The effectiveness of lesson study on mathematical knowledge for teaching

Nellie C. Verhoef, David O. Tall

University of Twente, Faculty of Behavioral Sciences, Department of Teacher Training, The Netherlands

University of Warwick, Mathematics Education Research Centre, United Kingdom

Those who can, do. Those, who understand, teach (Shulman, 1986, p.4)

Abstract

Keywords: Mathematical knowledge for teaching; Lesson study; Long-term development of mathematical thinking; Derivative

1. Introduction

The aim of the present study is to investigate the effectiveness of lesson study on the mathematical knowledge for teaching. Lesson study is a long-established practice of lesson preparation that has been in existence for one hundred and thirty years. It is a teaching improvement and knowledge building process that has origins in Japanese elementary education (Lewis, 2002). Japan and Asian countries are in contrast with the results of European countries in that they occupy numbers one, two, three and four in the ranking list of the Trends in International Mathematics and Science Study (TIMMS, 2007). This reports a remarkable percentage of students in Asian countries that reached the Advanced International Benchmark for mathematics, representing fluency on items involving the most complex topics and reasoning skills (Olson, Martin & Mullis, 2008). A salient difference with European countries is that these students work to develop the understanding of mathematics so that their success is not only maintained, but improved (Tall, 2008b). European governments establish guidelines for teaching and learning approaches that are controlled in more or less directive ways. As a consequence teachers are focused on preparing for the exams. Individual teachers may reflect on and improve their practice in the isolation of their own classrooms. The complexity of their daily work rarely allows them to converse with colleagues about what they discover about teaching and learning (Cerbin & Kopp, 2006). If it is possible to share their ideas about teaching and learning, it likely takes the form of knowledge they develop from their experiences in the classroom (Verhoef & Terlouw, 2007). This type of knowledge seems immediately useful, but it tends to be tied to concrete and specific contexts and is not always in a form that can be accessed and used by others (Hiebert, Gallimore & Stigler, 2002). This suggests that practitioner knowledge should preferably be made public, shareable, and verifiable so that it may become professional knowledge. It is recommend to improve teaching practice in their fields, leading to the formation of a professional knowledge base.

1.1 Mathematical knowledge for teaching

Mathematical knowledge for teaching is strongly related to Pedagogical Content Knowledge (PCK) which "identifies the distinctive bodies of knowledge for teaching. It represents the blending of content and pedagogy into an understanding of how particular topics, problems or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (Shulman, 1987, p.4). The work of Shulman and his colleagues expanded ideas about how knowledge might matter to teaching, suggesting that it is not only knowledge of content but also knowledge of how to teach such content that conditions teachers' effectiveness (Borko, Eisenhart, Brown, Underhill, Jones & Agard, 1992; Bruner, 1960; Grossman, 1990; Leinhardt & Smith, 1985; Magnusson, Krajcik & Borko, 1999; Schwab, 1961/1978; Wilson, Shulman & Richert, 1987).

Ball (1990; 1991) conceptualized PCK as mathematical knowledge for teaching. This type of knowledge allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems. Research findings indicate that teachers' mathematical knowledge for teaching positively predicted student gains in mathematics achievement during the first and third grade (Hill, Rowan & Ball, 2005; 2008). Moreover, this helps envision a new generation of process-product studies designed to answer questions about how teachers' mathematical behavior – in particular their classroom explanations, representations, and interactions with students' mathematical thinking – might affect student outcomes. If successful, efforts to improve teachers' mathematical knowledge through content-focused professional development and pre-service programs may work to improve student achievement, as intended.

The German COACTIV project specially concentrated on secondary mathematics teacher's PCK (Krauss, Baumert & Blum, 2008). They found that expert teachers view mathematics as a process and believe that it should be learned by means of self-determined active discovery including reflecting on one's errors (Baumert & Kunter, 2006). A solid basis of content knowledge appears to facilitate the construction of PCK.

These research findings indicate that mathematical knowledge for teaching develops in collaboration with colleagues as practical mathematics education experts as well as staff members of the university as mathematics science experts. This justifies the choice of a research approach based on the model of lesson study.

1.2 Lesson Study

A collaborative research framework generally in practice develops through the process of lesson study, which emphasizes teacher's leadership for learning and improving teaching, interaction between students in the classroom, and students' individual needs and learning differences (Matoba & Sarkar Arani, 2006; Matoba, Shibata, Reza & Arani, 2007). The approach of lesson study is typical small, but professionally scaled. A small team working together to design, teach, study, and refine a single class (research) lesson. This work culminates in at least two tangible products: (a) a detailed, usable lesson plan, and (b) an in-depth study of the lesson that investigates teaching and learning interactions, explaining how students responded to instruction, and how instruction might be further modified based on the evidence collected (Cerbin & Kopp, 2006). The design of that research lesson intended to "bring the goals to life" by modifying an existing lesson or starting anew (Lewis, 2000). The designing process is a process of learning because of the teachers consider how they will help students achieve the goals (Wiggins & Mc Tighe, 1998). In planning a research lesson, teachers predict how students are likely to respond to specific questions, problems and exercises. The primary focus of lesson study is not what students learn, but rather how students learn from the lesson. The framework of long-term mathematical thinking will be used to categorize aspects of students' learning processes (Tall & Mejia-Ramos, 2009). The starting point of this approach is the selecting of a course, topic and goals for student learning followed by a research lesson that addresses immediate academic learning goals (e.g., understanding specific concepts and subject matter) and broad goals for development of intellectual abilities, habits of mind and personal qualities.

