THE DEVELOPMENT OF MATHEMATICAL THINKING: PROBLEM-SOLVING AND PROOF

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Abstract: What is the nature of mathematical thinking, problem-solving and proof? In the book ***Thinking Mathematically*** *that John Mason wrote with Leone Burton and Kaye Stacey, the term 'proof' never appears. On enquiring the reason for this, John expressed the deep fear that the word 'proof' engendered in his Summer School students. In this paper I will reflect on the development of mathematical thinking in the individual learning mathematics over a life-time and relate the theory in Thinking Mathematically to a theory of the long-term development of mathematical thinking that includes the development of proof. This will be related to the work of Richard Skemp on mathematical knowledge and emotions and provide an overall template for the journeys which individuals take as they develop mathematical thinking over the longer term.*

INTRODUCTION

It was my privilege to use the book *Thinking Mathematically* for over a quarter of a century from its first publication in 1982 to my retirement in 2007. This was a life-changing experience. Before my encounter with this remarkable text I saw my objective as a mathematics educator to reflect on mathematical knowledge and present it to students in ways that would enable them to make sense of it. In my early career, I wrote books and course notes with this purpose in mind. On the publication of *Thinking Mathematically*, I chose to use the text as a course book for a course that I termed 'Problem Solving' for second and third year undergraduate mathematicians with a liberal sprinkling of computer scientists, mathematical physicists and others.

I remember my abject fear when I first met with these students. I was going to start with the first problem in the book, inviting the students to work out whether it was better to calculate a percentage discount before or after adding a percentage tax. My panic was noted by my secretary in those early days as I walked by her door looking nervous and she said, 'You're doing that problem-solving again, aren't you?'

My fear arose because these were very able mathematics students and it was quite likely that they would say, 'but you just multiply the two factors and multiplication is commutative.' But none of them did.

Place someone in an unusual context and present him or her with a problem and it is likely that they will initially lose all sense of direction and need to

Written to celebrate the life and inspiration of John Mason … the one and only genuine Professor of Mathematical Thinking.

build up their confidence. This happened to me and it happened to my students. Over time we developed confidence and an ability to anticipate what would happen. It turned the routine learning (or mis-learning) of mathematics into a dynamic act of self-construction and gave most of us concerned a deep sense of pleasure.

Each week we had a two-hour problem solving session with a class of forty to eighty students where I began by setting the scene with the objective of the class, using successive sections of the book each week, then leaving the students to solve a particular problem illustrating the objective of the day. I also announced a 'problem of the week' for students who finished the problem of the day to keep them occupied. Initially some competitive students (often male) would move on to the problem of the week fairly quickly, but often they hadn't solved the problem at all. The book suggested three levels of explanation:

convince yourself

convince a friend,

and

convince an enemy.

Often the students had a story that clearly convinced themselves and even convinced their friends in the group, but by acting as an enemy I was able to begin to help them be more reflective about what they claimed, so that, over time, they began to question their ideas as a matter of course.

It was my belief that I should not try to solve the problems in advance. It was a distinct advantage to be caring but non-directive in my relationships with the students. *Not* knowing the 'answer' meant that I could change my approach from someone who shows how to do things and gives hints into someone who emcourages the students to think for themselves. 'Are you sure?' 'What does this tell you?" 'Is there another way of looking at it?'

At the same time I introduced the students to Richard Skemp's theories of modes of building and testing and, more importantly, to his ideas of goals and anti-goals, to help the students reflect on their emotions to be able to reason why they felt as they did and use this knowledge to advantage.

Skemp's three modes of building and testing

In his book *Intelligence, Learning and Action*, Richard Skemp (1979, p. 163) made a valuable distinction between different modes of building and testing conceptual structures in table 1. He speaks of building and testing a personal 'reality' as opposed to the 'actuality' of the physical world. Mode (i) relates to the individual's conception of the world we live in ('actuality'), mode (ii) to the individual's relationships with others, and mode (iii) to the individual's relationship with mathematics itself. There is a strong relationship with the levels of *Thinking Mathematically* (convince yourself, convince a friend,

REALITY CONSTRUCTION	
REALITY BUILDING	**REALITY TESTING**
Mode (i) from our own encounters with actuality: *experience*	***Mode (i)*** against expectation of events in actuality: *experiment*
Mode (ii) from the realities of others: *communication*	***Mode (ii)*** comparison with the realities of others: *discussion*.
Mode (iii) from within, by formation of higher order concepts: by extrapolation, imagination, intuition: *creativity*	***Mode (iii)*** comparison with one's own existing knowledge and beliefs: *internal consistency*.

