use of computer applications, graphic calculators and using definitions as a starting point. His initial low priority for realistic practical situations gradually eases, with his final lowest preference being the relational use of sequential arguments.

The three teachers had different preferences for the use of technology, with Teacher A using a computer and his students using graphic calculators, Teacher B used no technology and teacher C used a digital white board. However, common to all three was the low priority for the use of computer applications and graphic calculators. This clearly inhibited the implementation of a locally straight approach to the calculus using dynamic computer software.

3.2 Changes in teachers' preferred starting strategies

The eight chosen statements for possible starting strategies were classified by the assessors as embodied or symbolic.

Four were considered embodied [e] because they included practical or visual aspects: 'different examples' (with a focus on difference); 'realistic examples' based on the Dutch realistic approach; 'practical situations'; and 'representations of a concept' (with a focus on different representations including graphs).

The other four starting strategies were classified as symbolic [s]: 'general abstract concepts', 'worked examples', 'definitions', and 'thinking models'. The symbolic classification may be seen as related more to Bruner's notion of ‘symbolic’, including both operational symbolism and verbal language. After the classification had been performed by the assessors, the authors subdivided the symbolic category [s] into operational symbolic [os] and verbal symbolic [vs] which revealed verbal symbolism as an initial step into the world of formalism. Thus ‘worked examples’ is classified as [os] while the others – ‘general abstract concepts’, ‘definitions’, ‘thinking models’ – are classified as [vs] representing the shift to verbal representations, placing the informal definition of the limit concept as an informal introduction to formal thinking.

Table 2 lists the top three and bottom three preferred starting strategies (out of eight) of teachers A, B and C. As in table 2, the first column labels successive applications of the questionnaire with the remaining columns showing the most and least preferred starting strategies for each teacher.

		Teacher A		Teacher B		Teacher C	
First three preferred starting strategies							
Pre-test	1	Different examples	[e]	Worked examples	[os]	Different examples	[e]
	2	Practical situations	[e]	Different examples	[e]	Worked examples	[os]
	3	Realistic examples	[e]	Thinking models	[vs]	Representations	[e]
Intermediate	1	Realistic examples	[e]	Different examples	[e]	Different examples	[e]
	2	Different examples	[e]	Representations	[e]	Worked examples	[os]
	3	Practical situations	[e]	Worked examples	[os]	Representations	[e]
Post-test	1	Definitions	[vs]	Different examples	[e]	Thinking models	[vs]
	2	Practical situations	[e]	Thinking models	[vs]	Representations	[e]
	3	Representations	[e]	Worked examples	[os]	Worked examples	[os]
Last three preferred starting strategies							
Pre-test	6	Thinking models	[vs]	Definitions	[vs]	Practical situations	[e]
	7	Definitions	[vs]	Representations	[e]	Realistic examples	[e]
	8	General concepts	[vs]	General concepts	[vs]	Definitions	[vs]
Intermediate	6	Definitions	[vs]	Worked examples	[os]	Realistic examples	[e]
	7	Thinking models	[vs]	General concepts	[vs]	Practical situations	[e]
	8	General concepts	[vs]	Definitions	[vs]	Definitions	[vs]
Post-test	6	Thinking models	[vs]	Realistic examples	[e]	Realistic examples	[e]
	7	General concepts	[vs]	Definitions	[vs]	General concepts	[vs]
	8	Realistic examples	[e]	General concepts	[vs]	Definitions	[vs]

Table 2: Teacher’s choice of statements with high or low priorities at the start of instruction

In the pre-test, all three teachers preferred to focus initially on examples of various kinds.

Teacher A’s preferred starting strategies were embodied at every stage, with the sole exception of ‘definitions’ which appeared in the final post-test linked to practical situations and representations.

His three least preferred strategies were all verbal symbolic with the sole exception of realistic examples, which were demoted from a preferred strategy in the pre-test and intermediate test to the least desirable strategy in the final post-test.

Teachers B and C both seek to combine examples in embodiment and operational symbolism with some references to verbal symbolic ‘thinking models’ as the study progresses.