This implies that a lesson study approach would be a start to make visible student's learning processes in a framework of long-term mathematical thinking - that means open to observation and analysis.

1.3 Long-term mathematical thinking

In the seventies Skemp (1976) distinguished relational understanding and instrumental understanding. Both of them had their own advantages and disadvantages. Hiebert and Carpenter (1992) distinguished procedural and conceptual thinking. The Dutch Van Hiele (1986) used comparable categorization in geometry. He formulated successive levels of thinking. Each one builds upon the previous one, but is more sophisticated and involves significant changes in meaning, so that the language used at the next stage is subtly different from the previous one, as the concepts become reorganized and more sophisticated (Verhoef & Broekman, 2005). He continued by saying that it is possible to have a different level in approach to algebra if it is formulated in terms of axioms and definitions (Van Hiele, 2002). In arithmetic and algebra, there is a change in level of thought that arises from a shift in thinking as operations at one level become mental objects at another. The idea of an operation that takes place in time becoming a thinkable concept that exists outside of a particular time and place is an ongoing theme in a wide range of theories (Dubinsky & McDonald, 2001; Gray & Tall, 1994) Additionally, Gray and Tall (1991) introduced the term *procept* to refer to the dual use of symbolism as process and concept in which a process (such as counting) is compressed into a concept (such as number), and symbols such as 3+2, 3a+2b, f(x), dy/dx operate dually as computable processes and thinkable concept. This framework has been developed into what Tall (2006) described as three mental worlds of mathematics: (i) the conceptual-embodied world (based on perception of and reflection on properties of objects); (ii) the proceptual-symbolic world that grows out of the embodied world through actions (such as counting) and symbolization into thinkable concepts such as number, developing symbols that function both as processes to do and concepts to think about (called procepts); and (iii) the axiomatic-formal world (based on formal definitions and proof) which reverses the sequence of construction of meaning from definitions based on known concepts to formal concepts based on set-theoretic definitions. In a pilot study this framework will be used to describe student's mathematical thinking.

The mathematical concept that will be focused on is the derivative at first, because of the general upper level high school curriculum program.

1.4 Derivative

Calculus in school is a blend of the world of embodiment (drawing graphs) and symbolism (manipulating formulae). According to Tall (2003), the *cognitive root* of the notion of derivative is ~~the~~ *local straightness*. The property of *local straightness* refers to the fact that, if we focus close enough on a point of a function curve (a point at which the function is differentiable) then this curve looks like a straight line. Actually, this 'straight line' is the tangent line of the curve at this point. This property is valid in all cases of tangent lines and its understanding could be facilitated by the use of technology (Giraldo & Calvalho, 2006; Habre & Abboud, 2006). On the other hand the early experiences of the circle tangent contribute to the creation of a *generic tangent* as a line that touches the graph at one point only and does not cross it (Vinner, 1991). Students conflicts are related to rates of changes and graphically to slopes (Kidron, 2008; Zandieh & Knapp, 2006). Furthermore, students perceive not generally valid properties related to the number of common points or the relative position of the tangent line and the graph as defining conditions for a tangent line. Different combinations of these properties create intermediate models of a tangent line (Biza, 2007; Biza, Christou & Zachariades, 2006). The derivative only grows from embodiment to symbolism through progressive process-object compression, and to formalism through definition and deduction (Inglis, Mejia-Ramos & Simpson, 2007). There is also a possible development from definition and deduction using definitions based on mental embodiments of the concepts or on symbolic manipulations that are performed (Godfrey & Thomas, 2008). Focusing on the relationship between the initial stages of the student's long-term thinking process and teacher's interventions result in the research question: *What is the effectiveness of lesson study on mathematical knowledge for teaching?*

2. Method

2.1 Participants

Five upper level high school mathematics teachers from different schools and five staff members of the University of Twente participated in a lesson study team. The university members consisted of one from the department of Applied Mathematics, a mathematics teacher trainer, a coordinator of school stages, a PhD of Lesson Study and a researcher of mathematical teacher education. The male school teachers tagged by capitals A, B, C, D and E indicated to be interested in personal professionalization.

A (age 56) attained a bachelor's certificate mathematics and a master's mathematics teacher educational certificate. He worked at a mathematics teacher for 17 years with lower level to upper level high school students. His mathematics school team consisted of 13 teachers. The goal of the team was to attain reliable mathematics education. A's school tried to realize collaboration with regional Christian schools.

