METAPHOR OR MET-BEFORE?
THE EFFECTS OF PREVIOUS EXPERIENCE ON THE PRACTICE AND THEORY OF LEARNING MATHEMATICS

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ABSTRACT: While the general notion of 'metaphor' may offer a thoughtful analysis of the nature of mathematical thinking, this paper suggests that it is even more important is to take into account the particular mental structures available to the individual that have been build from experience that the individual has 'met before'. The notion of 'met-before' offers not only a principle to analyse the changing meanings in mathematics and the difficulties faced by the learner—which we illustrate by the problematic case of the minus sign—it can also be used to analyse the met-befores of mathematicians, mathematics educators and those who develop theories of learning to reveal implicit assumptions that support theoretical frameworks in some ways but act as impediments in others.

Keywords: Metaphor, met-before, epistemological obstacle, embodiment, local straightness.

1. INTRODUCTION

The notion of metaphor is used widely in mathematics education to denote the way in which we think about mathematics in terms of physical and mental actions. Here we hypothesize that the general idea of metaphor, especially when used to perform an intellectual analysis of how concepts are conceived, does not necessarily give a complete view of how students learn. We suggest that, rather than just a general consideration of the metaphors involved, it is more insightful to consider what the students bring to their learning, both in terms of previous experience that is supportive and previous experience that may be problematic.

To represent the student's current knowledge structures and the effect of earlier experiences on learning, Tall (2004) proposed the term 'met-before'. This started out as a word play to describe 'what the individual thinks *now* as a consequence of experiences that have been met before'. However, as we shall see, it also has theoretical implications that are somewhat different from the term 'metaphor', as used in cognitive science and in mathematics education. A met-before focuses not on a top-down intellectual analysis of the concepts concerned and the possible cognitive and philosophical origins, but on the way in which a student, or a mathematician, or a mathematics educator, may interpret ideas biased by their previous experience. Inappropriate use of personal met-befores can lead to subtle difficulties in learning for the student. More seriously, we suggest that theoretical frameworks proposed by mathematicians and mathematics educators may also be implicitly affected by the met-befores of those who formulate the theory.

This paper is in three main parts. After a preliminary consideration of various influences in learning, the first main part lays out the context and the formulation of the notion of met-before. The second includes a specific example with empirical data relating to the changing meaning of the minus sign as students follow through successive ideas in a college mathematics course, along with examples of other met-befores that affect learning negatively. Finally we consider the wider implications of the concept of met-before in the broader frameworks of cognitive science and mathematics education. We find that the term gives new insight into such debates as the Math Wars and other areas of dispute in mathematics education, in particular, how theoreticians' met-befores affect the nature of the theories that they propose.

1.1 Influences in learning

Ausubel (1968) wrote: "the most important single factor influencing learning is what the learner knows. Ascertain this and teach accordingly." New experiences that build on prior experiences are much better remembered and what does not fit into prior experience is either not learned or learned temporarily and easily forgotten. When instructors understand what students know and how they think—and then use that knowledge to make more effective instructional decisions—significant increases in student learning occur (Black & William, 1998). Students' ability to modify their prior knowledge has been examined and the discontinuities that are encountered have been documented, particularly at the elementary school levels (Skemp, 1987; Fischbein et al., 1985; Behr et al., 1992) and at the undergraduate level (Davis & McGowen, 2007; McGowen & Davis, 2001; Tall, 2009).

Existing knowledge may not be appropriate in a new situation and so the learner needs to adapt their approach to cope with new knowledge. We suggest that this need for adaptation is a major factor in causing a range of difficulties for students learning mathematics. Sowder (2000) described the enormity of the changes in thinking and adaptations necessitated as students move "from operating on whole numbers to operating on signed numbers and rational numbers (that is, fractions and decimal numbers) and from a primary focus on addition and subtraction to multiplication and division as well." Hiebert and Behr (1988) noted that recognition of how the nature of the unit changes requires a shift in thinking that is "a fundamental change with far-reaching ramifications: a change in the nature of the unit." Sowder argues that, “when teachers do not understand the significance of these subtle changes in how numbers are used, their students can become very confused” and lists three examples of this change:

- Students move from singleton units to composite units when they multiply; that is, what counts as a number changes: A set of things can now be thought of as one whole; I have two sets of three pencils; I have four six-packs of Coke.
- Students create new types of unit quantities when they divide: Dividing 30 cookies by 6 children yields 5 “cookies per child.”
- Units are partitioned to form fractions. A number, for example 1/3, is now part of a whole but can be thought of as a unit itself, so that it makes sense to talk about multiples of one-third, for example, “two one-thirds is two-thirds”.

These are all examples of new things that children need to construct as they attempt to comprehend new mathematical ideas. Curriculum designers and mathematics educators usually focus on the positive aspects of learning, specifying those things that are required knowledge for the next stage in the curriculum. The notion of met-before includes both positive aspects that support new learning and prior experiences that are problematic in new contexts.