On the final post-test, all three teachers specify two verbal symbolic aspects alongside realistic examples amongst their least favoured items. Indeed, teacher C maintains realistic examples among his least favourite items throughout. This relates to an internal debate in the Netherlands in which realistic mathematics has been linked to the need for remedial courses in university to teach necessary symbolic techniques. (Craats, 2007; Tempelaar & Caspers, 2008; Werkgroep 3TU, 2006). While realistic mathematics has a strong international reputation in elementary mathematics, its role in the Netherlands is questioned by those requiring symbolic facility at a later stage.

All three teachers include ‘definitions’ in their three least favoured initial strategies on the pre-test, and this remains in the three least-favoured strategies in the intermediate and final responses, with the exception of teacher A, who switches in the final post-test to make definitions his highest priority. Now, however, it is coupled with interpreting the definitions through practical examples and different representations, linking the definitions to a range of conceptual embodiment and operational symbolism. This change in emphasis follows the cycle involving definitions of trigonometric functions that may be more amenable to such an approach than the limit concept in the calculus. It may also signify a change in interpretation of a definition classified as instrumental to reconsider it as relational, relating the verbal definition to embodied and symbolic meanings.

3.3 Changes in teachers’ free-associated aspects of mathematical thinking

Teachers’ characteristic free-associated aspects related to teaching the mathematical concept are listed in Table 3. In the columns, teachers A, B and C’s respective free associated aspects are displayed in order, relating to the pre-test and post-test data collection for both lesson study cycles.

Lesson study cycle 1: The derivative		
Teacher A	**Teacher B**	**Teacher C**
Pre-test		
Change Growth Average rate of change Difference quotient Differential quotient Velocity Slope Tangent line Equation of tangent line	Rate of change Help with more information on functions Difference in growth over an interval and at a point (concept of limit) Different notations (not at the start) Remarks: (1) I teach this abstractly because it develops into a skill; (2) I look for more practical aspects to use at the start.	Average rate of change Instantaneous rate of change General formula for the derivative (to take the limit?) Arithmetic rules for the derivative Equation: tangent line To calculate extreme values *Change*, a scheme with components: table, function, rule, symbol, physics symbolised in mathematics
Post-test		
Slope of a graph Difference quotient Limit Differential quotient Change Slope Tangent line (equation) Interval Maximum/minimum	Tangent line: what is it? Slope Difference quotient → differential quotient (what is the difference?) Rate of change – average change The concept of the limit Can we define the derivative as a function? What to do with the derivative? To calculate slope and extreme values	Rate of change Change at one moment (speed camera) Tangent line on the graph → understand this concept (visual) Be aware of the accompanying formula: see the problem (one point), work around (understanding $\Delta x \to 0$) Understand relation between drawing the graph and accompanying formulas Limit at the end