Figure 1: Modes of Building and Testing

convince an enemy), in terms of order of levels, but not in a one-to-one fashion. Whereas Mode (i) refers to the personal perceptions of the world based on experience and reflections on actual experiments, the act to 'convince yourself' can involve any personal ideas that the individual may bring to bear on the problem in hand. However, in both cases, the onus is on the individual to use their own resources. Meanwhile Mode (ii) involves relationships with others, which would include both friends and 'enemies', where the latter are doubters who demand a higher level of rigour. Skemp's beautiful Mode (iii) involves the relationship of the human mind and spirit with mathematics, through creativity and internal consistency.

In *Thinking Mathematically*, the role of Mode (iii) is formulated in terms of an 'internal enemy', in which the individual learns to criticise their own creative thinking to seek self-improvement and internal consistency. The full list of levels of explanation in *Thinking Mathematically* is therefore:

Convince yourself

Convince a friend

Convince an enemy

Develop an internal enemy.

Long-term this leads to the desire to think mathematically by producing arguments that may begin with personal insights, are made clearer by discussions with a friend, then with an enemy whose purpose is to challenge the ideas put forward and make the deductions more rigorous. The ultimate goal is a personal level of consistency corresponding to a mode (iii) relationship with the coherence of mathematical ideas themselves.

Mathematics and the emotions

Thinking Mathematically focuses on the role of the emotions in mathematics, particularly in dealing with the high of an 'Aha!' experience which should be enjoyed before subjecting the insight to further scrutiny, and being 'Stuck', requiring a positive approach to analyse what has happened and how this can help to suggest alternative approaches.

In the middle of the twentieth century, psychologists separated the cognitive and affective domains (as, for instance, Bloom's famous *Taxonomy of Educational Objectives* distinguished three distinct domains: cognitive, affective and psychomotor). Richard Skemp stood out from the crowd by relating the cognitive and affective domains in terms of his (1979) theory of goals and anti-goals. A goal is an intention that is desired. It may be a short-term simple goal, for instance, to add two numbers together, or it may be a long-term major goal, for example, to succeed in mathematics. On the other hand, an anti-goal is something that is not desired and is to be avoided. For instance, a child may wish to avoid being asked a question in class because of a fear of being made to seem foolish. In general terms a goal is something that increases the likelihood of survival, but an anti-goal is something to avoid along the way.

Children are born with a positive attitude to learning. They explore the world spontaneously, with great pleasure. But unpleasant experiences may cause them to avoid a repetition of that unpleasantness, which leads to the development of anti-goals.

In his theory of goal-oriented learning, Skemp formulated two distinct aspects of goals and anti-goals. One concerns the emotions sensed as one moves towards, or away from, a goal or anti-goal (represented by arrows in figure 2). The other concerns an individual's overall sense of being able to achieve a goal, or avoid an anti-goal (representing by the smiling faces for a positive sense and frowning faces for a negative).

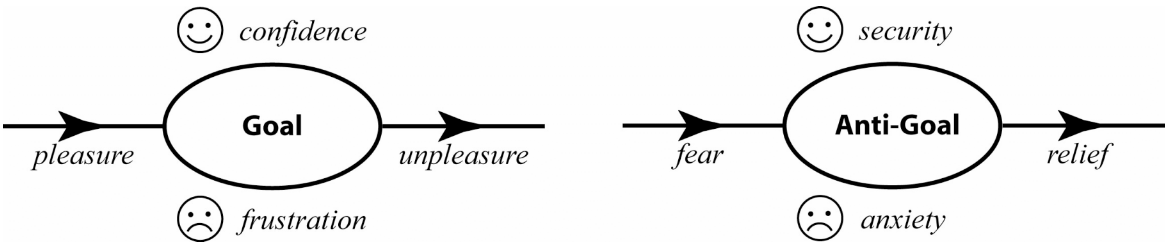

Figure 2: emotions associated with goals and anti-goals

The emotions related to goals and anti-goals are very different. Believing one is able to achieve a goal is accompanied by a sense of confidence, whilst being unable to achieve a goal is accompanied by frustration. Moving towards a goal gives pleasure, whilst moving away gives unpleasure, in the sense employed earlier by Freud. It is subtly different from the more usual, but not equivalent, term 'displeasure'. Drifting away from a desired goal may not be 'unpleasant' in the sense that it is distasteful, it may simply generate a feeling that one is going on the wrong path and intimate the need to reconsider one's options.