2. MET-BEFORE

The notion of met-before was introduced to focus on how new learning is affected by experiences that the learner has met before (Tall, 2004). At first it was a joke. It started out with a restatement of the term ‘metaphor’ as the phrase ‘met afore’, using the old English word ‘afore’ to emphasise that a metaphor relates new knowledge (the ‘target’) in term of existing knowledge (the ‘source’) developed from previous experience, so that the new ideas can be related to familiar knowledge already in the grasp of the learner. The idea at first fell on stony ground as many of those who heard it could not make sense of two words that sound the same but are spelled differently. Then the new word ‘met-before’ was used as a replacement and suddenly the change in a single syllable, from metAphor to metBefore clarified what was to become a new way of looking at the effect of previous experience, particularly those which cause impediments to learning.

The notion of ‘epistemological obstacle’ has long been a focus of attention since it was introduced by Bachelard (1938) to describe how the progress of science could be blocked by existing cognitive beliefs. It was used by Brousseau (1983) to denote a particular piece of knowledge that prevents the acquisition of new knowledge. For example the concept of a function being given by an expression may act as an epistemological obstacle to the appreciation of the general set-theoretic function concept. The study of epistemological obstacles is well grounded in the literature. However, it focuses on *impediments* that occur as learning shifts to new contexts. We believe it would benefit from being situated within a broader theory that complements both positive and negative aspects.

It is a theoretical construct that is owned by mathematics educators and possibly by teachers, but it is hardly appropriate to discuss the notion of 'epistemological obstacle' with a young child. Our quest is to produce a framework of thinking in which theoreticians, teachers and learners can work together, hand in hand. For instance, while a teacher might not discuss the notion of 'epistemological obstacle' with a learner, it could be quite natural to ask, 'what have you *met before* that makes you think that?' It may then be possible to relate to an earlier positive experience where the idea worked to build confidence in reflecting on what needs to be changed to handle the new situation.

The term met-before applies to *all* current knowledge that arises through previous experience, both positive and negative. It can be given a working definition as 'a mental structure that we have *now* as a result of experiences we have met before.'

Met-befores can be *supportive* where old ideas are used in new contexts in ways that make sense. For instance, knowing that 2+2 makes 4 continues to work if one is adding apples, pencils, whole numbers, lengths in metres, weights in kilograms and even in more sophisticated cases such as $320+20=340$, $2a+3b+2a=4a+3b$ or even $2+5i+2=4+5i$.

The iconic function machine representation is often used in the early stages of the curriculum—usually as a "guess my rule" problem, to guess the internal formula expressing the rule. However, it can be a supportive met-before when used in a different way to retain greater generality through everyday examples with functions given by a procedure rather than a simple formula. It can help students to organize their thinking and clarify their understanding of mathematical operations and symbolic notation (such as finding the additive inverse versus subtraction or squaring a negative number versus forming the negative of a number squared). It can also be used to encourage students to consider the differences between mathematical processes such as evaluating an expression and the solving of an equation (McGowen, 2006).

The idea of a function machine as an input-output machine is supportive in focusing on the relationship between input and output, rather than the specific procedures that occur internally. In this case the two procedures "double the result and add six" and "add three, then double the result" are often seen as being different (as sequences of actions) by students. By focusing only on the relationship between input and output of a function machine, it can encourage students in realising that the expressions $2x+6$

and $2(x+3)$ are different ways of carrying out the same underlying process and so represent the same function. Even here, as we will discuss later, some students still have difficulties dealing with various aspects of these procedures (Tall, McGowen & DeMarois, 2000).

Met-befores can also be *problematic*, for instance the term 'difference' is often used without a sign, for instance, the 'difference between 7 and 3 is 4' and 'the difference between 3 and 7 is also 4', so 'difference' in this case means 'take the smaller from the larger'. In the child's first experiences in arithmetic, this causes no problems, but when the child is asked to take a smaller number from a larger in a case with two-digit numbers, the problem to take 27 from 43 may be written down as in column arithmetic as

$$\begin{array}{r} 43 \\ -\ 27 \\ \hline 24 \end{array}$$

In the first column the student has found the difference between the 4 and the 2 to be 2, and in the units column the difference between 3 and 7 is 4. Part of this phenomenon involves supportive met-befores, such as accurate knowledge of relationships between numbers. However, the major problem is the problematic met-before that the old idea of difference no longer works in this context. This is not simply a question of the learner 'making a mistake'. Here the learner is using a well-formed conception that to find the difference, one takes the smaller number from the larger. It works with whole numbers, but not with place value. The learner is now in a context requiring support and reason to comprehend a new situation, not simply an opportunity to be told to 'correct their mistakes' and move on the slippery path towards becoming insecure about mathematics.

It is also important to note that met-befores may be supportive in some contexts and problematic in others. For instance, the idea that 'take away leaves less' is supportive for whole numbers and (positive) fractions, it is supportive in the context of finite sets where even Euclid claims it as a common notion that 'the whole is greater than the part'. However, it is problematic in dealing with negative numbers and also in cardinal numbers for infinite sets. This emphasizes the need to consider the roles of met-befores in different situations where sometimes they are helpful and sometimes they are not.

2.1 Emotional aspects of met-befores

Met-befores give rise not only to positive and negative effects on learning, they also cause emotional reactions to learning situations. On the one hand, supportive met-befores give confidence in handling any context in which they work. They may give pleasure from success, or at the very least, become part of the unconscious functioning of mathematical activities that proceed routinely in the background while the learner concentrates on essential decision-making. Supportive met-befores enhance the chances of making sense of new ideas, increasing the possibility of achieving the goal of conceptual understanding.