Lesson study cycle 2: The developmental shift of the conception of sine and cosine		
Teacher A	**Teacher B**	**Teacher C**
Pre-test		
Able to draw the unit circle, able to explain its role Ratio of sides: indicate the sides of a right-angled triangle Convert between degrees and radians The concepts: period, amplitude, phase-shift and zero crossing, as well as frequency, wavelength Reproduce the standard values of sine, cosine and tangent Calculate sides and angles in a triangle, eventually using sine and cosine rules Able to solve equations using sine, cosine and tangent Determine a function rule in relation to a sine/cosine graph Draw a graph given by a function rule	Coordinates of a point on the unit circle Standard graphs of $\sin x$ and $\cos x$ followed by transformations of these to $y = a + b \sin c(x-d)$, using concepts such as amplitude, phase. Ratios of sides in right-angled triangles (soh/cah/toa) Applications: (1) resonances in physics, therefore knowledge is needed to calculate rules related to sine and cos, $\sin 2x = \ldots$, $\cos 2x = \ldots$, $\sin (x+y) = \ldots$, etc.; (2) sometimes polar coordinates; or (3) parametric curves, Lissajous figures.	Memorize sin/cos/tan in right-angled triangles, formula soh/cah/toa Unit circle, height h, width b, angle x; Deduce $h = \sin x$, $b = \cos x$. Walk around a circle. The distance is x. Circumference = $2\pi r$ (2π in unit circle). Relation angle α and distance x. Function: $h = \sin x$ graph, with x on the x-axis and h on the y-axis. The function $y = d + a \sin b(x-c)$, $p = \pi/b$ (period). (1) concept of parameters (2) from parameter to graph (& back) (eventually derivative of $\sin x/\cos x$) Tan x → unit circle $\tan x = h/b$, function relation in unit circle, $\tan x = \sin x/\cos x$, $\sin^2 x + \cos^2 x = 1$ (Pythagoras) Sum/difference, double angle formulas
Post-test		
Standard triangles Ratios of side Unit circle Degrees, Radians Graph of $f(x)=\sin x$, $f(x)=\cos x$, zero points, domain, range Derivative → calculate max-min Formulas: $\sin 2x = \ldots$, $\sin^2 x + \cos^2 x = 1$ Parametric functions, sine and cosine Calculate a combined period of two sine functions	Ratios of sides within rectangular triangles (soh/cah/toa) Drawing a unit circle and developing the graph of sine Radians Concepts: amplitude, period, translation ($y=a + b \sin c(x-d)$) In a later phase trigonometric formulas like $\sin 2a = \ldots$, $\sin (a+b) = \ldots$ Using 1-1-√2 & 1-2-√3 triangles or 'the unit circle' or better *both*	Geometry: soh/cah/toa in a triangle Need for the unit circle: look! $\sin x$ = height, $\cos x$ = width, x is distance walking around unit circle From the graph of $\sin x$ and/or $\cos x$ to sin and cos function Relationship between radians & degrees Afterwards the derivative of sin and cos functions and area under them Then relations between sin, cos, tan; formulas for these relationships

Table 3: Teachers' free associated aspects of the mathematical conceptions

None of the teachers mention the notions of slope and local straightness in their freely associated written comments. After the initial study lesson, they focused on the items required in the Dutch curriculum. This specifies two distinct courses: Mathematics A (mostly focused on contexts) for arts and humanity students, and Mathematics B (more symbolic) for science students.

Teacher A followed an intended relational embodied approach with comments reproduced from the Dutch textbook. In the pre-test, the first three remarks – change, growth and average rate of change – are taken from Mathematics A and the rest from Mathematics B, written in the order that they appear in the curriculum. He did not mention the limit concept explicitly.

In the observed study lesson he focused on visualizing the derivative embodied as the changing slope of the graph (as in Tall & Ramos, 2004) followed by a translation to the symbolism of the difference quotient, then the notion of limit followed by standard elements from the Dutch curriculum.

In the second lesson study cycle, Teacher A's free associated aspects remain broadly the same from pre-test to post-test, while changing in order and becoming expressed more concisely. They remain focused on a relational blending of visual representations using right-angled triangle and circles, together with the reproduction and use of the formulae of trigonometry. Table 1 reveals his consistent preference for relational understanding over instrumental understanding. However, his preferred starting strategies in table 2 shift definitions from one of his bottom three preferences to the highest priority in the final post-test. While the assessors considered 'definitions as a starting point' classified in table 1 as being instrumental, his final preferences now suggest that the (verbal) symbolic 'definition' is now conceived as a basis for relational links with 'practical situations' and 'representations'.

Teacher B begins with rate of change, information on functions and then the 'difference in growth over an interval and at a point', introducing the limit concept. He realises that he should consider different notations (though not at the start) and remarks that he teaches 'abstractly because it develops into a skill' while 'looking for more practical aspects to use at the start'.

After the first lesson study cycle, he returns to a consideration of the tangent line and 'what is it?', shifting to a questioning approach that is affected by the needs of discussion in a lesson study approach. He continues by asking about the difference between differential quotient and differential quotient, referring to 'rate of change – average change' leading to the concept of limit. He questions 'can we define the derivative as a function? and asks what we can do with the derivative, leading to 'calculate slope and extreme values'. This lesson expands a traditional sequence by posing questions to the students but does not explicitly mention the global idea of local straightness to 'see' the changing slope of the graph.

