WHO INVENTED DIRAC'S DELTA FUNCTION?

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Abstract. The Dirac delta function has solid roots in 19th century work in Fourier analysis by Cauchy and others, anticipating Dirac's discovery by over a century.

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1. Introduction

The specialisation of the scientific disciplines since the 19th century has led to a schism between the scientists' pragmatic approaches (e.g., using infinitesimal arguments), on the one hand, and mathematicians' desire for formal precision, on the other. It has been the subject of much soul-searching, see, e.g., S.P. Novikov [18].

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In an era prior to such a schism, a key interaction between physics and mathematics was foreshadowed in a remarkable fashion, exploiting infinitesimals (see Appendix C), in Cauchy's texts from 1827 (see Section 3).

2. Dieudonné's question

J. Lützen [17] traces the origins of distribution theory (traditionally attributed to S. Sobolev and L. Schwartz), in 19th century work of Fourier, Kirchhoff, and Heaviside. For an accessible introduction to this area, see J. Dieudonné [5].

Dieudonné is one of the most influential mathematicians of the 20th century. A fascinating glimpse into his philosophy is provided by his review of J. Lützen's book. At the outset, Dieudonné poses the key question:

> One [...] may well wonder why it took more than 30 years for distribution theory to be born, after the theory of integration had reached maturity.

This remark is a reflection of a pervasive myth, to the effect that the physicists invented the delta function, and the theory of distributions legalized them after the fact many years later. Thus, M. Bunge, responding to Robinson's lecture *The metaphysics of the calculus*, evoked the physicist's custom of

> refer[ring] to the theory of distributions for the legalization of the various delta 'functions' which his physical intuition led him to introduce [21, p. 44-45]; [22, p. 553-554].

Meanwhile, D. Laugwitz [13, p. 219] notes that probably the first appearance of the (Dirac) delta function is in the 1822 text by Fourier [6].

In his review of J. Lützen's book for Math Reviews, F. Smithies notes:

> Chapter 4, on early uses of generalized functions, covers fundamental solutions of partial differential equations, Hadamard's "partie finie", and many early uses of the delta function and its derivatives, including various attempts to create a rigorous theory for them.

At the end of his review, Smithies mentions Cauchy: "In spite of the thoroughness of his coverage, [Lützen] has missed one interesting event—A. L. Cauchy's anticipation of Hadamard's 'partie finie' ", but says not a word about Cauchy's delta functions.

3. Cauchy's delta function

Dieudonné's query, mentioned at the beginning of section 2, is answered by Laugwitz, who argues that objects such as delta functions (and their potential applications) disappeared from the literature due to the elimination of infinitesimals, in whose absence they could not be sustained. Laugwitz notes that

> Cauchy's use of delta function methods in Fourier analysis and in the summation of divergent integrals enables us to analyze the change of his attitude toward infinitesimals [13, p. 232].

A function of the type generally attributed to P. Dirac (1902–1984) was specifically described by Cauchy in 1827 in terms of infinitesimals. More specifically, Cauchy uses a unit-impulse, infinitely tall, infinitely narrow delta function, as an integral kernel. Thus, in 1827, Cauchy used infinitesimals in his definition of a "Dirac" delta function [2, p. 188]. Here Cauchy uses infinitesimals α and ϵ, where α is, in modern terms, the "scale parameter" of the "Cauchy distribution", whereas ϵ gives the size of the interval of integration. Cauchy wrote [2, p. 188]:

> Moreover one finds, denoting by α, ϵ two infinitely small numbers,

$$\frac{1}{2}\int_{a-\epsilon}^{a+\epsilon} F(\mu)\frac{\alpha\, d\mu}{\alpha^2+(\mu-a)^2} = \frac{\pi}{2}F(a) \tag{3.1}$$

(Cauchy's 1815-1827 text is analyzed in more detail in Appendix A). This passage from Cauchy is reflected in secondary literature (see Laugwitz's 1989 paper *Definite values of infinite sums* [13, p. 230]). The expression

$$\frac{\alpha}{\alpha^2+(\mu-a)^2}$$

(for real α) is known as the *Cauchy distribution* in probability theory. The function is called the probability density function, and the parameter α is called the *scale parameter*.