By using Skemp's theoretical framework while working with the book *Thinking Mathematically*, I found it possible to have discussions about individuals' emotional reactions to mathematics, to recognize the different emotional signs and to use them to advantage. For instance the subtle difference between frustration and anxiety in being unable to solve a problem reveals the difference between a goal one desires positively and an anti-goal one wishes to avoid. Once the source of the problem is identified, it becomes possible to take action to move in a more appropriate direction.

Proof Anxiety

The one important word missing from *Thinking Mathematically* is 'proof'. In a private conversation, John told me that this was because of the reaction of students to the word in his summer schools working with Open University students. If the idea of 'proof' was mentioned, they froze. In Skemp's terminology this seems to be anxiety arising from a sense of not being able to avoid an anti-goal. Proof seems to be something that these mature students had difficulty with, and they had long since seen it as a topic that they wished to avoid. If guided towards it, they felt a sense of fear, which could only be relieved by moving away from it again.

Thinking Mathematically is designed to give positive encouragement to students through strategies that are likely to lead to the pleasure of success and build confidence in the art of problem solving as a goal to be achieved, rather than an anti-goal to be avoided. So what is it that causes proof to become an anti-goal? To gain insight into this, it is helpful look at the long-term development of mathematical thinking.

Cognitive development of mathematical thinking

In a number of recent papers (e.g. Tall, 2008), I have followed the path of development of human thinking from mental facilities *set-before* birth and the subsequent experience *met-before* in our lives that affect our current thinking as it matures. Long-term we develop through refining our knowledge structures, coming to terms with complicated situations by focusing on important elements and naming them, so that we can talk about them and build ever more sophisticated meanings. Mason's insight of a *delicate shift of attention* plays its part in switching our thinking from the global complications to the essential aspects that turn out to be important. More generally it is the *discipline of noticing* that is important to seek to focus on essential ideas and gain insight into various problematic situations.

The framework that I have developed centres on the way in which we use words and symbols to *compress* knowledge into thinkable concepts, such as compressing counting processes into the concept of number or the likenesses of triangles into the principle of congruence in Euclidean geometry. Through experience and reflection, we build thinkable concepts into knowledge structures (schemas) that enable us to recognise situations when we attempt to solve new problems. Problem Solving arises when our knowledge structures are not sufficient to recognise the precise problem, or, if we have recognised it, to have the connections immediately available to solve it. To be more effective in mathematical thinking we therefore need to be aware of how our knowledge structures operate and how they develop over time.

As a pupil of Richard Skemp, I was taken by his simple analysis of the way the human mind works through *perception*, *action* and *reflection*, which gives us input through perception, output through action and makes mental links between the two through reflection. Skemp took his theory forward by suggesting that the mind operated at two levels, delta-one with physical perception and action, and delta-two with mental perception and action, linked together by reflection. I reflected on this structure and came to the conclusion that the distinctions between what we *perceive* through our senses and what we *conceive* in our mind are not as clear as we might wish them to be. So, rather than a two-stage theory, I saw a developing mental structure focusing on the complementary nature of perception and action and how it shifts from physical perceptions and actions to mental structures.

Quite recently (February 2008 to be more precise) I realised, to my astonishment, that our mathematical thinking could be seen to develop from just *three* mental facilities that are set-before our birth and which come to fruition through our personal and social activities as we mature. I termed these three set-befores: *recognition*, *repetition* and *language*. Recognition is the human ability, which we share with many other species, of recognising similarities and differences that can be *categorised* as thinkable concepts. Repetition is the human ability, again shared with other species, of being able to to learn to repeat sequences of actions in a single operation, such as see-grasp-suck, or the human operations of counting or solving linear equations. This is the basis of *procedural* knowledge. However, language enhances the set-befores of recognition and repetition. Recognition can be extended to give successive levels of thinking: forming thinkable concepts, then using those concepts as mental objects of attention to work at higher levels. Repetition can be compressed subtly through *encapsulation* of operations as thinkable concepts, denoted by symbols that can evoke either the underlying operation to perform, or the thinkable concept itself to be manipulated in its own right. These thinkable concepts that act dually, ambiguously and flexibly as process and concept are named *procepts*. As thinking processes become more sophisticated, language itself becomes increasingly powerful, leading to new formal ways of forming concepts through *definition* and mathematical *proof*.

