COGNITIVE AND SOCIAL DEVELOPMENT OF PROOF THROUGH EMBODIMENT, SYMBOLISM & FORMALISM

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Proof is a construct of mathematical communities over many generations and is introduced to new generations as they develop cognitively in a social context. Here I present a practical framework for this development in simple terms that nevertheless has deep origins. The framework builds on an analysis of the growth of mathematical ideas based on genetic facilities set-before birth. It unfolds a developmental framework based on perception, action and reflection that leads to distinct ways to construct mathematical concepts through categorization, encapsulation and definition, in three distinct mental worlds of embodiment, symbolism and formalism, which provide the foundation of the historical and cognitive growth of mathematical thinking and proof.

INTRODUCTION

Mathematical proof in today's society uses a formal approach based on axioms and definitions, constructing a formal framework by proving theorems. Mathematics educators research the process, to analyze how mathematicians and students think mathematically and to provide a theoretical framework to improve the teaching and understanding of the subject. In doing so we build on the work of others. However, the human species thinks using a biological brain and the growth of knowledge depends on how this biological entity makes sense of the world. The ideas that we share depend on the concepts developed by our predecessors, our genetic inheritance and our personal experiences in society.

As we reflect on the nature of mathematical proof, we find a process that every mathematician claims to adhere to, yet none can formulate precisely without appealing to implicit meanings shared by the mathematical community. The central question here is to seek the essential foundations of mathematical thinking and proof as it grows within society and within the individual.

Mathematicians speak of 'intuition' and 'rigour', seeing intuition as a helpful personal insight into what might be true, but requiring a rigorous mathematical proof to establish the insight as proven. However, the intuition of a mathematician with a rich knowledge structure (that Fischbein, 1987, calls 'secondary intuition') is clearly more sophisticated than that of a child. It is therefore important to build a framework that takes account of the developing nature of individual mathematical thinking.

THE GROWTH OF MATHEMATICAL THINKING

In this section I outline a framework for mathematical thinking based, on the one hand, on the biological foundation of human thinking and, on the other, on mathematics as developed in our mathematical communities.

I define a 'set-before' as ' a mental structure that we are born with, which make take a little time to mature as our brains make connections in early life,' and a 'met-before' as 'a structure we have in our brains *now* as a result of experiences we have met before'. It is the combination of set-befores that we all share to a greater or lesser extent and the personal met-befores that we use to interpret new experiences that lead to the personal and corporate development of mathematical thinking. In particular, mathematicians come into the world as newborn children, so all of us go through a process of personal cognitive development within society as a whole.

After long periods of reflection, I was surprised to find that just three set-befores form the basis of mathematical thinking. The first is the set-before of *recognition* that enables us to recognize similarities, differences and patterns. The second is *repetition* that enables us to practice a sequence of actions to be able to carry it out automatically. The third is the capacity for *language* that gives *Homo sapiens* the advantage of being able to name phenomena that we recognize and to symbolize the actions that we perform to build increasingly sophisticated ways of thinking.

From these three set-befores, three different forms of concept construction are possible. First, recognition supported by language enables us to *categorize* concepts as formulated by Lakoff and his colleagues (e.g. Lakoff & Nunez, 2001). Repetition allows us to learn to perform operations procedurally by rote, but in mathematics we can also symbolize operations and *encapsulate* these processes as mental objects (as formulated by Piaget (1985), Dubinsky (Cottrill et al, 1996), Sfard (1991) and others), which Gray & Tall (1994) called *procepts*. Language allows us first to *describe* objects for categorization purposes and then to give *verbal definitions*, but a huge shift occurs when we use set-theoretic language to *define* objects to give formal axiomatic structures in advanced mathematical thinking.

Building on these set-befores gives three major ways in which mathematical thinking develops which I term *three mental worlds of mathematics:*

> A world of (conceptual) *embodiment* that begins with interactions with real-world objects and develops in sophistication through verbal description and definition to platonic mathematics typified by euclidean (and also non-euclidean) geometry.
>
> A world of (procedural-proceptual) *symbolism* that develops from embodied human actions into symbolic forms of calculation and manipulation as procedures that may be compressed into procepts operating dually as process or concept
>
> A world of (axiomatic) *formalism* based on axioms for systems, definitions for new concepts based on axioms, and formal proof of theorems to build coherent theories.

In each of these worlds, various phenomena are noted, given a name (which may be any part of speech) and then refined in meaning to give a *thinkable concept* that can be spoken or symbolized with varying levels of rich internal structure, and then connected together in *knowledge structures* (schemas). When thinkable concepts are analyzed in detail, they may be seen as knowledge structures, in a manner that Skemp (1979) described in his 'varifocal theory' where concepts may be seen in detail as schemas and schemas may be named and become concepts. This shift between knowledge structure and thinkable concept is, in John Mason's phrase, achieved simply by 'a delicate shift of attention'. Further details of the three worlds can be found in published papers available for download from my website: http://www.davidtall.com/papers.

