CRYSTALLINE CONCEPTS IN LONG-TERM MATHEMATICAL INVENTION AND DISCOVERY

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In this paper [1], I formulate a cognitive concept that offers a simple unifying foundation for the most sophisticated level of development in the growth of mathematical thinking while also having implications for mathematics at all levels. It relates to my formulation of three distinct worlds of mathematical thinking (Tall, 2004), which in turn links naturally to a wide range of theories of cognitive development [2], some of which I discuss in this article. The idea offers common ground for all these theoretical frameworks, relating the development of human thinking to the structure of mathematical concepts. It enables us to respond to deep questions about the nature of mathematics itself in an essentially simple, structured way.

For example, "Is mathematics discovered or is it invented?" "Does it exist out there in some pure platonic world, or is it constructed by the embodied mind of Homo Sapiens?" Controversy over the possible responses to these questions has continued over the years and is more than just a philosophical argument between intellectuals, for it affects all our students in the way in which mathematics is taught and learnt.

"Is mathematics something pure and abstract that we transmit to our children, or is it something that they construct for themselves?" "Is mathematical truth to be presented as an abstract logical argument or does it require a meaningful representation?" These problems have a relevance at all stages of mathematical learning, from the explorations of the young child, the mathematical techniques taught in school, open-ended problem-solving activities, the growing sophistication in mathematics as the learner meets new situations, through to mathematical research at the frontiers of knowledge.

We have a conundrum. Mathematics is considered by some to be a human creation, building from sensori-motor origins to higher levels of mathematical research (Lakoff & Núñez, 2000). Others see mathematics as an abstract ideal, with a truth and beauty that mere mortals only discover, not invent (Jaffe & Quinn, 1993). These two apparently opposing views need not be in total conflict, as we may realize by considering the nature of the common mathematics that we share.

While each of us builds our own mental constructions, the concepts that we develop, say in counting, one, two, three, … with each number followed by a new number, different from all of those that came before, the structure we obtain is essentially unique. It can be formulated axiomatically as the Peano Postulates, although such a formality is not necessary to realize that just as two and three make five, taking three from five will always leave two. Because we all work with essentially the same system, the underlying structure that we 'construct' is the same for us all, and therefore may also be seen as being independent of a particular individual.

This is not a phenomenon that occurs only at the highest levels of mathematical thinking, it enters the lives of every learner as they realize that mathematical operations are not arbitrary, but have inevitable relationships that work in a particular context. It occurs when a child notes a symmetry in an isosceles triangle cut out of paper and folds it down its middle to see that not only do the sides coincide, the base angles are also equal. It occurs when a Brazilian street child uses practical mathematics to sell cigarettes and give the correct change.

In Tall (2004), I formulated the theoretical framework of 'three worlds of mathematics' that offers a long-term view of conceptual development as the individual matures from child to adult. It is based on human embodiment, building from sensori-motor origins, developing in parallel through sensory perceptions of shape and space on the one hand and motor activities such as counting and sharing, leading to arithmetic and algebra on the other. Later in development, mathematical thinking can be transformed into a formal mode of operation based on axiomatic definition and mathematical proof.

This framework reveals three quite different long-term developments of mathematical ideas, one through refinement of ideas about space and shape that are verbalized and later transformed into various forms of geometric inference including Euclidean proof, one through encapsulating mathematical processes as manipulable concepts in arithmetic and algebra, and one that develops much later in specialist pure mathematics from formal definitions and deducing theorems using mathematical proof.

Now I had *three* ways of developing mathematical thinking when I was seeking a single underlying foundation. As I reflected on each of these, one single idea emerged that offered the key to the long-term sophistication of mathematical thinking, to the fundamental nature of mathematics that reveals its essential quality. It is the notion of a *crystalline concept.*

This is a phenomenon that we construct as a thinkable concept based on our human perception and action, which develops an increasingly sophisticated structure as we compress knowledge and link ideas together. It is widely found in all mathematics and is best introduced initially in terms of examples.