This offers a framework for the development of mathematical knowledge structures, building on recognition, repetition and language, with compression into thinkable concepts through categorisation, encapsulation and definition, evolving through three distinct but interrelated mental worlds of mathematics that I term conceptual embodiment, proceptual symbolism and axiomatic formalism. Within the confines of this framework I usually compress the names to single words: embodiment, symbolism and formalism, while acknowledging that these terms have very different meanings in other theories.

This enables me to put the names together in new ways, such as formal embodiment, embodied symbolism, or formal symbolism. Indeed, the meanings of the two word phrases themselves depend on the direction travelled. Arithmetic arises from counting, adding, taking away, sharing as embodied operations that shift into symbolic embodiment. Representing number systems on the number line shifts back to give an embodied symbolism.

For instance, algebra builds from embodiment to symbolism through generalised arithmetic operations of combining, taking away, sharing, distributing, and so on. The reverse direction takes us from algebraic expressions and functions to graphs. These are quite different activities and, as we shall see later, there are a number of problematic aspects of these relationships.

The cognitive development of proof in the embodied world

We now turn our attention to see what the framework of embodiment, symbolism and formalism tells us about students' growing appreciation of proof.

In the embodied world of geometry, building on perception of figures and actions to make constructions gives us more specific insight into the nature of these figures. We already have the analysis of van Hiele to chart the development over the years. Give a child a plastic triangle, with equal sides and the child sees it as a whole and can touch and explore it to sense its corners, its sides and its angles. At one and the same time, it has three equal sides *and* three equal angles. From this beginning, were a figure has simultaneous properties, the child moves through successive van Hiele levels where the meanings and relationships change in conception. I choose to describe these successive levels as:

Perception: recognising shapes

Description: verbalising some of the properties

*Definitio*n: prescribing figures in terms of selected properties

Euclidean Proof: using constructs such as congruent triangles to build up a coherent theoretical framework of Euclidean geometry

Rigour: Formulating other geometric structures in terms of set-theoretic axioms.

In school mathematics, we are mainly concerned with the first four levels up to the development of Euclidean proof. My major focus of attention is the shift from Description to Definition. It seems innocuous. One simply moves from specifying certain properties of a figure to giving a more focused definition. However, cognitively, there is a *huge* shift in meaning. The plastic triangle that the child describes as being equilateral with its three equal sides and three equal angles is now defined as having three equal sides. Full stop.

The child can *see* that an equilateral triangle also has three equal angles, but now it becomes necessary to *prove* that an equilateral triangle, as defined, really does have three equal angles, *as a consequence* of having three equal sides. The method of proof is quite technical. It goes like this. First establish the meaning of congruent triangles. (Two triangles are congruent if they have three corresponding properties: three sides, two sides and included angle, two angles and corresponding side, or right-angle, hypotenuse, one side).

Effectively the notion of congruence depends on embodied actions. If two triangles ABC, XYZ have two sides equal $AB = XY$, $AC = XZ$ and included angle equal, $\angle A = \angle X$, then pick up triangle ABC and place it on triangle XYZ with vertex A placed on X, side AB placed on XY and angle A over angle X. Then, because the angles are equal, the side AC will lie directly over XY and, because the side-lengths are equal, point C will be coincident with X and point B will be coincident with Y. It follows that all the other corresponding aspects must be equal, including all corresponding angles, all corresponding sides and even the midpoints of the respective sides, the angle bisectors, and so on.

Now take a triangle ABC with equal sides AB, BC and, by constructing the midpoint M of the base AC, form two triangles ABM and CBM. These have corresponding sides equal, $AB=CB$ (given), $AM=CM$ (by construction), BM (common), so the triangles are congruent and, in particular, $\angle A = \angle C$. Q.E.D. Apply the same argument again, and if a triangle has three equal sides, *then* it has three equal angles.

There are some who appreciate the need for proof and get great pleasure out of the beauty of many aesthetic ideas in Euclidean geometry, such as the circle theorems where two angles subtended by the same chord in a circle are equal. But the vast majority of learners have connections in their minds that *tell* them such things as the fact that an equilateral triangle has equal sides and equal angles, and so, *why do they need to 'prove' it*. The shift from description to definition and deduction is mystifying for many and forms an obstacle causing fear and anxiety. Indeed, the only way to cope with the problem is to use the met-before of repetition to learn the proofs as procedures by rote. It addresses the goal of passing examinations without attending to the goal of understanding.