Problematic met-befores, however, impede learning and can frustrate the learner in making sense of new ideas. As Skemp (1979) asserted in his goal-oriented learning theory, while goals are related to pleasure as they are achieved and frustration when they are elusive, if the goal becomes increasingly distant then the opposite emotional effect occurs. An anti-goal is formed, which is a situation to be avoided. Avoiding an anti-goal can bring relief but failure to avoid it leads to anxiety.

Students suffering from the effects of anti-goals have a very different experience in learning, which can turn into mathematics anxiety. One way of addressing this situation is to change the goal. Maintaining a goal that is unachievable can only lead to increasing frustration. Now an alternative goal may come to the fore. Instead of the goal of conceptual learning (knowing *why*), the goal may change to a more pragmatic procedural learning (knowing *how*).

Learning procedures by rote and being able to do them gives a new goal of learning to pass examinations, or being able to use the procedures for practical purposes in applications. If learning defaults to the goal of knowing how, it can be successful. However, if it is accompanied by a lack of conceptual meaning so that mistakes occur, it can become fragile and more likely to fail in the longer term. At this stage the problems may proliferate as the student becomes confused as to which rule to use, where to use it, and how to interpret it.

This latter condition often describes the state of students in college who return to study the mathematics that they failed to learn at school. They are already damaged by the frustrations of not being able to understand and the anxiety of not being able to avoid failure so that their knowledge structures have become increasingly fragmented and lacking in coherence.

The solution may not be simply for the teacher to be enthusiastic and work towards positive attitudes. Without addressing the problematic met-befores that remain under the surface, any chance of conceptual understanding is suppressed and the only way forward is to focus the student on the techniques required to succeed in performing correct methods of getting the answer. This leads to the widespread use of rote-learning that may give brief respite to get through routine examinations but it may also exacerbate the problem in the longer term.

In curriculum design, the focus is usually on supportive methods—teaching pre-requisites that are required in a supportive role in later learning. Less often is the role of problematic met-befores made explicit. If they are discussed at all, it is often in the role of 'misconceptions' and 'misunderstandings'.

Such terminology is not helpful. It suggests a deficiency in the thinking of the child that may cause negative feelings and possible anxiety. One cannot have a misconception without first having a conception. These met-befores are *pre*-conceptions in the sense that they are aspects that were experienced *before* the need for a new conception and were helpful in the earlier context. It is surely more productive to accentuate the positive (what worked before) and consider how to look at new ideas in a different way that is more appropriate for the new situation, rather than accuse learners of making errors when they need carefully directed support to help them attempt to make sense of complicated new ideas.

In the USA there is a profound and widespread fear of mathematics. Burns (1998) claims that almost two thirds of all American adults have a hatred and deep fear of mathematics. Even at college level, a study of over 9,000 American students found that one in four had a moderate to high need for help with their mathematical anxieties (Jones, 2001).

The proposals to address mathematical anxiety given by the NCTM (1989, p.233) encourage:

- Confidence in using math to solve problems, communicate ideas, and reason;
- Flexibility in exploring mathematical ideas and trying a variety of methods when solving problems;
- Willingness to persevere in mathematical tasks;
- Interests, curiosity, and inventiveness in doing math;
- Ability to reflect on and monitor their own thinking and performance while doing math;
- Focus on value of and appreciation for math in relation to its real-life application, connections to other disciplines, existence in other cultures, use as a tool for learning, and characteristics as a language.

These hugely worthy aspirations have a common element. They all accentuate the positive. Nowhere do they address the negative. Much of the research on anxiety in mathematics in the United States places the emphasis of mathematical anxiety on external factors: negative images of mathematics from teachers, parents and others, social deprivation, disturbing previous experiences in mathematics classes, poor teaching based on learning rules that are not understood, poor preparation for tests, anxiety at being asked to do mathematical problems in front of the class, fear of failure, poor self-image, poor memory (Tobias, 1978; Betz, 1978; Ashcraft & Kirk, 2001).

Notice that virtually all of these involve the *symptoms* of mathematics anxiety, not the deeper cause. In particular, there are rarely any references to problematic met-befores that may arise from the nature of the mathematics itself.

For this reason, the example that follows will often focus on problematic met-befores that contribute to the difficulties that students experience when encountering a new situation. It relates to a widespread problem that occurs as the student shifts from arithmetic, to algebra, and on to evaluating functions.

3. THE PROBLEMATIC DEVELOPMENT OF THE MINUS SIGN

The minus sign arises in mathematics with several different but related meanings. It is usually met first in childhood as "take-away" in practical situations such as '5 apples take away 3 apples leaves two apples', written later as $5 - 3 = 2$. In this context, '3 apples take away 5 apples' makes no sense, so that—as a met-before—this can cause a later obstacle when handling negatives. We have already noted that the word 'difference' is often used non-directionally, so that the difference between 2 and 5 is the same as the difference between 5 and 2, which is 3. The implied principle is that you always 'take the smaller from the larger.'