Choices and consequences in mathematics

In mathematics we can make choices. The Egyptians chose to represent whole numbers with icons for 1, 10, 100 (𓏺,𓎆,𓍢) and so on; the Babylonians made marks on clay tablets in base 60; the Greeks used the 24 letters of their alphabet, plus three phoenician letters to give 27 symbols for the numbers 1 to 9, 10 to 90 (in tens) and 100 to 900 (in hundreds); the Romans used a different system with symbols in fives and tens, (I, V, X, L, C, D, M); Mayans worked in base 20, the Chinese and the Hindus developed base ten systems, with the Hindu-Arabic numerals becoming the chosen system in modern western civilizations.

Each civilization introduced not only different notations but also different ways of computation—such as the method of duplication for multiplication and division in ancient Egypt, or the various forms of abacus in many countries around the world.

However, despite this variety of choice, when it comes to arithmetic, whatever system of representation is used, the arithmetic is always the same. Two and two always makes four. No one can choose the result to be five.

The flexible properties of numbers and arithmetic are enshrined in the notion of procept (Gray & Tall, 1994), which recognizes that the symbols operate dually as process and concept in precise ways. It is not just that 2 + 2 is 4, but that if 2 is taken from 4 then 2 remains. The number 4 is also 3 + 1 and 1 + 3, or 2 [illegible] or 8 divided by 2. As we make mental calculations, to calculate 8 and 6, a child may know that 8 + 2 is 10 and that 2 from 6 is 4, so 8 + 6 is 10 + 4 which is 14. This tight knowledge structure is known as a *procept*. It connects together arithmetic relationships in a precise manner that is a consequence of the natural relationships between numbers.

The notion of procept is our first example of a crystalline concept. A crystalline concept may be given a working definition as "a concept that has an internal structure of constrained relationships that cause it to have necessary properties as a consequence of its context." Flexible arithmetic is powerful because of the internal crystalline structures within and between number concepts.

Making choices in various realms of endeavor

The choices made in mathematics are different from choices in other forms of human activity. For instance a Japanese Haiku consists of three lines with 5 syllables, 7 syllables and 5 syllables. It is a framework designed to give a certain balance and shape to a poem that is euphonious but arbitrary.

An American blues format consists of 12 bars with a specific chord sequence underlying music that is endlessly varied at the choice of the performer. A popular song has a chorus of 32 bars where the first, second and final set of 8 bars have similar formats and the third set (the "middle eight") is a contrasting melodic idea. But that does not stop creative people doing something different, especially when there is an established convention that is expected and something different is sensed as intriguing and creative.

A waltz is an elegant three beats in a bar, but Tchaikovsky wrote a flowing waltz movement in his Fifth Symphony with a flexible five-beat rhythm of two beats plus three. Popular music regularly uses four beats in a bar, but Dave Brubeck's *Take Five* sounds innovative with its rhythmic groups of five. The Beatles *All You Need Is Love* has its basic four-beat rhythm interpolated with extra phrases with a different rhythmic pattern. Hungarian folk dancing has intricate rhythmic groupings that feature in the music of Bartók.

Art changes over the years as painting techniques are developed from essentially flat representations to a three-dimensional sense using parallel lines meeting in a vanishing point. Perspective changed in the modern works of Picasso painting new images with simultaneous views from different directions.

Poetry, music and art demand fresh approaches as creative individuals invent new ways that break old rules. Mathematics too invents new ways of working that break old rules. The limited solutions of algebraic equations representing lengths and areas are transformed into a wider context allowing negative or complex solutions. The Euclidean notion of parallelism is complemented by the non-euclidean geometries of Bolya and Lobachevsky. The dynamic concepts of continuous variation in calculus are transformed into the epsilon-delta logical approach in analysis. Infinitesimals are banned by Cantor and yet are later validated in the non-standard analysis of Robinson.

Each of these involves a new way of conceiving ideas in a new context. At each stage the new way of thinking evolves its own crystalline structure of inevitable relationships. Mathematicians may *choose* what they wish to study and how they formulate their ideas, but they then *discover* that the new context entails implications that follow inexorably from their assumptions.

Christopher Zeeman, who proved new theorems about concepts in higher dimensions, including his theorem on unknotting spheres in five dimensions was asked whether he thought that mathematics was discovered or invented. He replied:

> Both. Sometimes you invent it; sometimes you discover it. You have to invent maths to get a solution to a problem but, in the process, I often discover a whole lot more which I didn't expect. (Arnot, 2005, pp. 20-21)

As research mathematicians shift to work in a new context, the nature of the concepts they consider may take on different forms, but those forms follow from the axiomatic framework of definitions and theorems.