Laugwitz notes that formula (3.1) is satisfied when $\epsilon \geq \alpha^{1/2}$ (as well as for all positive real values of $\epsilon > 0$). Cauchy's formula extracts the value of a real function F at a real point a by integrating F against a kernel given by a (Cauchy-)Dirac delta function. Furthermore, Laugwitz documents Cauchy's use of

> an explicit delta function **not** contained under an integral sign [13, p. 231] [emphasis added—authors],

contrary to a claim in Dieudonné's text.[1] Such an occurrence of a delta function in Cauchy's work is discussed in Appendix B.

Felix Klein points out that the

> naïve [perceptual] methods always rise to unconscious importance whenever in mathematical physics, mechanics, or differential geometry a preliminary theorem is to be set up. You all know that they are very serviceable then.

On the other hand, Klein is perfectly aware of the situation on the ground:

> To be sure, the pure mathematician is not sparing of his scorn on these occasions. When I was a student it was said that the differential, for a physicist, was a piece of brass which he treated as he did the rest of his apparatus [12, p. 211].

Additional remarks by Klein, showing the importance he attached to this vital connection, may be found in Appendix D.

4. Heavyside function

Dieudonné's review of Lützen's book is assorted with the habitual, and near-ritual on the part of some mathematicians, expression of disdain for physicists:

> However, a function such as the Heavyside function on $\mathbb{R}$, equal to 1 for $x \geq 0$ and to 0 for $x < 0$, has no weak derivative, in spite [sic] of its very mild discontinuity; at least this is what the mathematicians would say, but physicists thought otherwise, since for them there *was* a "derivative" δ, the Dirac "delta function" [the quotation marks are Dieudonné's—authors].

Dieudonné then proceeds to make the following remarkable claims:

> Of course, there was before 1936 no reasonable mathematical definition of these objects; but it is characteristic that they were never used in *bona fide* computations except *under the integral sign*,[2] giving formulas[3] such as
>
> $$\int \delta(x-a)f(x) = f(a).$$

[1] See footnote 2.

[2] This claim is inaccurate; see main text at footnote 1.

[3] Here we have simplified Dieudonné's formula in [5, p. 377], by restricting to the special case $n = 0$.

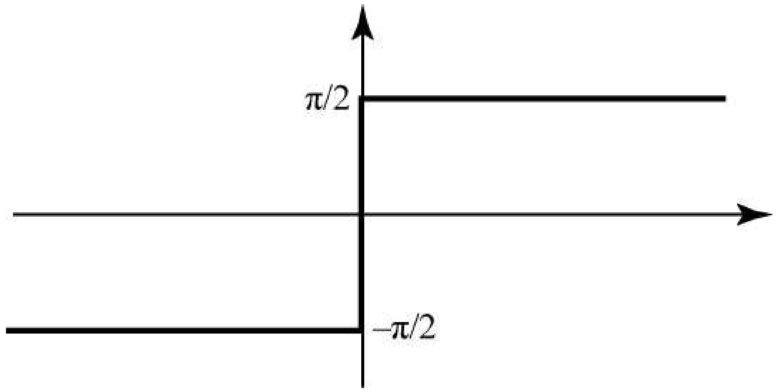

FIGURE 4.1. Heavyside function

Are Dieudonné's claims accurate? Dieudonné's claim that before 1936, delta functions occurred only under the integral sign, is contradicted by Cauchy's use of a delta function *not* contained under an integral sign, over a hundred years earlier (see Appendix B).

Are the physicists so far off the mark, mathematically speaking? Is it really true that there was no reasonable mathematical definition before 1936, as Dieudonné claims?

Consider the zig-zag $\mathcal{Z} \subset \mathbb{R}^2$ in the (x, y)-plane given by the union

$$\mathcal{Z} = \left(\mathbb{R}_- \times \left\{-\tfrac{\pi}{2}\right\}\right) \cup \left(\{0\} \times \left[-\tfrac{\pi}{2} \tfrac{\pi}{2}\right]\right) \cup \left(\mathbb{R}_+ \times \left\{+\tfrac{\pi}{2}\right\}\right),$$

thought of as the physicist's Heavyside function, see Figure 4.1.