In another context, the minus sign may be used to indicate temperatures lower than zero, or on a (horizontal) number line it indicates values to the left of the origin. In these cases, the minus sign is part of the notation for the number; it always indicates a negative value. This met-before can cause a major problem in algebra where $-x$ is the additive inverse of x and, for negative numerical values of x, the value of $-x$ will then be positive. It proves to be problematic in college algebra courses where experience shows that, when students are asked what they think of when they see the minus sign, they will often first say, "subtraction," and then, "negative number." They rarely mention the new interpretation, "finding the additive inverse." Later, additional interpretations such as the inverse of a function contribute even more to their confusion.

This is illustrated by a post-test interview with a student, MD, that is typical of the belief held by many students, that a variable with a minus sign in front of it has a negative value.

I: What does it mean "to square" a number?
MD: Squaring...multiplying a number times itself, like –5 times –5.
I: What is negative five squared?
MD: Twenty-five. [She responds quickly and confidently.]
I: [showing the student a response she gave on a test problem: –25] What comes to mind?
MD: Square negative five.

I: How did you get the answer –25?
MD: Oh! On the calculator. I just entered the problem exactly as written.
I: You told me a few minutes ago that "to square" means multiplying a number times itself.
MD: Yes.
I: ... and that squaring negative five gives an answer.
MD: of 25. Yes.
I: If you square a negative number, what is the sign of the answer?
MD: (very quickly) Positive! It's always positive.
I: Then what process was used by the calculator to produce an answer of –25?
MD: Oh! Square five, then take the opposite of the answer.

[In the course attended by the students, the term 'opposite' was used to denote the negative inverse, with the intention of attempting to clarify the different meanings of 'negative'.]

I: What does it mean, "to take the opposite"?
MD: Change the sign of the number or answer.

The quickness of her responses and her confidence in her answers suggests that MD understands what it meant to square a number, whether it be positive or negative. However, when asked to interpret the meaning of $-f(x)$, she reveals the met-before of the minus sign denoting a negative number:

I: How do you interpret this? [writes down $-f(x)$].
MD: negative output. The answer is negative.
I: and [writes down $f(-x)$]?
MD: The input is negative.
I: How do you know the answer is negative? [Interviewer points at $-f(x)$].
MD: I don't—I just assumed it was negative. The minus sign is in front of f.
I: And in $f(-x)$?
MD: Negative, the input is negative.
I: How do you know…
MD: I just assumed it would be negative because the minus sign is in front of x.
I: [writes down –5 and $-x$.] and asks MD: Does it make a difference if the minus sign is in front of a number or in front of a variable?
MD: Being in front of a variable, it would be a negative answer. And negative five is just that, negative five.

MD was requested to take out her graphing calculator and was asked:

I: Use the Y1 = key and enter [writes down: $-3x$, thus avoiding a verbal interpretation].

[MD enters $-3x$, correctly using the opposite (–) key (as it was referred to in the course), not the subtract operator key on the TI-83 calculator.]

I: If you substitute 2 for x, what answer would the calculator display?

MD: (answering quickly) "negative six," then verifying her answer by substituting the value 2 for x.

I: And if you substitute negative one for x?

MD: (quickly) three, again verifying her answer by substituting the value for x on the calculator.

I: And if you substituted zero?

MD: Zero.

I: How did you get the answer negative six?

MD: I multiplied 2 by negative 3.

I: And the answer 3?

MD: I multiplied negative one by negative 3.

I: Would you review your answers displayed on the calculator? Reviewing your substitutions and the results, was the answer for $-3x$ always negative?

MD: No, only if x was a positive number—Oh! The minus sign doesn't always mean a negative answer!

3.1 LARGER VERSUS SMALLER

Students' prior arithmetic operational experiences include other met-befores that can cause difficulty when variables are introduced. These include the met-before of larger versus smaller (a) in the case of $2y$ versus y when y is a negative value and (b) subtracting variables, such as $x - y$ compared with $y - x$, when x is a positive value and y is a negative value.

Student responses to the following question (Bright & Joyner, 2003) provide additional evidence of ideas that were perfectly satisfactory in their original arithmetic context but are now recalled as met-befores that interfere with construction of new knowledge. Students were given several pairs of variables representing positive or negative values according to their respective positions on the number line pictured below and asked the following survey question (overleaf) to determine which had the greater value.

The numbers 0, 1, x, y, and $-z$ are marked on the number line below.

Figure 1: quantities on a number line

For each pair of numbers circle the number that has the greater value.
If the two numbers are equal, circle both numbers.

A.	y	$-y$	D.	$x-y$	$y-x$
B.	z	$-(-z)$	E.	$\lvert y\rvert$	$-y$
C.	$2y$	y	F.	$x+y$	$x-y$

A majority of 128 college freshmen recently surveyed claimed that $2y$ was larger than y (Part C). The two most common explanations for this choice were "$2y$ is greater because it is two times more than y" and "$2y$ because it has a number in front of the letter." Here the possible met-before is the idea that the product of whole numbers is generally bigger than either (except the isolated case when one of the numbers is 1). One student added "and because it has more variables" (again perhaps because the more (whole) numbers you multiply together, the more you get.)