B (age 48) attained a bachelor's certificate mathematics and a master's certificate mathematics teacher education. He worked as a mathematics teacher from 1988 mostly with upper level high school students. B's math school team consisted of eight teachers (none of them with a master's mathematics teacher educational certificate). The goal of the team was to improve students' arithmetic skills. B's school aspired to create a deeper sense of the meaning of education.

C (age 48) attained a bachelor's certificate technique. He worked as a staff member of the university of Twente for seven years. He attained his master's mathematics teacher educational certificate. He worked as a mathematics teacher with mostly upper level high school students recently (August 2009). His mathematics school team consisted of four teachers (all of them with a master's mathematics teacher educational certificate). The goal of the team was to inspire students for mathematics. C's school tried to stimulate school participation of parents.

D (age 55), with a PhD Chemistry, attained a master's mathematics teacher educational certificate. He worked as a mathematics teacher with mostly upper level high school students from August 2008. D did not have a mathematics school team of teachers. D's school aspired to implement human collaboration.

E (age 35) attained a master's certificate applied mathematics and a master's mathematics teacher educational certificate. He worked as a mathematics teacher for 10 years, mostly with upper level high school students. His mathematics school team consisted of 12 teachers (three of them with a master's mathematics teacher educational certificate). The goal of the team was to link students' mathematics with science. E's school focused on student centered education.

2.2.1 Preparation for Lesson Study

Each participant was given a research paper to study and to present the ideas to their colleagues in a seminar. Teacher A got the paper of 'Student Perspectives on Equation: The Transition from School to University', from Godfrey and Thomas (2008). Teacher B got the paper 'Exploring the Role of Metonymy in Mathematical Understanding and Reasoning: The Concept of Derivative as an Example', from Zandieh and Knapp (2006). Teacher C got the paper 'The Transition to Formal Thinking in Mathematics', from Tall (2008). Teacher D got the paper 'Abstraction and consolidation of the limit procept by means of instrumented schemes: the complementary role of three different framework', form Kidron (2008). Teacher E got the paper 'Students' conceptual understanding of a function and its derivative in an experimental calculus course', from Habre and Abboud (2006).

2.2 Data gathering instruments

Pretest. The pretest protocol contained three main questions:

(1) Teachers' goals of mathematics education contrasting the understanding of mathematical concepts using visual, graphical or numerical representations with the use of procedures to solve problems using algorithms and calculations,

(2) the objectives of the teaching method *at the start* of the instruction subdivided into abstract concepts and procedures to solve abstract problems using recognition, exact calculations and reasoned arguments; or situated worked examples with meanings and contexts,

(3) aspects to give attention in relation to the concept of the derivative.

Posttest. The posttest protocol contained three main questions:

(1) Teachers' goals of mathematics education contrasting the understanding of mathematical concepts using visual, graphical or numerical representations with the use of procedures to solve problems using algorithms and calculations,

(2) the objectives of the teaching method *at the start* of the instruction subdivided into abstract concepts and procedures to solve abstract problems using recognition, exact calculations and reasoned arguments; or situated worked examples with meanings and contexts,

(3) aspects to give attention in relation to the concept of the derivative.

All the teachers' textbooks were partly based on a situational instructional approach. The school management allowed them three hours free time weekly to participate in this lesson study team. Their commitment was to professionalize themselves by designing one lesson in collaboration with others in the research as well as taking part in selected research activities.

2.3 Material

Participants designed an observation and an evaluation list, research activities to describe and to evaluate students' mathematical thinking processes with regard to the introduction of the derivative. The executed lessons were video-taped. This developmental process was characterized in the approach of lesson study. The expectation was that this designing process in collaboration with colleagues at school and the lesson study team would stimulate teachers' mathematical knowledge for teaching. The observation and evaluation lists were based on the framework of long-term mathematical thinking. The formulations were typified in terms related to Skemp's (1976) instrumental and relational understanding and Tall's (2008b) embodied and symbolic worlds. The results of the observation and evaluation lists were discussed with colleagues at school and the lesson study team.

2.4 Procedure

The worked out pretest was used to match teachers with the same goals into two subgroups. A student assistant worked out the recorded lesson study team meetings at the university. Each video-taped lesson at school was observed and evaluated by one school colleague and two staff members of the university. The posttest was finalized by an exit semi-structured interview. Only three of the original five participating teachers A, B and C, remained till the last meeting. The other two teachers D and E were not given release time by their school management for this later activity.

2.5 Data analysis

Reliability. The pretest and the posttest results were labeled into goal statements, chosen teaching methods at the start, and aspects in relation with the concept of the derivative. The posttest results were finalized by exit interview questions about teacher's personal professionalization, subdivided into: (1) scientific literature; (2) three weeks lesson study team meetings with discussions at the university; or (3) school classroom practices (Johnson & Johnson, 1999).