In Euclidean geometry, the relationship between concepts is formulated through Euclidean proof. Constructions that focus on certain aspects necessarily imply others. For instance, if two straight lines in a plane never meet, then other properties necessarily follow: corresponding angles where a line cuts the two parallels are equal, alternate angles are equal, and interior angles add up to 180°. This leads to a *relational* view of these concepts as the various properties are seen to be related to each other (see Figure 1).

These various meanings for parallelism have an even more sophisticated meaning. The four given properties of parallel lines are all *equivalent*. Any one could be taken as

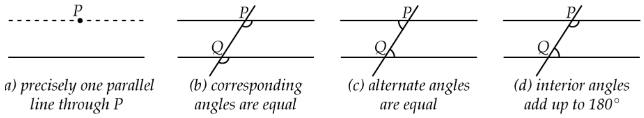

Figure 1. Equivalent properties of parallel lines.

the basic definition and then the others all follow. A higher level of thinking is then possible. Not only are these properties equivalent, they are simply *different aspects of the same underlying crystalline concept:* the concept of parallel lines in Euclidean geometry.

The same sequence of development in meaning occurs with other geometric concepts such as triangles, squares, rectangles, polygons, circles, spheres, polyhedra and so on. Children sense these as whole gestalts with various simultaneous properties. They learn to draw figures by freehand, describe their properties and begin to construct more precise figures using ruler and compass. A construction of a figure, such as an isosceles triangle with two equal sides, necessarily has other properties; for instance it will have two equal angles and be symmetric about an axis through the vertex meeting the base at right angles. Ideas such as congruent triangles can be verbalized and used to show that if figures have certain properties, then other properties must follow. Some properties may be seen to be equivalent and then the whole thing may crystallize into a pure abstraction in the form of a platonic object.

Alternative possibilities occur in other geometries. The concept of "parallelism" is not relevant in projective geometry where two distinct lines always meet in a single point, so there are no parallel lines. In spherical geometry, a "line" is a great circle that offers the shortest distance between two points. In the absence of parallelism, other familiar concepts of Euclidean geometry no longer hold. For instance, the notion that the sum of the angles of a triangle is 180° is no longer true for spherical triangles (made up with three great circles) where the sum of the angles in radians equals $\pi + \Delta / r^2$ where Δ is the area of the triangle and r the radius of the sphere.

This underpins the idea that a crystalline structure *depends on the context.* In different kinds of geometry the mathematical concepts have a specific structure with inferential links that are context-dependent.

Crystalline concepts throughout mathematics

We have now seen crystalline concepts as procepts in arithmetic and platonic objects in geometry with structure dependent on the context. The same occurs in axiomatic formal mathematics where a mathematician may choose to define an axiomatic system to have whatever axioms are desired, but then the consequences of those definitions follow from the context specified by those axioms.

The choices for axioms are not entirely arbitrary. One might, if one had nothing better to do, propose a new axiomatic system by choosing at random, say a certain number of axioms from the list of axioms for an ordered field. However, the theorems that one might prove may or may not have any wider value. Axiomatic systems are chosen more usually to reflect a particular area of interest, to select appropriate properties as a foundational list from which other properties may be deduced and connected together in a coherent knowledge structure based on the selected definitions and mathematical proof. For this reason, axiomatic systems that are chosen for study invariably incorporate a crystalline structure of deduced relationships.

Often the particular axioms chosen will involve flexibility in choice. For example, in the context of an ordered field, the axiom of completeness may be defined in various equivalent ways, by declaring that "an increasing sequence bounded above has a limit less than or equal to any upper bound", "a decreasing sequence bounded below has a limit greater than or equal to any lower bound", "a non-empty set bounded above has a least upper bound"' "a non-empty set bounded below has a greatest lower bound', "a cauchy sequence has a limit".

All of these equivalent axioms are instances of the same underlying concept. Furthermore, not only are these different expressions of completeness equivalent within the context of an ordered field, the concept of complete ordered field is itself crystalline. Any two axiomatic systems satisfying the axioms for a complete ordered field are isomorphic. In other words, they may be conceived as a single underlying crystalline concept. This leads to more powerful ways of thinking at a higher level, so that, in making deductions, one does not need to always refer back to a specific axiom, but may use any equivalent form instead. At this higher level of thinking, mathematics is not only more complex, it is also becomes more simple in operation.