Nearly one-third of the students responded that the two expressions $x-y$ and $y-x$ (Part D) were equal. Reasons given included: "same problem, just switched around," "because they are both subtracting a variable," and "because you don't know if the variable was an opposite or not." These responses may relate to the notion of 'difference' being independent of order for whole numbers or to the additional difficulty encountered with the procedure to subtract a quantity by changing its sign. Less than one-third of the students indicated that $x-y$ is greater than $x+y$ (Part F) and were able to provide a valid reason. The most common response was: "$x+y$ because the numbers were added," which may relate to the met-before with whole numbers that the sum is bigger than the difference. One out of every five students responded that "$x+y = x-y$." Reasons given for this choice included: "because adding a positive and a negative is the same as subtracting a positive and a negative number" and "because in both equations you are really adding the numbers." Note, in this case, that y is visibly negative, so subtracting a negative means changing its sign and adding the positive.

The student difficulties include not only evident met-befores such as 'difference is larger minus smaller', 'multiplication makes bigger', 'addition gives a bigger result than subtraction' and problems coping with the minus sign, there are also more complicated errors relating to mis-remembering rules learnt by rote. This is consistent with the hypothesis that changes in context lead to difficulties in making sense of the mathematics and consequent learning by rote may result in more fragile knowledge that is likely to break down. There may be a further unintended consequence. By failing to help students make sense of the new ways of thinking appropriate for a new context, we may give ourselves a proliferation of even more new difficulties that make later teaching and learning even more complicated.

3.2 Ambiguity of Language and Symbols

The difficulties that face the student in making sense of algebra are made more problematic by the need to handle several distinct conventions that are used, often implicitly, in interpreting algebraic symbols. The process of reading from left to right produces a particular met-before that is sometimes supportive, but more often problematic. This met-before is compounded by several distinct forms of ambiguity that students must deal with in the various conventions used in arithmetic and algebra. For instance the question, "What is 2 more than 3 times 4?" can be interpreted in two quite different ways, depending on how it is spoken or read. The student who considers the question as "What is two more than three (pause) times four?" is likely to respond "20." The student who considers it as "What is two more than (pause) three times four?" is more likely to answer "14." These ambiguities are addressed in algebraic notation using brackets. However, their meanings need careful linkage to everyday language.

A second form of ambiguity is that symbols in arithmetic and algebra have a dual use as evoking a process (such as addition in 2+3) or as a concept (the sum 2+3). Gray and Tall (1994) defined a symbol that operates dually as process and concept to be a *procept* and used the same term when different procedures give the same overall effect, so that $2x + 6$ and $2(x + 3)$ represent the same procept even though the procedures—'double a number and add 6', 'add 3 to a number and double the result'—are different.

The symbol -3 is an example of a procept that can be interpreted in different ways, depending upon the context. It could be interpreted as (1) the concept negative three or (2) the unary process of taking the additive inverse (a process requiring one input) or (3) the binary process of subtracting 3 from some unknown number (a process requiring two inputs). When used in combination with other operations, such as -3^2, the need to distinguish between "minus the square of three" and "the square of negative 3" lead to new conventions and more possibilities for confusion, as we will see in the next section.

3.3 Arithmetic knowledge of grouping symbols in algebra

In the application of concepts in arithmetic, a number of situations require flexibility to deal with both process-object ambiguity and notational ambiguity. The precedence of division and multiplication over addition and the precedence of powers over taking the additive inverse are frequently ignored by students who compute using the met-before of reading from left to right (or right to left in some languages). This order is such a strong met-before that knowledge of how grouping symbols change the order of operations may fail to make a strong impression on many learners.

A recent pre-test of 140 undergraduate students documented how met-befores from prior experience in arithmetic involving grouping symbols and the minus symbol becomes problematic in algebraic situations. Asked to evaluate $f(x) = x^2 - 3x + 5$ given $x = -3$, only 21 of 140 undergraduate students gave a correct response. Most errors in responses to this item were the result of the students' substitution of -3^2 and the subsequent evaluation of -3^2 as -9 instead of $(-3)^2$.

The failure to recognize that a negative value is being squared is so refined and stable that its selection and retrieval is automatic for many students, who wrote $f(-3) = -3^2 -3(-3) + 5$, followed by $-9+9+5=5$. Some students believe that they have "used up" the negative sign. One student, after writing $y = -3^2 -3(-3) + 5$ commented, "I have to do parentheses first." Beneath his initial work of $-3^2 -3(-3) + 5$, he wrote $9 + 5$. Pointing at the first term, -3^2, he said, "Now I have to do this but I can't remember if it's negative nine or just nine. I never can remember which to use." He wrote down -9 and stopped. "There's no sign in front of this (pointing at $9 + 5$), so I need to multiply," and wrote: $-9(14) = 136$.

The following two items reveal further problematic aspects when verbal statements read from left to right need translating into algebraic symbolism that requires strategic use of brackets to represent the operations.

Item A: Given the input/output machine, write an algebraic expression for this machine.

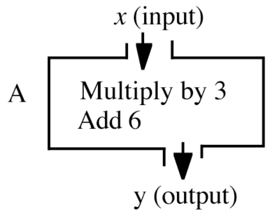

Figure 2: multiply by 3 and add 6

Item B: Given the input/output machine, write an algebraic expression for this machine.