The cognitive development of proof in the symbolic world

The symbolic world of arithmetic and algebra develops out of embodied actions of counting, adding, taking away, making a number of equal-sized groups, sharing, and so on. These are then symbolised and there is a shift of attention away from specific embodiments and towards the relationships between the symbols.

In the embodied world of counting, it is not initially obvious that addition is commutative. If a child is at a stage of 'count-on' then 8+2 by counting on two after 8 to get 9, *10* is much easier than count-on 8 after 2 to get 3, 4, 5, 6, 7, 8, 9, *10*. The realisation that it is possible to perform the shorter count and get the same answer can be a pleasurable moment of insight.

Over time, experience shows that addition and multiplication are independent of order, and do not depend on the sequence in which the operations are performed, so that 3+4+2 can be performed as 3+4 is 7 then 7+2 is 9, or as 4+2 is 6 and 3+6 is also 9. These are formulated as 'rules', though they are not rules that are to be imposed on numbers, but observations that have been noticed. Then there is the associative law that says that $3\times(4+2)$ is the same as $3\times 4+3\times 2$ which gets more interesting in sums like $20-3\times(4-2)$ being the same as $20-3\times 4+3\times 2$.

At this stage the learner has to deal with a range of principles in using the notation of arithmetic and how they operate in practice. These principles are then employed in algebra.

To 'prove' the formula for the difference between two squares, it is usual to start with $(a+b)(a-b)$ and to multiply it out using the 'distributive law' then use commutativity of multiplication to reorganise the expression and cancel *ba* and *–ab* to get the final result:

$$\begin{aligned}(a+b)(a-b) &= a(a-b)+b(a-b)\\ &= a^2-ab+ba-b^2\\ &= a^2-b^2\end{aligned}$$

The problem here is to know what is 'known' and what needs to be 'proved'. The 'laws' being quoted (if they are indeed spoken explicitly) depend on experience and build on all kinds of met-befores that are implicit within the mind. While it may be appropriate in the more sophisticated axiomatic formal world to build proofs on definitions and deductions, for the teenager struggling with algebra it may cause nothing but confusion.

My own view is that the shift from embodiment to symbolism that operates in whole number arithmetic is not as evident in the shift from embodiment to algebra. For the learner who has a flexible proceptual view of symbolism, algebra may be an easy, even essentially trivial, application of generalised arithmetic. But for the learner who is already struggling with arithmetic and operates more in a time-dependent, procedural manner, it is likely to be highly complicated.

Letters may be used to represent unknown numbers in an equation such as $3x+5=5x-7$ or as units as in $120\text{ cm}=1.2\text{ m}$. The famous 'students and professors problem' relating the number of students (S) to the number of professors (P) when there are 6 students for each professor should be written as

S = 6P using the algebraic meaning of letters. However, it is often interpreted as 1P = 6S in the units sense that 1 professor corresponds to 6 students.

The met-before that every arithmetic expression, such as 3+2, 3.14×4.77, or $\sqrt{2}+1$, 'has an answer' is violated by algebraic expressions such as $3+2x$ that has no 'answer' unless x is known. So now the student who is bewildered by expressions that cannot be worked out is asked to manipulate them as if he or she knows what they are, when they have no meaning.

The interpretation of letters as objects which may help the student simplify $3a+4b+2a$ to $5a+2b$ by thinking of a as 'apple' and b as 'banana', but it fails to give a meaning to the expression $3a-5b$ (how can you take away 5 bananas when you only have 3 apples?)

The idea that an equation such as $5x+1=3x+5$ is a balance between 3 things and 5 on one side and 5 things and 1 on the other is seen as being widely meaningful to many students (Vlassis, 2002). Take $3x$ off both sides to get $2x+1=5$, now take 1 off both sides to get $2x=4$ and divide both sides by 2 to get the solution $x=2$. But change the equation slightly to $3x+5=5x-7$ and suddenly it has no embodied meaning. How can you imagine a balance in which one side is $5x-7$? How can you take 7 away from $5x$ when you don't yet know what x is?

In so many ways, the shift from embodiment to symbolic algebra is a minefield of dysfunctional met-befores for so many learners. This does not lead to the goal of making sense of algebra to develop power in formulating and solving equations. Instead, algebra becomes a topic to be avoided at all costs, an anti-goal provoking fear and a sense of anxiety as one attempts to find any method possible to avoid failure. For so many it leads to dysfunctional ways of learning procedures to cope with the difficulties: the English use of BODMAS to remember the order of precedence of operation (Brackets, Of, Division, Multiplication, Addition, Subtraction), the American acronym FOIL to multiply out pairs of terms in brackets (First, Outside, Inside, Last), operations to solve equations such as 'change sides, change signs; divide both sides by shifting the quantity to the other side and put it underneath.' For so many, algebra is an anti-goal to be avoided at all costs.