3. Results

The teachers prioritize the goal statements given on a scale of 1 to 12. Afterwards they declared that their middle choices were arbitrarily. As a consequence only teachers' extreme characteristic goal statements were gathered in Table 1. The first column represents the scale numbers of teachers' priorities. The second column represents teachers' choices of goals of mathematics education. Not-italic goal statements in the cells belong to the pretest. Italic goal statements in the cells belong to the posttest. Teacher A executed the first lesson, teacher B executed the second lesson and teacher C executed the third lesson in the lesson study approach.

Table 1

Teacher's characteristic goal statements with high or low priorities

Priority	Goal statement		
	Teacher A	Teacher B	Teacher C
1	Structures as a basis for thinking *To learn to understand mathematical concepts*	To learn procedures to solve problems *To apply mathematical techniques in practice*	To learn to understand mathematical concepts *Structures as a basis for thinking*
2	To be able to relate mathematical concepts *Structures as a basis for thinking*	To apply mathematical techniques in practice *To learn to argue deductively (stepwise)*	To be able to relate mathematical concepts *To learn to argue deductively (stepwise)*
3	To learn to argue deductively (stepwise) *To learn to argue inductively (not stepwise)*	To learn to argue deductively (stepwise) *To learn procedures to solve problems*	Structures as a basis for thinking *To learn procedures to solve problems*
4	To learn to argue inductively (not stepwise) *To learn to argue deductively (stepwise)*	To be able to execute calculations correctly *To solve realistic practical situations*	To learn procedures to solve problems *To be able to execute calculations correctly*
9	To be able to execute calculations correctly *To learn procedures to solve problems*	To learn to understand abstract concepts *To learn to argue inductively (not stepwise)*	To use the graphic calculator adequately *Axioms and definitions as a starting point to learn*
10	To use the graphic calculator adequately *To use the graphic calculator adequately*	Axioms and definitions as a starting point to learn *Axioms and definitions as a starting point to learn*	To use computer applications *To solve realistic practical situations*
11	To use computer applications *To use computer applications*	To use computer applications *To use computer applications*	To solve realistic practical situations *To use computer applications*
12	To learn procedures to solve problems *Axioms and definitions as a starting point to learn*	To learn to argue inductively (not stepwise) *To use the graphic calculator adequately*	Axioms and definitions as a starting point to learn *To use the graphic calculator adequately*

Table 1 indicates that the teachers emphasize the same choices of goals of mathematics education in the pretest and the posttest in general. All of them indicate low priorities to the use of computer applications and the use of the graphic calculator (GRM).

Teacher A emphasizes understanding as the goal of mathematics education. A's highest priority - 'to learn to understand mathematical concepts' - in the posttest was based on the observation that students did not understand the concept of the tangent as an essential component of the derivative. A prioritizes low 'axioms and definitions as a starting point to learn' in the posttest based on the evaluation that his students did not have any comment in practice.

Teacher B emphasizes procedures to solve problems as the goal of mathematics education. B's highest priority in the posttest - 'to apply mathematical techniques in practice' was based on the evaluation that the students were not being able to solve problems correctly. The use the GRM was in-adequate. B prioritizes low the use of the GRM in the posttest

Teacher C emphasizes understanding as the goal of mathematics education. C's highest priority in the posttest - 'structures s a basis for thinking' - was based on the evaluation that students were not being able to relate the assumed algebraic and graphical representations. C had prohibited the use of the GRM based on the experiences of B. C prioritizes low the use of the GRM in the posttest.

The observations and evaluations make teachers aware of assumed thinking steps which students did not make in the practice. The teachers prioritizes high 'to learn to argue' as a goal of mathematics education.

The teachers prioritize the start of instruction to attain goals on a scale from 1 to 8. Afterwards they declared that their middle choices were arbitrarily. As a consequence only teachers' extreme characteristic statements related to the start of instruction were gathered in Table 2. The first column represents teachers' priorities. The second column represents teachers' choices of a teaching method. Not-italic statements belong to the pretest. Italic statements belong to the posttest. Teacher A executed the first lesson, teacher B executed the second lesson and teacher C executed the third lesson in the lesson study approach.

Table 2

Teacher's characteristic statements of start of instruction with high or low priorities

Priority	Start of instruction		
	Teacher A	Teacher B	Teacher C
1	Start with different examples *Start with a realistic worked example*	Start with a practical worked example *Start with different examples*	Start with different examples *Start with different examples*
2	Start with the result of a practical situation *Start with different examples*	Start with different examples *Start with representations of a concept*	Start with worked examples *Start with worked examples*
3	Start with a realistic worked example *Start with the result of a practical situation*	Start with a thinking model as a referential framework *Start with a practical worked example*	Start with representations of a concept *Start with representations of a concept*
6	Start with a thinking model as a referential framework *Start with axioms and definitions*	Start with axioms and definitions *Start with worked examples*	Start with the result of a practical situation *Start with a realistic worked example*
7	Start with axioms and definitions *Start with a thinking model as a referential framework*	Start with representations of a concept *Start with a general abstract concept*	Start with a realistic worked example *Start with the result of a practical situation*
8	Start with a general abstract concept *Start with a general abstract concept*	Start with a general abstract concept *Start with axioms and definitions*	Start with axioms and definitions *Start with axioms and definitions*

Table 2 indicates that teachers emphasize the same choices of a start of instruction in the pretest and the posttest in general. All of them indicate a high priority to a start with different examples. They indicate a low priority to a start with axioms and definitions.