Relationships with other theoretical frameworks

The term "crystalline" first arose in a conversation with Anna Sfard in my home in 1990, as we discussed her use of the term "condensation" to refer to the ability to deal with a given process in terms of input/output without necessarily considering its component steps (Sfard, 1991). I suggested that the metaphor of "condensation" changes a gas into a liquid that could be put in a container and poured. I put forward the idea that this metaphor could be extended to speak of "crystallization" into a solid object that could be manipulated. At the time, there were already competing terms for the process of transforming mental processes into objects, including "reification" (Sfard, 1991) and "encapsulation" (Dubinsky, 1991), so a further term was hardly welcome.

I now use the term "crystalline" with a far wider meaning than the encapsulation of a process as an object, just as Dubinsky (Czarnocha *et al.,* 1999) recognized that, in his APOS theory, an object could be encapsulated not only from a process, but also from a schema. More generally, I see the term "crystalline" to apply to a mathematical context that constrains a particular knowledge structure to have necessary properties that can be determined from the situation. It does not need to be the result of an encapsulation of a process: it can be any coherent situation in which the structure of relationships is entailed by the context. In particular, it applies to the nature of a mathematical structure in any of the three worlds of mathematics, be it the categorization of a specific object or relationship in geometry, the computational and manipulative relationships in arithmetic and algebra, or the defined concepts of formal mathematics.

The relationship of the three worlds of mathematics to various other theoretical frameworks has already been intimated in an earlier article in *For the Learning of Mathematics* (Tall, 2004). While process-object encapsulation theories are related to the development of the proceptual symbolic world of arithmetic, algebra and subsequent symbolic developments, my own personal development in thinking of the conceptual embodied world of space, shape and thought experiment was inspired by the work of van Hiele (1986).

Subsequent research (*e.g.*, Gutiérrez *et al.,* 1991) has shown that the van Hiele levels are not a discrete sequence of developmental stages, but this does not invalidate the broad sweep of van Hiele's theory as the child becomes engaged with increasingly sophisticated ways of thinking, from initial recognition of shapes, through verbal descriptions of properties, definitions and practical constructions, then on to Euclidean proof. Van Hiele goes on to a higher level of rigour, which relates within the three-world model to a shift to the axiomatic formal world, while other more advanced forms of geometry, such as spherical geometry and projective geometry can be performed by construction either in embodiment, symbolism or a blend of both without necessarily being reformulated in terms of axioms and definitions.

The SOLO Taxonomy of Biggs and Collis (1982) started out as framework to judge the Structure of the Observed Learned Outcomes, in which responses to assessment are seen to be unistructural (one aspect), multistructural (several aspects), relational (aspects related together), extended abstract (a single coherent whole). This development is seen to occur in a sequence of stages (sensori-motor, ikonic, concrete symbolic, formal, post-formal) after the fashion of Piaget and Bruner, but differing in certain details. Furthermore, each cycle was claimed to develop through successive unistructural, multistructural, relational (UMR) levels before leading to the extended abstract level that becomes the foundation for the next stage.

Pegg and Tall (2005) reorganized the SOLO framework in which the UMR cycle applied not to each stage, but more directly to the developing meaning of individual concepts. From this viewpoint, the construction of crystalline concepts fits in a system of increasingly sophisticated knowledge including:

- a *unistructural* situation in which ideas are sensed as single wholes with observed properties that may be sensed but may not as yet be verbalized or connected,
- a *multistructural* situation in which many different aspects are described and occur simultaneously,
- a *relational* structure in which these aspects begin to be related one to another,
- an *equivalence* where two-way relationships reveal the same general structure expressed in different ways,

and

- a *crystalline concept* where all these equivalent ideas are seen as different aspects of the same underlying conceptual entity.

For instance, in Euclidean geometry, the van Hiele levels follow this pattern as an overall arch, with level one (perception) seeing figures as a gestalt wholes, level two (description) describing properties of individual figures that occur at the same time, then level three (definition) introduces verbal definitions that enable figures to be constructed by ruler and compass constructions and level four (Euclidean proof) proves theorems that are built into a coherent framework. At an even higher level, the figures may be conceived as platonic objects that have many properties all of which are interrelated by Euclidean proof.