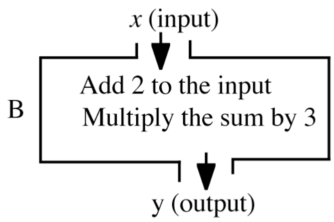

Figure 3: add 2 and multiply the result by 3

Answering both Items A and B correctly requires students to understand how the presence or absence of grouping symbols modifies the computation. Item B requires a bracket, but item A does not. On a post-test at the end of the course, 101 of the 140 students (72%) answered Item A correctly, but only 26 students (19%) correctly answered Item B. These results are similar to those found in an earlier study of developmental algebra students from four different community colleges where 71 students correctly answered Item A on the post-test but only 38 students correctly answered Item B (DeMarois, 1998),

4. THEORETICAL ASPECTS OF THE CONCEPT OF MET-BEFORE

The discussion so far has focused on the met-befores that cause students difficulty as they study college algebra and have the problem of coping with the minus sign and grouping symbols. We can all stand and nod our heads as we, the experts, offer our assistance to help students understand subtle changes in mathematical meaning. However, the notion of met-before has implications beyond the problems encountered by students.

It may seem initially as though the concept of met-before is nothing more than a study of epistemological obstacles using a misspelling of the word 'metaphor'. Nothing could be further from the truth. Research mathematicians, mathematics educators, cognitive scientists, and other participants in the learning and teaching of mathematics have their own met-befores, in addition to the students and teachers that they study. This affects the theories that are used to interpret student experiences.

For instance, in writing *Where Mathematics Comes From*, Lakoff and Núñez present a beautifully argued theoretical framework in which mathematics is embodied, building from fundamental sensori-motor actions of the human brain and body, constructing mathematical ideas through metaphors, starting from grounding metaphors based on sensori-motor human experience, forming linking metaphors to connect them together, and then re-definitional metaphors to specify formal mathematical concepts. It is a compelling theory and sheds new light on how humans are able to think mathematically.

However, two aspects weaken the argument. The first is the top-down nature of the 'idea analysis' proposed by the authors, which in practice refers to general metaphors in learning and offers intellectual reasoning from the viewpoint of linguistics, philosophy and cognitive science rather than the development of the learner. The second is that the authors themselves are part of this embodied cognition. Therefore, they too will have an embodied background and benefit or suffer from their previous experience, a factor that is not made explicit in their argument.

As an instance, consider the Lakoff and Núñez notion of the 'romance of mathematics' in which they carry out an idea analysis of mathematical thinking based on metaphors. These metaphors are based on an intellectual analysis of the four grounding metaphors of arithmetic (as object collection, object construction, measuring stick, motion along a path) and travelling through a succession of linking and re-definitional metaphors until they reach the 'formal reduction metaphor', which links sets and symbols to mathematical ideas (p.369). On the way they attack what they term 'the romance of mathematics' (p.338), which they see as a myth: that mathematicians live in a world of mathematics with various attributes including the idea that 'mathematical objects are real' and 'mathematics is universal, absolute and certain' (p.339). This is used as a weapon to contrast the 'beautiful story of mathematics' (p.340) with 'the sad consequences' (p.341) that 'it intimidates people. It makes mathematics beyond the reach of even intelligent students with other primary interests and skills. It leads many students to give up on mathematics as simply beyond them.'

This is a touching story and makes a powerful claim for the idea of embodied mathematics, which we share, resonating with the experience of so many mathematical learners who find difficulty with formal mathematics. However, the notion of met-before does not reveal the full picture. Another essential aspect is to study the met-befores of the various participants to understand not only 'where mathematics comes from' using an *intellectual* viewpoint, but where it comes from in the actual experience and cognitive development of every individual involved. This may give new insights, worthy of the highest ideals of mathematics education to see ourselves not only as part of the solution but also as part of the problem.

We begin by looking at the experiences of research mathematicians. They live in a world where they prove theorems that apply to *any* context in which the assumptions of the theorems are true. They do not rely on a single embodiment, but on a formal argument that applies to any embodiment already experienced and any others to come that satisfy the given axioms and definitions.

Formal mathematics as practiced by most research mathematics is a difficult area of struggle with complicated situations requiring deep reflection on ambiguities, conflicts and paradoxes (Byers, 2007) which in turn require the blending of conflicting ideas to produce emergent ideas that were not part of their earlier thinking (Fauconnier and Turner, 2002).

This involves inventing new formal definitions as a basis for formal arguments and the exploration of the logical consequences of these definitions. This exploration can lead to entirely new insights that the mathematician had never conceived before. In such circumstances, the mathematics ceases to feel as if it is their own creation. What they have found are new concepts that have such a purity and inevitability that they now take on a role of existence, as if they are beyond the personal sphere of the researcher and belong to the wider sphere of the mathematical community in what is, for them, the true 'romance' of mathematics.

This does not remove the thinking from the embodied mind of the mathematician to some platonic sphere; it remains embodied, but now the embodiment is based on logical deduction from formal definitions expressed using sophisticated language. Furthermore, as Hadamard (1945) noted, one of the developments of building theorems in a theory is the proof of specific *structure* theorems that act as 'staging posts' in further development. As Tall (2008) observes, these structure theorems, proved logically and verbally, lead to structures that again enable the mathematician to think in embodied terms that are now no longer naïve embodied constructions, but are now formally linked together in a logical manner.