Now consider the graph of $\arctan(x)$ in the (x, y)-plane, and compress it toward the y-axis by means of a sequence of functions $\arctan(nx)$, or $\arctan(x/\alpha)$ where $\alpha = \frac{1}{n}$. Their derivatives $F_\alpha(x)$ satisfy

$$\int_{-\infty}^{\infty} F_\alpha = \pi$$

by the fundamental theorem of calculus. In an infinitesimal-enriched continuum (see Appendix C), we can consider an infinitesimal α. Then the graph

$$\Gamma_{\arctan(x/\alpha)}$$

of $\arctan(x/\alpha)$ is "appreciably indistinguishable" from the zigzag $\mathcal{Z}$.[4] Instead of attempting to differentiate the zigzag itself with the physicists, we differentiate its infinitesimal approximation $\arctan(x/\alpha)$, and note that we obtain precisely Cauchy's delta function appearing in formula (3.1), against which F is integrated.[5]

5. Conclusion

The customary set-theoretic framework that has become the reflexive litmus test of *mathematical rigor* in most fields of modern mathematics (with the possible exception of the field of mathematical logic) makes it difficult to analyze Cauchy's use of infinitesimals, and to evaluate its significance. We will therefore use a conceptual framework proposed by C. S. Peirce in 1897, in the context of his analysis of the concept of continuity and continuum, which, as he felt at the time, is composed of infinitesimal parts, see [8, p. 103]. Peirce identified three stages in creating a novel concept:

> there are three grades of clearness in our apprehensions of the meanings of words. The first consists in the connexion of the word with familiar experience. . . . The second grade consists in the abstract definition, depending upon an analysis of just what it is that makes the word applicable. . . . The third grade of clearness consists in such a representation of the idea that fruitful reasoning can be made to turn upon it, and that it can be applied to the resolution of difficult practical problems [19] (see [8, p. 87]).

The "three grades" can therefore be summarized as

(1) familiarity through experience;
(2) abstract definition aimed at applications;
(3) fruitful reasoning "made to turn" upon it, with applications.

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[4]In modern notation, this relation would be expressed by the fact that the standard part "st" of the graph $\Gamma_{\arctan(x/\alpha)}$ *is* the zigzag:

$$\mathrm{st}\left(\Gamma_{\arctan(x/\alpha)}\right) = \mathcal{Z} \subset \mathbb{R}^2.$$

Here the internal function $\arctan(x/\alpha)$ is the mathematical counterpart of the physicist's Heavyside function. Of course, Cauchy did not have the notion of a standard part function, to express the idea that an error term is infinitesimal. Instead, he used the expression *sensiblement nulle* (sensibly nothing), see [13, p. 231].

[5]Giorello [7], Lighstone and Wong [15], and later Péraire [20] have developed this theme further. Yamashita [26] provides a bibliography of articles dealing with non-standard delta functions.

To apply Peirce's framework to Cauchy's notion of infinitesimal, we note that grade (1) is captured in Cauchy's description of continuity of a function in terms of "varying by imperceptible degrees". Such a turn of phrase occurs both in his letter to Coriolis of 1837, and in his 1853 text [4, p. 35].[6] At Grade (2), Cauchy describes infinitesimals as generated by null sequences (see [1]), and defines continuity in terms of an infinitesimal x-increment resulting in an infinitesimal change in y. Finally, at stage (3), Cauchy fruitfully applies the crystallized notion of infinitesimal both in Fourier analysis and in evaluation of singular integrals, by means of a "Dirac" delta function defined in terms of a (Cauchy) distribution with an infinitesimal scaling parameter.

It emerges that, contrary to Dieudonné's claim, Cauchy did have a *reasonable mathematical definition* of a, Dirac, delta function. What was lacking is an explicit formalisation of an infinitesimal-enriched continuum where Cauchy's definition could be made operative.

Appendix A. Cauchy's Note XVIII

Cauchy's lengthy work *Théorie de la propagation des ondes à la surface d'un fluide pesant d'une profondeur indéfinie* was written in 1815. The manuscript was published in 1827 as a 300-page text, with a number of additional Notes at the end. The running title used throughout is *Mémoire sur la théorie des ondes*.