Teacher A's highest priority - 'start with a realistic worked example' - in the posttest was based on the observation that students did not make any comment in practice. A prioritizes low 'start with a general abstract concept' in the posttest.

Teachers B’s highest priority in the posttest ‘start with different examples’ was based on the evaluation, that students were not being able to use the same techniques in different cases. B prioritizes low ‘start with axioms and definitions’ in the posttest.

Teacher C does not change his priorities. He prioritizes high ‘start with different examples’ and low start ‘with axioms and definitions’ based on his experiences in practice.

The observations and evaluations make teachers aware of a difference between the assumed and the actual present knowledge.

Teachers’ characteristic remarks about aspects of the derivative were gathered in Table 3. The first column represents the choices of a teaching method. The rows are related to goals of mathematics education. Not-italic remarks in the cells belong to the pretest. Italic remarks in the cells belong to the posttest. Teacher A executed the first lesson, teacher B executed the second lesson and teacher C executed the third lesson in the lesson study approach.

Table 3

Teacher's characteristic remarks about aspects of the derivative

Understanding			
Teaching method	Teacher A	Teacher B	Teacher C
Abstraction	Average and instantaneous degree of change Relation between velocity, acceleration and distance Graph: table, function, symbol; mathematics and physics *Slope of a graph* *Differential quotient, difference quotient* *Interval, max/min*	Average and instantaneous degree of change Concept of the limit *Degree of change – average change* *Tangent line on the graph* *Slope of a graph* *Differential quotient, difference quotient* *The derivative of a function*	Mathematical derivation in general *Degree of change – average change* *Tangent line on the graph* *Relation between the graph and the formula has to be understood* *Concept of the limit at the end* *Local: tangent line in one point, work-around*
Procedures to solve problems			
Situated worked examples	To draw an equation of the tangent To draw differential quotient, difference quotient *To draw an equation of the tangent line*	Help to get more information about functions Different notations Addition of practical issues *What to do whit the derivative? To calculate the slope and the extreme values*	Applications in practice, relation with physics and economy, also min/max Rules of calculation, trick To draw an equation of the tangent line

Table 3 indicates that teachers emphasize aspects of the derivative with regard to goals of understanding with the use of procedures to solve problems in relation with abstraction and situated worked examples as a start of instruction to attain these goals.

Teacher A's remarks in the posttest, concentrate on the mathematical sense of the derivative.

Teacher B's remarks about the concept of the limit only limits to the pretest. B emphasizes calculations and applications.

Teacher C indicates the concept of the limit at the end in the posttest. C emphasizes the mathematical sense of the derivative with the focus on next deductive mathematical concepts.

All of the teachers concentrate on the tangent line in the posttest. None of them associate the concept of the derivative in a non-mathematical context. They indicate general meanings of the derivative in the pretest. They consider velocity as an application of the derivative. The results show teacher's awareness of the introduction of the derivative in a pure mathematical context.

The impact of the lesson study approach, including three different executions of the research lesson, make teachers aware of students' mathematical thinking processes. The results of the field notes show that students are not being able to relate algebraic and graphical representations of the tangent. Students do not make any comment in the first two lessons. As a consequence the observation and evaluation lists were unusable. The teachers realize that they do not be able to stimulate students to think about loudly. They are not being able to create time to discover and to puzzle. They realize that their textbooks consist of assignments in a stepwise deductive approach. Teachers are aware that this approach do not stimulate students'' reflection processes. The textbook assignments hinder students to explore mathematical thinking. The teachers choose alternative research instruments in the third lesson. They design a question list consisted of written questions with regard to the concept of the tangent line, subdivided into a focus on the formula and the graph.

The final exit semi-structured interview results into personal preferences of professionalization. The priority differs between: the delivered literature (teacher A), the discussions in the lesson study team (teacher B) and the classroom practice (teacher C).

In summarize the results of this study show that teachers' mathematical knowledge for teaching developed in teachers' goal priorities during the pilot period in a lesson study context. Teacher's awareness with regard to student's thinking processes has grown in relation with mathematical sense. Teacher A focused on understanding concepts as a goal of mathematics education. A did not indicate axioms and definitions as a starting point to learn. A wanted to introduce a mathematical concept using realistic worked examples. A emphasized mathematical sense based on scientific literature. Teacher B focused on applying mathematical techniques in practice as a goal of mathematics education. B wanted to introduce the derivative using different examples. B did not want to introduce the derivative using axioms and definitions. B emphasized calculations and applications based on discussions in the lesson study team. Teacher C focused on structures as a basis for thinking as a goal of mathematics education. C discovered the relevance of deductive reasoning. C did not want to introduce the derivative using axioms and definitions. C emphasized the mathematical sense of the derivative with the focus on next deductive mathematical concepts based on classroom practice. The teachers emphasized practical tips to improve lessons. They indicated an increase enjoyment in teaching.