Now we are beginning to build up a picture of what may be happening in school as children learn arithmetic, then algebra. For so many, the initial embodiments of putting together and sharing have a meaning in the actual world in which they live. But the many successive compressions in meaning from operation to flexible procept work for some but impose increasing pressures on others. Eddie Gray and I called this 'the proceptual divide' in which the flexible thinkers have a built-in engine to derive new facts from old based on their rich knowledge of relationships between numbers, while others see increasing complication in all the detail and fall back on attempts to learn procedures by rote to cope with the pressures of testing.

Learning procedures by rote can be supporting in being able to perform routine calculations but procedural learning alone makes it more difficult to imagine flexible relationships between compressed concepts that are required in more sophisticated problem solving. As mathematics becomes more complicated for those who lack the rich flexible meanings, mathematics itself becomes an anti-goal to be avoided, creating a sense of anxiety and fear. More generally, mathematical proof, which requires a coherent grasp of ideas and how they are related, becomes problematic, both in geometry and in algebra.

Generating confidence through *Thinking Mathematically*

Given the relationship between cognitive success and emotional reactions, it becomes likely that one might attempt to improve students' abilities to think mathematically through organising situations in which they may experience success. Having experienced the good feelings generated in an open-ended problem-solving course myself, I was fortunate to be joined by Yudariah binte Mohammad Yusof, a university teacher from Malaysia who was concerned by the concentration on procedural learning in her students and the lack of a problem-solving ethic, other than that of becoming highly proficient at solving specific problems that would feature on the university examinations.

She took part in the Problem Solving course at Warwick University and trialled a questionnaire investigating student attitudes towards various aspects of mathematics and problem-solving. She then returned to Malaysia to teach the course and to research its effect on the students. (The details are given in Yusof & Tall 1996.) Half way through the course she telephoned me to express concern that her students continued to ask her what she wanted them to do, so that they could do well on the course. All I could say to her was that she should maintain the objective that the students needed to take control of their own working using the framework of *Thinking Mathematically*.

By the end of the course attitudes had changed dramatically. To identify what was meant by a 'desirable change', she asked the students' lecturers to fill in the questionnaires twice, once to indicated what they *expected* the students to say, once to say what they *preferred* the students to say. The direction of change from expected to preferred was taken to be a 'positive' change. In general *all* the changes in students' attitudes during the problem-solving course were positive, but when they returned to their normal mathematics lectures and were asked again six months later, the changes generally went back in the opposite direction. In other words, the problem-solving course took the students' attitudes in the direction desired by the staff, but when the staff themselves did the teaching, the attitudes of the students changed to the opposite direction.

My experience to date, through the work of research students carrying our studies in other countries and through my own links with communities around the world, is consistent with the global concern about the learning of mathematics. Some societies try to encourage meaningful learning through problem-solving, some teach by rote to encourage proficiency with many variations in between. Everywhere the pressure to compete and succeed in tests is driving the policies of governments. Surely our job as mathematics educators is not just to increase percentages passing examinations but a wider and deeper concern to understand the nature of mathematical thinking, to identify precisely why it is so difficult for many and how it can be improved for each individual.

Reflections

The analysis given here shows the power of *Thinking Mathematically* to improve students' attitudes and improve students' self-confidence and pleasure in doing mathematics and thinking for themselves. However, this occurs in a context in which so many older students have anxieties in dealing with the most central of all mathematical concepts, the notion of *proof*. The analysis given here in terms of the development of mathematical thinking through increasingly sophisticated embodiment and symbolism reveals transitions that are required to make sense of increasingly sophisticated mathematical thinking. The apparently innocuous shift from description to definition in geometry violates earlier beliefs in the properties of figures that are 'known' as part of a global perception but now must be 'proved' from the selected definitional properties. The shift from arithmetic to algebra involves a range of met-befores where established beliefs need to be changed to make sense of the new ideas.

John Mason has led a personal crusade for everyone to think about mathematics in new ways and his methods have yielded success. Clearly the way forward is to increase students' confidence by giving them genuine experiences of successful thinking, for only then will they face new problems as a challenge rather than a source of anxiety and fear. There is still much to be done by future generations to extend the pathways already trodden.

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