Note XVIII, entitled *Sur les intégrales définies singulières et les valeurs principales des integrales indeterminées*, starts on page 288. Cauchy recalls the notion of a singular definite integral, describing it in terms of an integrand that becomes "infinite or indeterminate". He continues by denoting by ε an "infinitely small number" (note Cauchy's use of term "number" rather than "quantity"), and by a, b two positive constants. On page 289, after choosing an additional "infinitely small number" α, Cauchy writes down the integral

$$\frac{1}{2}\int_{a-\epsilon}^{a+\epsilon} F(\mu)\frac{\alpha\, d\mu}{\alpha^2+(\mu-a)^2} = \frac{\pi}{2}F(a)$$

(already reproduced as formula (3.1) above), which he denotes by (2). Cauchy proceeds to point out that, since the integrand of his equation (2) is *sensiblement égale à zéro* [essentially equal to zero] for all values of μ *qui ne sont pas très rapprochées de* a [which are not too

[6]Note that both Cauchy's original French "par degrés insensibles", and its correct English translation "by imperceptible degrees", are etymologically related to *sensory perception*.

close to a], it follows that the integrals appearing in his earlier Note VI reduce to singular integrals determined by his equation (2).

Note XVIII then proceeds to discuss principal values and to offer alternative derivations of a number of earlier results, and is concluded on page 299.

Appendix B. Cauchy's 1827 *Mémoire*

An additional occurrence of a delta function occurs in Cauchy's brief 1827 text *Mémoire sur les développements des fonctions en séries périodiques* [3]. The text contains an (*a priori* doomed) attempt to prove the convergence of Fourier series under the sole assumption of continuity. What concerns us here is his, correct, use of infinitesimals at a certain stage in the argument. Cauchy opens his *mémoire* with a discussion of the importance of what are known today as Fourier series, in *a large number of problems of mathematical physics* [3, p. 12]. On page 13, Cauchy denotes by ε *un nombre infiniment petit* [an infinitely small number], and lets $\theta = 1 - \varepsilon$, and lets x be between 0 and $a = 2\pi$. On page 14, he points out that the expression

$$1 + \frac{1}{e^{-i(x-\mu)} - \theta} + \frac{1}{e^{i(x-\mu)} - \theta} \tag{B.1}$$

(his notation is slightly different) "will be essentially zero, except when μ differs very little from x". Note that the expression (B.1) appearing on Cauchy's page 14, does *not* occur under the integral sign (it was exploited as a kernel in the last formula on the previous page 13).

Cauchy then sets $\mu = x + iw$ and concludes that the integral will be essentially reduced to

$$f(x) \cdot \int_{\frac{-x}{\varepsilon}}^{\frac{2\pi - x}{\varepsilon}} \left(\frac{1}{1 + iw} + \frac{1}{1 - iw} \right) dw = 2\pi f(x)$$

(Cauchy writes a for 2π).

Appendix C. Rival continua

A Leibnizian definition of the derivative as the infinitesimal quotient

$$\frac{\Delta y}{\Delta x},$$

whose logical weakness was criticized by Berkeley, was modified by A. Robinson by exploiting a map called *the standard part*, denoted "st",

FIGURE C.1. Taking standard part

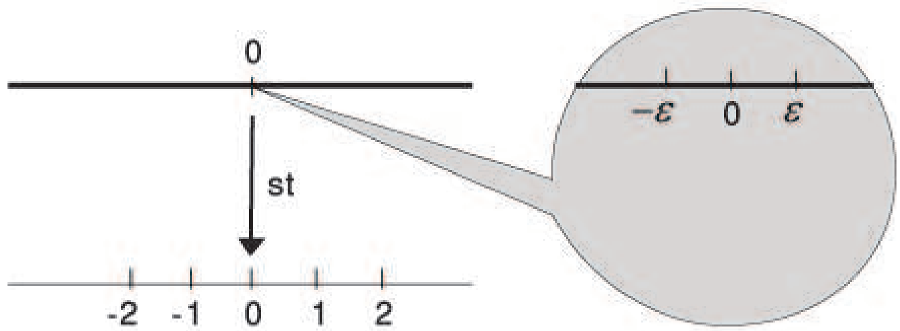

FIGURE C.2. Zooming in on infinitesimal ϵ

from the finite part of a B-continuum, to the A-continuum, as illustrated in Figure C.1.[7]

We illustrate the construction by means of an infinite-resolution microscope in Figure C.2.