4. Discussion

The results of the study show that teachers label the use of the computer and the use of the graphic calculator as a goal of mathematics with low priority. They underpin their choices by typifying the use of a computer in their classrooms as a teaching aid and not as a goal in itself. This underpinning is related to the situation with a great difference between financial possibilities of school organizations. Most of them have sufficient computer facilities, but these facilities are not equal. Some of the school organizations have classrooms with enough computers and smart boards, other school organization only have one classroom with computer facilities as a result of the school policy. Every school is free to make a choice. Each Dutch student has a graphic calculator. The use of graphic calculators is completely integrated in the Dutch mathematics curriculum.

However, the graphic calculator is of limited value in exploring ideas in the calculus with its button-pressing interface and its need to make changes in various inputs in different windows to set up a required interaction. There is also the need for the participants to have some experience of ways of using information technology to link embodied ideas with the symbolism and formalism of calculus and analysis. The calculator offers the symbolic derivative of a function given as a combination of standard functions. However, the underlying problem is to link the perceptual ideas of slope and local straightness with corresponding symbolic ideas calculating numeric and symbolic slopes and the formal idea of the limit.

The goal of the lesson study approach was to uncover students' thinking processes about the concept of the derivative. Teachers, not familiar with terms based on research literature, designed observation and evaluation lists to attain this goal. They categorized in collaboration the students' possibly statements into terms of Skemp's (1976) instrumental or relational understanding and Tall's (2008a) embodied and symbolic worlds. In total three lessons occurred in the lesson study approach.

Firstly, teacher A used an applet with the intention to demonstrate local straightness as being most meaningful to understand the derivative. After A's short introduction A concentrated on the ratio $\Delta y/\Delta x$ with the intention to connect with the textbook. His lesson was crowded because of his enthusiasm and his intended goals. He deleted students' possible barriers in advance by integrating these barriers in his act. As a consequence A's students did not have any questions. Teacher A was convinced of students' understanding of local straightness. A, B and C supposed student's possibly not relational understanding of the tangent line after A's lesson evaluation.

Secondly, teacher B decided to focus on the concept of the tangent line before introducing the derivative. B started by activating students. Each student was given a squared graph of $y=x^2$ on squared paper and were asked to draw a tangent at a point that was not placed on a crossing of the gird line. As a consequence, the tangent lines they drew were slightly different and gave small differences in the numerical slope of the tangent. B’s plenary discussion focused on the concept of the tangent, but also ended in the ratio dy/dx, because of his strict textbook guidelines. Once again, the students did not ask any questions. Teacher B was convinced of students’ understanding of the tangent. Because of not happened student’s questions,

C ended the lesson study approach by introducing written questions to try to expose students’ thinking processes in his version of the lesson in the lesson study approach. Teacher C had a bad history in the lesson study approach. C’s two colleagues in the subgroup were gone. These two teachers did not get time from their school management. Afterwards C joined A, and B. C focused on the formula of the tangent line and $\Delta x \to 0$. He had the following concept of the limit in his mind without naming the limit concept itself. The observers noted that students were not amazed at all when their practical approach to the tangent produced different tangent lines with different slopes as compared with the graphic calculator that produced a single formula. C realized that his written questions were unclear: he asked instrumental understanding and he expected answers based on relational understanding. The resulting answers to these questions proved to be difficult to analyze. A, B, and C’s developmental thinking processes ended in amazing and passion for mathematical knowledge for teaching.

Finally, the results of this study show that experienced teachers have developed their teaching methods in balance with textbook guidelines. Experienced teachers tend to teaching methods which they are familiar with, executed in their colleagues’ groups. External stimuli, like scientific literature, discussions in a lesson study team and reflection on classroom practices, are aware of students’ learning processes. The lesson study approach only realizes awareness as a first developmental step of mathematical knowledge for teaching. Long-term mathematical development is a balance between students’ learning processes and teachers’ teaching processes. Teacher’s leaning methods need to be based on a framework for human growth of knowledge that respects both the learner and the mathematical ideas (Masami & Tall, 2007). Curriculum rules and assessment methods limit teachers to develop their mathematical knowledge for teaching.

The question arises as to the source of these difficulties. In this experiment, the participating researchers encouraged the teachers to read the literature and to form their own opinions and design their own lesson. Although the teachers empathized with a 'locally straight' approach to the calculus, they all gave a low priority to the use of computers in learning. They tried to interact with the students in any activity to zoom in on a graph. They therefore missed the opportunity to take enough time to encourage the students to focus on the essential idea that a differentiable function gets less curved as one zooms in until it is visually straight. With this essential idea, the students might have been encouraged to 'look along the graph' to *see* its changing slope and to conceptualize the graph of the changing slope as the derivative of the function. Such an approach would involve a different lesson sequence in which experience of local straightness and the investigation of the changing slope of standard functions could lead to the need to compute the derivative in a precise symbolic way that naturally leads to the limit concept. The teachers' perceived need to carry out the specifics of the existing curriculum meant that an alternative approach was not incorporated.