It is a surprise to us that a book written in part by a linguist does not interpret the role of language in general and mathematical language in particular as a human endeavour that builds on embodiment to lead through formal mathematics to more sophisticated mathematical forms of embodiment. For it is just this form of thinking that leads to the creation of new theories that take us beyond our initial naïve conceptions as human beings. This 'romance', berated by Lakoff and Núñez, is the stuff of mathematical thinking that is shared by mathematicians with mathematical met-befores that they build on to produce new theories.

The consequences are evident. The met-befores of research mathematicians include the experience of manipulating mental entities that have a reality *for them* and lead to new constructs that seem too perfect to simply be invented by the instigator(s) of the new theory. That is their privilege. The 'sad consequence' is that mathematicians sometimes err by believing that their sophisticated met-befores have a form that can be shared by learners who lack their experience. It is argued vigorously in 'the Math Wars' that children should be taught logically how to make sense of mathematics, when educators are attempting to tailor their teaching to fit their perceived views of how children think. Mathematicians could do well to have some sympathy with the growing minds of children if they wish to encourage new generations of mathematicians to mature to continue their enterprise.

Meanwhile, educators have a responsibility to view more than the positive met-befores that are seen to be pre-requisites for learning new mathematics. It is essential that they also consider problematic met-befores that impede learning and enter into a dialogue with students to encourage them to develop new ways of working in a new context. Even better, it may be helpful to look at problematic met-befores in a positive light: that they operated satisfactorily in an earlier context to give the students confidence in their previous knowledge and to seek positive new ways of addressing the new situation.

The inventors of theories also need to reflect on their own met-befores. An example of an implicit met-before that apparently goes unnoticed in the book '*Where mathematics comes from'* is the notion of a 'naturally continuous function.' This has properties that can be embodied, such as the fact that its graph can be imagined as the motion of a point, it can be drawn physically as a continuous stroke of a pencil that always remains on the paper with no gaps, it moves smoothly, always with a tangent. Graphs that violate natural properties are 'monsters' (Lakoff and Núñez, p.307.)

The fact is that 'monsters' do not violate 'natural' ideas: they violate *met-befores*. They are problematic for individuals who have not developed the requisite knowledge structures to make sense of them. Broadly speaking, mathematicians before the introduction of axiomatic formalism at the end of the nineteenth century, most of today's students and teachers, and apparently certain thinkers in cognitive science, all share a common met-before. They studied calculus where the functions are initially given by formulae composed of standard functions that can be differentiated. Their total experience is of functions that are not only continuous, they are everywhere differentiable, except possibly at a few points that require ingenuity to think up a formula that might have different left and right derivatives, or some problem involving multiple oscillations, or some other monstrous property.

It is even difficult in analysis courses in university to get beyond relatively simple 'monstrosities' like $\sin(1/x)$ or $x\sin(1/x)$ at the origin, or oddities like $|\sin x|$ that has different left and right derivatives at multiples of π. So the met-before builds that 'most' functions are differentiable except, possibly at a finite number of exceptional points. Indeed, in most university courses, examples of functions in calculus are usually given, at worst, as piecewise continuous functions where each piece is a differentiable formula, with possible problems occurring at the points where two pieces meet. This reinforces the met-before.

This seems to be a universal experience, an epistemological obstacle, that cannot be avoided. However, this is false. It is what the French School term a 'didactical obstacle' occurring because of the way that calculus is taught. An alternative 'locally straight approach to the calculus' (Tall, 1985, 2009) uses the new technology of computer software to allow us to 'zoom in' on a graph to see it look less and less curved before our embodied eyes, and even magnify a small portion of the graph to see its slope change as one traces along the graph. This characterizes a differentiable function as one that is 'locally straight' under suitably high magnification and the derivative is the visible change in slope as one traces along the graph. This is a natural approach that builds on the student's embodied experience of drawing a graph continuously with a stroke of a pencil and looking along a curve with a dynamic shift of the eye. Continuity in this case is related to *perceptual* continuity using the attributes of our human senses.

If experiences of drawing and analyzing graphs are extended to include the possibility of functions that are 'highly wrinkled' then the situation changes. This can be done using a function such as the 'blancmange' function (Takagi, 1903, Tall, 1982) which is formed by adding together successively half-size saw teeth to produce a fractal that looks wrinkled at any magnification. To calculate $s_1(x)$, find the whole number *n* such that $n \leq x < n+1$ and calculate the decimal part $d = x - n$, then, if $d \leq \frac{1}{2}$ define $s_1(x) = d$, otherwise $s_1(x) = 1 - d$.

To calculate the blancmange function itself, calculate a succession of smaller sawteeth where

$$s_n(x) = s(2^{n-1}x)/2^{n-1}.$$

The nth approximation to the blancmange function is

$$b_n(x) = s_1(x) + \ldots + s_n(x).$$

and the blancmange function itself, *bl*(*x*) is the limit of these approximations. (Figure 4.)