We will denote such a B-continuum by the new symbol $\mathbb{R}$. We will also denote its finite part, by

$$\mathbb{R}_{<\infty} = \{x \in \mathbb{R} : \ |x| < \infty\}.$$

The map "st" sends each finite point $x \in \mathbb{R}$, to the real point $\mathrm{st}(x) \in \mathbb{R}$ infinitely close to x:

$$\begin{array}{c} \mathbb{R}_{<\infty} \\ \big\downarrow{\scriptstyle \mathrm{st}} \\ \mathbb{R} \end{array}$$

[7]In the context of the hyperreal extension of the real numbers, the map "st" sends each finite point x to the real point $\mathrm{st}(x) \in \mathbb{R}$ infinitely close to x. In other words, the map "st" collapses the cluster of points infinitely close to a real number x, back to x.

Robinson's answer to Berkeley's logical criticism (see D. Sherry [23]) is to define the derivative as

$$\mathrm{st}\left(\frac{\Delta y}{\Delta x}\right),$$

instead of $\Delta y/\Delta x$. For an accessible introduction to the hyperreals, see H. J. Keisler [10, 11].[8]

In addition, one can modify the map from $\mathbb{R}_{<\infty}$ to $\mathbb{R}$ using an arbitrary linear map $f(x) = ax + b$ where $a, b \in \mathbb{R}$, to 'reveal' not only distinctions between points that are an infinitesimal distance apart (e.g., points a and $a + \epsilon$) using the map $\frac{x-a}{\epsilon}$, but also the ability to 'see' in a finite real picture the distinction between any two points a, a', where $a - a'$ is finite, infinitesimal or infinite. This can be done using the map

$$m(x) = \mathrm{st}\left(\frac{x-a}{a'-a}\right)$$

defined on the 'field of view' including a and a', where the image $m(x)$ is finite. Thus the entire B-continuum $\mathbb{R}$ can be seen with detail of a given order of infinitesimal or infinity, with points differing by a lower order being represented as identical after applying "st", and points differing by a higher order being too far off to be visible. This allows one to visualize both asymptotic behaviour and infinitesimal behaviour, see Tall [24, 25].

Appendix D. Klein's remarks on physics

Here we present Klein's discussion of infinitesimal oscillations of the pendulum in [12, p. 187]. Klein presents the derivation of the pendulum law by pointing out that

> it follows from the fundamental laws of mechanics that the motion of the pendulum is determined by the equation $\frac{d^2\phi}{dt^2} = \frac{g}{\ell}\sin\phi$.

Here g is the gravitational constant, while ℓ is the length of the thread by which the pendulum is suspended, and ϕ is the angle of deviation from the normal. Klein continues:

[8] Note that both the term "hyper-real field", and an ultrapower construction thereof, are due to E. Hewitt in 1948, see [9, p. 74]. The transfer principle allowing one to extend every first-order real statement to the hyperreals, is due to J. Łoś in 1955, see [16]. Thus, the Hewitt-Łoś framework allows one to work in a B-continuum satisfying the transfer principle. More advanced properties of the hyperreals such as saturation were proved later, see Keisler [11] for a historical outline. A helpful "semicolon" notation for presenting an extended decimal expansion of a hyperreal was described by A. H. Lightstone [14].

> For small amplitudes we may replace $\sin\phi$ by ϕ without serious error. This gives for the so called infinitely small oscillation of the pendulum $\frac{d^2\phi}{dt^2} = \frac{g}{\ell}\phi$.

Klein proceeds to write down the general solution $\phi = C\cos\sqrt{\frac{g}{\ell}}(t-t_0)$, and points out that the duration of a complete oscillation, i.e., the period $T = 2\pi\sqrt{\ell/g}$, is independent of the amplitude C. Reflecting upon the teaching practices at the time, Klein muses over the incongruity of

> the curious phenomenon that one and the same teacher, during one hour, the one devoted to mathematics, makes the very highest demands as to the logical exactness of all conclusions. In his judgment [...] his demands are not satisfied by the infinitesimal calculus. In the next hour, however, that devoted to physics, he accepts the most questionable conclusions and makes the most daring applications of infinitesimals [12, p. 187].

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