Japanese Lesson Study requires more than the design of a single lesson, it requires a coherent approach to a series of lessons. This study reveals the significance of the complex reality of school practice in reference with the powerful claim of curriculum guide lines, study guides based on textbooks, and the attaining of high exam results.

More research in a lesson study approach in the context of complex school practices are needed to investigate teachers' individual professionalization.

References

Ball, D. L. (1990). The mathematical understanding that prospective teachers bring to teacher education. *Elementary School Journal, 90,* 499-466.

Ball, D. L. (1991). Teaching mathematics for understanding: What do teachers need to know about subject matter? In M. Kennedy (Ed.), *Teaching academic subjects to diverse learners* (pp. 63-83). New York: Teachers College Press.

Baumert, J., & Kunter, M. (2006). Stichwort: Professionelle Kompetenz von Lehrkräften. *Zeitschrift für Erziehungswissenschaft, 9*(4), 469-520.

Biza, I. (2007). Is the tangent line tangible? Students' intuitive ideas about tangent lines. In D. Kuchemann (Ed.), *Proceedings of the Conference of the British Society for Research into the Learning of Mathematics* (Vol. 27(1), pp. 6-11). London.

Biza, I., Christou, C., & Zachariades, T. (2006). Students' thinking about the tangent line. In J. Novotna, H. Moraova, M. Kratka & N. Stehlikova (Eds.), *Proceedings of the 30th PME International Conference,* Vol. 2, pp. 177–184. Prague, Czech Republic.

Borko, H., Eisenhart, M., Brown, C. A., Underhill R. G., Jones, D., & Agard P. C. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education, 23*(3), 194-222.

Bruner, J. (1960). *The process of education.* Cambridge, MA: Harvard University Press.

Cerbin, W., & Kopp, B. (2006). Lesson Study as a model for building pedagogical knowledge and improving teaching. *International Journal of Teaching and Learning in Higher Education, 18* (3), 250-257.

Crowley, L. & Tall, D. (1999). The roles of cognitive units, connections en procedures in achieving goals in college algebra. In: *Proceedings of the 23rd conference of PME, Israel, 2,* 225-232.

Dubinsky, E., & McDonald, M.A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In Derek Holton et al. (Eds.), *The teaching and learning of mathematics at University level: An ICMI study* (pp. 273-280). The Netherlands: Kluwer.

Giraldo, V., & Calvalho, L. M. (2006). Generic organizer for the enrichment of the concept image of derivative. In J. Novotna, H. Moraova, M. Kratka & N. Stehlikova (Eds.), *Proceedings of the 30th PME International Conference* (Vol. 3, pp. 185–192). Prague, Czech Republic.

Godfrey, D., & Thomas, M.O.J. (2008). Student perspectives on equation: the transition from school to university. *Mathematics Education Research Journal, Vol. 20,* 2, 71-92.

Gray, E. M. & Tall, D. O. (1994). Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetic, *Journal for Research in Mathematics Education, 26*(2), 115–141.

Grossman, P. L. (1990). *The making of a teacher: Teacher knowledge and teacher education.* New York: Teachers College Press.

Kidron, Y. (2008). Abstraction and consolidation of the limit precept by means of instrumented schemes: the complementary role of three different frameworks. *Educational Studies of Mathematics, 69*, 197-216.

Habre, S., & Abboud, M. (2006). Students' Conceptual Understanding of a Function and Its Derivative in an Experimental Calculus Course. *Journal of Mathematical Behavior, 25*(1), 57-72.

Hiebert, J. & Carpenter, T. P. (1992), Learning and Teaching with Understanding. In D. Grouws (Ed.), Handbook of Research on Mathematics Teaching and Learning (pp. 65–97), MacMillan, NY.

Hiebert, J., Gallimore, R., & Stigler, J. (2002). A knowledge base for the teaching profession: What would it look like and how can we get one? *Educational Researcher, 31*(5), 3-15.

Hill, C., Rowan, B., & Ball, D.L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement, heather. *American Educational Research Journal, 42*(2), 371-406.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372-400.

Inglis, M., Mejia-Ramos, J.P., & Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics, 66*(1), 3-21.

Johnston, D., & Johnston, R. (1999). *Learning together and alone: cooperative, competitive, and individualistic learning.* Boston: Allyn and Bacon.

Kidron, Y. (2008). Abstraction and consolidation of the limit precept by means of instrumented schemes: the complementary role of three different frameworks. *Educational Studies in Mathematics, 69*, 197–216.

Krauss, S., Baumert, J., & Blum, W. (2008). Secondary mathematics teachers' pedagogical content knowledge and content knowledge: validation of the COACTIV constructs. *ZDM, 40*(5), 873-892.

Leinhardt, G., & Smith, D.A. (1985). Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology, 77*(3), 247-271.