In the second lesson study cycle he begins by mentioning points on the unit circle and graphs of sin$x$ and cos$x$, generalising them to $y = a + b\sin c(x - d)$, associated with concepts such as amplitude and phase, only then mentioning the ratios of the sides in a right-angled triangle. He refers to applications in physics and the standard rules in the order $\sin 2x$, $\cos 2x$, $\sin(x + y)$, … and moves on to polar coordinates, parametric curves and Lissajous.

After the cycle, he starts more fundamentally with the ratios of sides in a right-angled triangle, then emphasizes the underlined comment '<u>drawing a unit circle and developing the graph of sine</u>'. He moves on to radians and various concepts such as amplitude, period, translation related to more general formulae such as $y = a + b\sin c(x - d)$. He finishes by returning to the relationship between special examples of 1-1-√2 and 1-2-√3 triangles using standard triangles, the unit circle, or 'better BOTH.' This is in line with his subtle changes in preference in table 1, from a focus on instrumental understanding to a relational blending of representations and symbolism and his shift in table 2 starting from a combination of examples and operational symbolism to a preference for 'thinking models', 'representations' and 'worked examples'.

Teacher C begins with the traditional approach to the derivative, from 'average rate of change' to 'instantaneous rate of change' and refers to the 'general formula for the derivative', questioning whether to take the limit. He follows with the 'arithmetic rules for the derivative' and the 'equation [of a] tangent line, moving on 'to calculate extreme values'. He ends by speaking of 'change' as 'a scheme with components table, function, rule, symbol' with reference to 'physics symbolised in mathematics'.

In the post-test he begins with 'rate of change' and the realistic embodied idea of a 'speed camera', blending the visual concept of the tangent line and being 'aware of the accompanying formula'. He speaks of seeing 'the problem (at one point)' and 'work around understanding $\Delta x \to 0$'. He emphasizes the idea of 'understanding [the] relation between drawing the graph and accompanying formulas' with 'the limit at the end'.

This relates to his blending of relational understanding of concepts and instrumental acquisition of skills in table 1 and his subtle shift from combining embodiment and symbolism to a more integrated desire to blend thinking models, representations and worked examples in table 2.

In the second cycle, teacher C emphasizes memorizing trigonometric formulae in right-angled triangles, deducing trigonometric relationships in a unit circle and the dynamic embodied idea to 'walk round the circle' to relate the distance travelled to the angle turned. He continues with symbolism and a mix with embodiment. This leads to the idea of a trigonometric <u>function</u>

(underlined) and to more generalised ideas including parameters, and relational ideas involving the visual and symbolic expressions of the theorem of Pythagoras and related trigonometric properties.

In the post-test, he expresses the same broad development in more concise terms.

His overall development relates to his comments in table 1 where his initial preferences for understanding and relating concepts and acquiring skills is extended to include learning to argue generally and in table 2 where his preference for the blending of embodied and symbolic examples is expanded to include thinking models.

4. Discussion

The most immediate observation from the data is that successive teachers used their single study lesson in a sequence of ways that did not lead to the Japanese form of lesson study. Nor did they use the embodied ideas of continuity and local straightness to inform their fundamental teaching of the notion of derivative. Instead they followed the Dutch curriculum design in which the textbooks in combination with the pressure of common study guidelines impeded their desire to design the research lessons as part of a longer-term lesson-study module (Verhoef & Tall, 2011).

However, when the designed curriculum materials did not result in the students actively participating in the lesson, the teachers realised that they needed to focus more deeply on the students' thinking processes. They decided to begin by uncovering students' thinking processes to prepare the introduction of what was, for them, a totally strange and new teaching approach. That meant that their interactions with students were limited to short plenary discussions between students and teacher.

The teachers avoided interactions between students. They focused more and more on students' written responses, which provided them with aspects to analyze and to categorize. The research outcomes confirm that the teachers tended to use teaching methods with which they and their colleagues were familiar.