I do not find the idea of straight-jacketing conceptual development as a sequence of discrete stages that must be taught in sequential order to be very helpful. Human learning is more complex than that. The natural development of the brain over time involves making new mental links as successful neural connections are strengthened and previously disparate ideas are connected together to resonate in more stable structures. This means that, even though one may not insist on a rigid sequence of discrete stages, there is an underlying neurological development in the brain of each of us that offers the possibility of increasingly compressed and connected knowledge structures.

In this way, it is possible to see a broad spectrum of development of a structure of knowledge where a *unistructural conception* involves a broad sense of a structure as a whole, which may result in the reporting of some isolated property in a test. A *multistructural conception* involves a range of properties that can be described but are not yet linked together. As these properties begin to be distinguished and verbalized, they can be used to describe situations in an increasingly *relational* manner. As I see it, the final *extended abstract* development expressed in the SOLO taxonomy can now be seen to grow out of the relational stage, first where two-way relationships lead to the recognition of *equivalent* concepts in which specific instances are seen as different examples of the same general concept. There is then the possibility of a more compressed stage in which these equivalences are seen to represent an underlying *crystalline* structure, be it a platonic figure, a procept or a formally defined concept.

In my own approach to learning (for instance in a "sensible approach to calculus", Tall, 1985, 2010), rather than an APOS approach in which the derivative is seen as the limit of encapsulating a process as an object, I see a global embodied view of the graph and use the human sense of moving the eye along the graph to *see* the changing slope and record its value as a new graph. To do this I first introduce the idea of a locally straight function f as one which, when its graph is magnified, looks less curved until, at suitably high magnification, it continues to look straight and so has a specific numeric slope $(f(x+h)-f(x))/h$ which, for small values of h stabilises to be seen as the *slope function.* I denote the slope function of f as Df, where the D is the operation of looking along the graph to see the stabilized slope graph. This starts with an *object* — the graph of a (locally straight) function f — and produces a new *object*, the graph of Df. The problem now is not to prove that the new object 'exists' as it can be *seen*, embodied as an object on a visual display. The problem is to *compute* it numerically as accurately as desired or, better still, symbolically as a formula. For instance, if I look along the graph of $f(x) = x^2$, then the slope function has the formula $2x + h$ and for small values of h, it stabilizes on the slope function $Df(x) = 2x$.

The important thing here is not the search for an elusive philosophical limit concept that is only potentially realizable as a potentially infinite limiting process, but as an imaginative mental object—the rate of change of the function f drawn as Df.

More generally, I see mathematical activity operating at many different levels, in different cultures, at different levels of sophistication. In introducing the term "crystalline concept", I seek common ground that has relevance across a wide range of theories.

Relationship with other theories

My original submission for this paper was interpreted by one reviewer as being "driven by an idea of universalizing truth where mathematical statements refer to something absolute — a substantialist idea of mathematics [...] written within the well-known rhetoric of formalizing epistemologies that recent works in anthropology, sociocultural approaches, and ethnomathematics have proven as partial and biased." This reviewer is welcome to express a personal opinion of my purpose; however, this opinion also carries its own form of bias and misconstrues my intentions.

My personal experiences come from being a mathematician who became interested in how people think mathematically and who has worked with children and students of all ages in a vast range of situations and cultures around the world. I care very much about how individuals in different circumstances work to make sense of mathematics. I see locally held opinions expressed strongly that have a compelling sense of validity. These include the use of a particular context of practical mathematics that works in a given situation.

I do not personally hold a Platonist view of mathematics as being independent of human thought, because I can see how Platonism arises naturally from human thought processes as the child matures in imagination by focusing on particular properties of space and shape, to imagine concepts such as points with position but no size, or lines as length without breadth. I see regular mathematical ideas building naturally from sensori-motor foundations with its dual function of making sense of *perceptions* through conceptual embodiment and learning to build *operations* out of motor skills such as counting and manipulating symbols to construct more compressed forms of mathematical thinking. Later I see new forms of mathematics based on linguistic skills and logical forms of argument that develop new ideas that are no longer the same as the "natural" ideas of our perceptions and actions, yet remain based in human mathematical thought.