Lewis, C. (2000). *Lesson study: The core of Japanese professional development.* Paper presented at the American Educational Research Association Meeting, New Orleans.

Lewis, C. (2002). *Lesson study: A handbook of teacher-led instructional improvement.* Philadelphia: Research for Better Schools, Inc.

Magnusson, S., Krajcik, J., & Borko, H. (1999). Nature, sources, and development of pedagogical content knowledge for science teaching. In J. Gess-Newsome & N. G. Lederman (Eds.), *Examining pedagogical content knowledge* (pp. 95-132). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Masami, I., & Tall, D. O. (2007). Long-term development of Mathematical Thinking and Lesson Study. Prepared as a chapter for a forthcoming book on Lesson Study.

Matoba, M. & Sarkar Arnani, M.R. (2006). Ethnography for teachers' development: Japanese approach of investigation on classroom activities. In N. Popov, C, Wolhutter, C. Heller, & M. Kysilka (Eds.), *Comparative education and teacher training* (pp. 116-125). Sofia: Bulgarian Comparative Education Society.

Matoba, M., Shibata, Y., Reza, M., & Arani, S. (2007). School-university partnerships: a new recipe for creating professional knowledge in school. *Educational Research for Policy and Practice, Vol 6*(2), 55-65.

Olson, J.F., Martin, M.J., & Mullis, I.V.S. (Eds.). (2008). TIMMS 2007 Technical Report. Chestnut Hill, MA: TIMMS & PRILS International Study Centre, Boston College, http://timms.bc.edu/TIMMS2007/techreport.html

Schwab, J. (1978). *Science, curriculum, and liberal education: Selected Essays.* Chicago: University of Chicago Press. (Original work published in 1961).

Shulman, L.S. (1986). Those who understand, knowledge growth in teaching, *Educational Researcher, 15*(2), 4-14.

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. Harvard *Educational Review, 57*, 1-22.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77,* 20–26.

Tall, D. O. (Ed.). (1991). *Advanced Mathematical Thinking.* Dordrecht, Holland: Kluwer.

Tall, D. O. (2003). Using Technology to Support an Embodied Approach to Learning Concepts in Mathematics. In L. M. Carvalho & L. C. Guimaraes (Eds.), *Historia e Tecnologia no Ensino da Matematica,* (Vol. 1, pp. 1–28), Rio de Janeiro, Brasil.

Tall, D. O. (2006). *A life-time's journey from definition and deduction to ambiguity and insight*. Retirement as Process and Concept: A Festschrift for Eddie Gray and David Tall (pp. 275-288). Prague. ISBN 80-7290-255-5.

Tall, D. O. (2008a). The Transition to Formal Thinking in Mathematics. *Mathematics Education Research Journal, 20*(2), 5-24.

Tall, D. O. (2008b). Using Japanese Lesson Study in Teaching Mathematics. *Scottish Mathematical Council Journal, 38,* 45-50.

Tall, D.O., Smith, D., & Piez, C. (2008c). Technology and Calculus. In M. K. Heid & Gl. M. Blume (Eds.), *Research on Technology and the Teaching and Learning of Mathematics, Volume I: Research Syntheses*, 207-258.

Tall, D. O, & Mejia-Ramos, J.P. (2009). The Long-Term Cognitive Development of Different Types of Reasoning and Proof. In G. Hanna, H.N. Jahnke & H. Pulte (Eds.), *Explanation and proof in mathematics: Philosophical and educational perspectives.* New York: Springer.

Van Hiele, P. M. (1986). *Structure and Insight.* Orlando: Academic Press.

Van Hiele, P.M. (2002). Similarities and differences between the theory of learning and teaching of Richard Skemp and the van Hiele levels of thinking. In D. Tall & M. Thomas (Eds.), *Intelligence, learning and understanding in mathematics. A tribute to Richard Skemp.* Post Pressed, Flaxton (Australia).

Verhoef, N.C., & Broekman, H. G. B. (2005). A process of abstraction by representations of concepts, Volume 4 (pp. 273-281). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*. Melbourne, Australia.

Verhoef, N.C., & Terlouw, C. (2007). Training mathematics teachers in a Community of Learners (CoL), Volume 4 (pp. 256-264). *Proceedings of the 7th Europian Society for Research in Mathematics Education.* Cyprus: Greece.

Vinner, S. (1991). The role of definitions in the teaching and learning of Mathematics. In D. O. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 65-81). Dordrecht, The Netherlands: Kluwer.

Wiggins, G., & McTighe, J. (1998). *Understanding by design.* Alexandria, VA: Association of Supervision and Curriculum Development.

Wilson, S. M., Shulman, L. S., & Richert, A. E. (1987). '150 different ways' of knowing: Representation of knowledge in teaching. In J. Calderhead (Ed.), *Exploring teachers' thinking* (pp. 104-124). London: Cassell.

Zandieh, M, & Knapp, J. (2006). Exploring the role of metonymy in mathematical understanding and reasoning. The concept of the derivative as an example. *Journal of Mathematical Behavior, 25,* 1- 17.