The question arises as to the source of these difficulties. In this study, the participating researchers encouraged the teachers to read the literature and to form shared opinions and design their lesson. Although the teachers empathized with a 'locally straight' approach to the calculus, they all gave a low priority to the use of computer activities. They therefore missed the opportunity to take enough time to encourage the students to focus on the essential idea that a differentiable function gets less curved as one zooms in until it is visually straight. With this essential idea, the students might have been encouraged to ‘look along the graph’ to *see* its changing slope and to conceptualize the graph of the changing slope as the derivative of the function. Such an approach would involve a different lesson sequence in which experience of local straightness and the investigation of the changing slope of standard functions could lead to the need to compute the derivative in a precise symbolic way that naturally leads to the limit concept. The teachers’ perceived need to carry out the specifics of the existing curriculum meant that an alternative approach was not incorporated.

Japanese lesson study requires more than the design of a single lesson, it requires a coherent approach to a series of lessons. This study reveals the significance of the complex reality of Dutch school practice, driven by the powerful claim of curriculum guidelines, study guides based on textbooks, and the attainment of high exam results. The response of the teachers in this study suggest that the dual attempt to introduce both a new mathematical approach (local straightness) and a new teaching approach (lesson study) puts great demands on teachers, already constrained to follow a tightly defined schedule preparing students for specific examinations.

As we authors reflected on the outcomes of this first year of study, we realised that there were precedents for the phenomena observed. For instance, in the original doctoral study of Tall (1986), it was found that the focus on a locally straight approach resulted in impressive changes in conceptual understanding, for instance, the experimental students were able to *look* at a graph and *see* its changing slope to sketch the derivative while the control students following a traditional course tried to guess the formula of the graph to differentiate the formula to calculate the derivative symbolically. However, when asked how they might explain the ideas of the calculus to a student with no previous experience, none of those following the experimental course mentioned local straightness; instead they focused on their subsequent experience of finding tangents using differentiation as required on the examination syllabus. Yet, when they were faced with interpreting the tangent in unusual cases, such as $|\sin x|$ at the origin, they were able to respond conceptually by saying there was no tangent because the graph was not locally straight or even that it had different left and right tangents.

The subtle relationships involving embodiment and symbolism have very different long-term consequences. Whereas embodiment can give conceptual insight into the initial meaning of the slope of a curve, symbolism becomes essential to formulate and solve problems involving combinations of functions. As a consequence it may be natural for teachers pressed to get their students to do well in assessment to focus as soon as possible on the symbolism. Such a phenomenon has resulted in national debates around the world.

In the USA, the ‘Math Wars’ arose between those who seek constructive meaning in student learning and those who require powerful computational skills. In the Netherlands a similar difference arises between those who seek to build a realistic mathematical approach and those who require students to acquire techniques appropriate for more advanced courses. In Japan there is a corresponding divide between those who use lesson study to encourage meaningful learning in primary schools and those who encourage technical fluency through practice in out-of-hours Juku schools.

We do not believe that the two goals are necessarily at odds, rather that they play complementary roles in the development of mathematical thinking. Conceptual embodiment such as local straightness can give fundamental meaning to subtle concepts such explaining what it means for a function to be *not* differentiable (it can have corners or be wrinkled in various ways). It can also provide a fundamental human insight into the potentially infinite limit limit process of finding the derivative, because the slope function can be *seen*, by tracing the changing slope and it is simply required to calculate a perceived function using operational symbolism rather than proving that an as-yet-unknown function exists. For those going on to study formal analysis at university, the underlying blend of conceptual embodiment and operational symbolism offers a natural foundation for the axiomatic formalism of mathematical analysis (Tall & Mejia Ramos, 2004; Tall, 2010).

However, the experience of this first year of a four-year project to introduce Lesson Study in Dutch Schools has revealed how underlying cultural forces can privilege a familiar approach to the curriculum and impede the development of a new approach that builds from the insight of conceptual embodiment to the power of operational symbolism. The teachers in the study sensed the need to help students perform well in assessments while having little time to reflect and think through new ways of teaching and learning.

The second year of the study is now in progress. The original team has been expanded with new members including those with experience of working with GeoGebra, which has led to the development of a GeoGebra environment that enables the students to explore local straightness and the process of tracing along the curve to see its changing slope giving rise to the graph of the slope function that will stabilize on the derivative. The existence of this tool to support dynamic visualisation still needs to be used to encourage the sense-making link between dynamic embodied meaning and operational symbolism. We await developments with interest.

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