Underlying any mathematics, in any situation, are structures that arise which operate in ways that are a consequence of the context. As soon as anyone begins to see a pattern or a necessary consequence of a mathematical situation, they are beginning to sense an underlying structure, perhaps one that is not yet even articulated.

I remember very clearly a lecture given by John Mason in which he expressed a beautiful idea that continues to resonate in my mind not just as a phrase but as the intention of his sincerity of expression to describe a precise but elusive idea. He spoke of a learner, before being able even to articulate an idea, having "a *sense* of a concept". He invested the word 'sense' with powerful emotion expressing a profound awareness of the possibilities in the situation.

I do not believe that the notion of a crystalline concept is something that only occurs at stratospheric heights of platonic existence. I do hypothesize that anywhere that an individual has a sense of properties that are constrained by a given situation then they are experiencing an awareness of the underlying patterns and consequences of mathematics. It is this deeper structure that I term a crystalline concept that "has an internal structure of constrained relationships that cause it to have necessary properties as a consequence of its context."

One may ask if the underlying product is *crystalline* (having certain platonic properties) or *crystallized* (through the thinker constructing the concept in the mind). Here we return to the dilemma as to whether mathematics is invented or discovered.

The response of the practical mathematician, Christopher Zeeman, reveals a pragmatic solution: some aspects are *invented* to be able to formulate and study a particular problem and then other aspects are *discovered* because of the crystalline nature of mathematics. It is therefore both crystallized by the individual and shared as a crystalline concept by appropriately sophisticated members of a particular community, be they expert mathematicians or young children exploring a sense of pattern.

Reflections

In this journey we see ideas generated from perceptions of the complications of our real world, through the unistructural gestalt observations of sensual ideas, the multistructural observations of situations with many properties, a relational stage of forging links between properties, then a stage of equivalence in which different instances are seen as having the same general properties and finally, mathematical thinkers arrive at crystalline concepts that seem to have a perfection of their own. These concepts are related to their context, be it in Euclidean or non-euclidean geometry, be it in arithmetic or algebra, or in the formal mathematics of axiomatic definition and deduction, or in any other aspect of mathematical endeavor.

What is essential seems to be that each of us goes through a process of maturation in which perceptions and actions are formulated in increasingly sophisticated ways that are partly chosen by us to study and then have a crystalline structure dictated by the context. The strong inferential links in crystalline concepts give mathematics its pristine clarity and inevitability.

At the same time, with the compression of complicated structures into concepts as single entities, thinking processes are essentially *simplified* to give subtle, rich, thinkable concepts that can be easily manipulated in the mind using the wealth of flexible relationships in the particular context. For individuals who can focus on the underlying simplicity, mathematics becomes more powerful and more connected. For individuals who remain with the detail without compression, it can soon become increasingly complicated and even impossible to grasp its increasingly disparate aspects.

For those of us who teach mathematics to learners, whether we see it as our purpose to introduce them to the wonders of mathematics or to inspire them to discover mathematics by their own efforts, we surely need to encourage them to think in ways that gives them power in operation and pleasure in success.

This involves not only being aware of their current development and how they might profit by exploring new ideas in ways that are appropriate for them at the time, but also to seek a broader understanding of the crystalline structures of mathematics itself.

Notes

1. This paper was stimulated by the discussion in the preparation of a chapter on the cognitive development of proof (Tall *et al.,* to appear). I wish to thank my co-authors for their contributions: Boris Koichu, Walter Whiteley (who provoked the idea of crystalline concept through his analysis of the concept of isosceles triangle), Margo Kondrieteva, Ying-Hao Cheng and Oleksiy Yevdokimov.

2. Such theories include those of Piaget (1970), Bruner (1966), Dienes (1960), Fischbein (1987), the SOLO Taxonomy of Biggs and Collis (1982), theories of process-object encapsulation found in the work of Dubinsky (Dubinsky & MacDonald, 2001), Sfard (1991), Gray and Tall (1994), the van Hiele levels in geometry (Van Hiele, 1986), the embodied linguistic theory of Lakoff and Núñez (2000), advanced mathematical thinking (Tall, 1991), and the nature of mathematical research (*e.g.* MacLane, 1994).

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