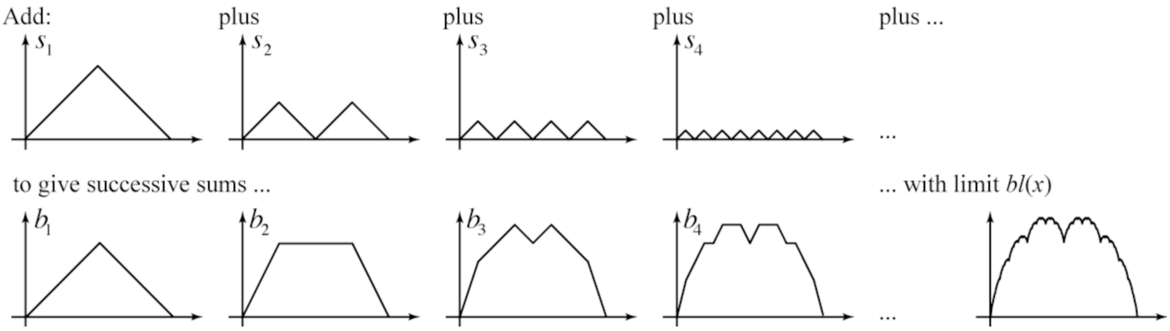

Figure 4: Getting a good-enough picture of the blancmange function

Not only is it possible to 'see' a wrinkled function that is not differentiable anywhere because it never magnifies to 'look straight' as the error function $bl(x) - b_n(x)$ is just a $1/2^n$ scale copy of the blancmange function itself. Therefore, wherever we look, at high magnification we see a tiny blancmange growing on a line segment, so the graph is *nowhere* locally straight. Once the blancmange function is seen as an everywhere continuous, nowhere differentable function, it opens up a wealth of new embodied ideas. For example, since the blancmange function has values between 0 and 1, a small blancmange such as $n(x) = bl(1000x)/1000$ is everywhere smaller than 0.001 and drawing it to a normal scale it is too small to see.

If we draw a graph of any differentiable function $f(x)$ on a computer screen, say between –5 and 5, we won't be able to distinguish between the graph of $f(x)$ and the graph of $f(x) + \lambda n(x)$ where λ is any real number between 0 and 1. This 'natural' idea shows that for every differentiable function $f(x)$, there is an *infinite* number of nowhere differentiable functions that are indistinguishable from it when drawn to a normal scale. The wrinkles only appear on higher magnification. With this met-before, there is a *huge* distinction between a 'naturally continuous function' that can be drawn with a pencil and a differentiable function that is locally straight. (Figure 5.)

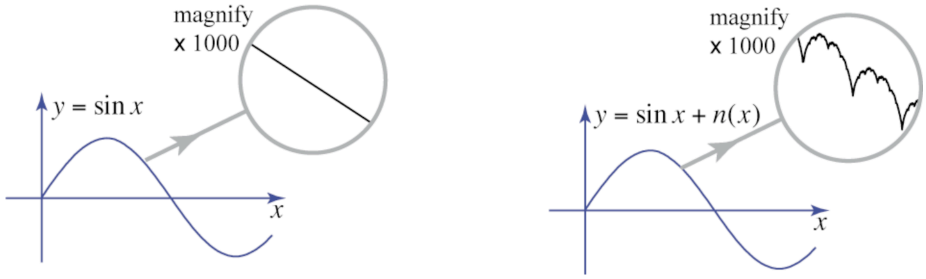

Figure 5: Two graphs that look the same at one level but are very different magnified

The moral of this example is that 'naturality' depends on previous experience: on our met-befores.

One amazing example of this is reported in Tall (1993) where student teachers attending an undergraduate course on mathematical analysis were shown a non-technical locally straight approach that included a proof of the fundamental theorem that the integral $I(x)$ of a continuous function $f(x)$ is differentiable and $I'(x)$ is equal to $f(x)$. The students were then shown a computer simulation of the calculation of the area under the graph of the blancmange function $bl(x)$. As the area function was drawn dynamically on the screen, a member of the class, who performed capably without being otherwise exceptional, suggested that the area function is differentiable once but not twice.

It was a stunning moment. It led to a further discussion of what happened when this function was integrated again and again, to find that if the blancmange function were integrated 27 times, then the resulting function would itself be differentiable 27 times but not 28. Is this a 'natural' occurrence? To the class of student teachers it certainly appeared to be so.

The moral of this particular story is that the notion of 'natural' in terms of embodiment depends on the met-befores of the individual concerned. As most of our current community studying the calculus have only the experience of functions that are everywhere differentiable, or, in more modern approaches, only piecewise differentiable, then the 'natural' ideas of the community have a certain form. It is a form presumably shared by the authors of *Where Mathematics Comes From* (Lakoff & Núñez, 2000) and also by most members of our current community. But it is not the only form of naturality.

We are already aware that mathematicians in different areas of mathematics have a variety of personal embodied notions of mathematical concepts based on embodied images, symbolic calculations and logical definitions and deductions. An applied mathematician or a theoretical physicist will treat problems in calculus and analysis in very different ways from a pure mathematician proving theorems in mathematical analysis. A study of how mathematicians think and how students learn to think mathematically deserves the widest possible view of how ideas develop over time.

The notion of met-before provides a theoretical construct in which greater insights can be found in the thinking of mathematicians, mathematics educators, cognitive scientists, mathematics teachers and in the learning of our students. As we develop our theories of how people think mathematically, we should all—including the authors of this paper—remember the saying:

> "Why do you look at the speck that is in your brother's eye, but do not notice the log that is in your own?"[1]

What is required is a theory of mathematics education that takes note of our own met-befores affecting the ways in which we think, in order to better understand the perceptions and problems of our students.

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[1] New American Standard Bible. St Matthew, chapter 7, verse 3.

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