THE COGNITIVE DEVELOPMENT OF MATHEMATICAL PROOF

The development of proof in mathematical thinking matures over a lifetime. The young child experiments with the world, making a grab at something seen, and after practice, developing the action-schema of 'see-grasp-suck.' Initially the child develops mentally by experiment. Literally, 'the proof of the pudding is in the eating.' As the child grows more sophisticated, proof develops in various ways based on the three set-befores and the individual learner's met-befores.

Figure 1 shows the hypothesized cognitive development of the child in the lower left hand corner, developing through perception, action and reflection.

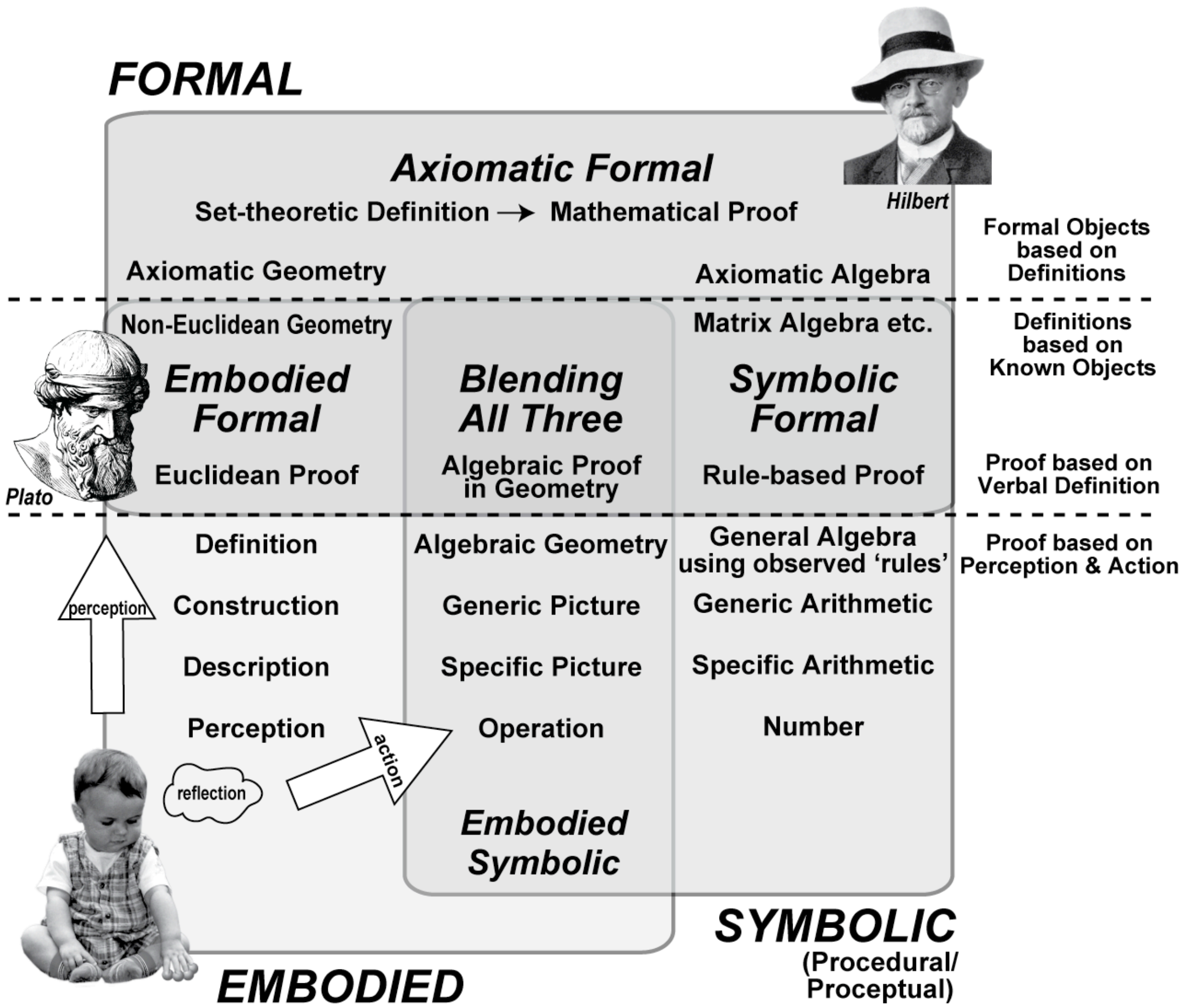

Figure 1: the cognitive development of proof through three mental worlds of mathematics

Perception develops in the embodied world through description, construction, and definition, leading to Euclidean proof represented by the bust of Plato. Even non-euclidean geometry is embodied, being based on mental embodiments of space that have different definitions from Euclidean geometry.

In parallel, the actions performed by the child, in terms of embodied operations such as counting, are encapsulated as symbolic thinkable concepts (procepts) such as number. Arithmetic develops through the compression of counting operations (count-all, count-on, count-on-from-larger) to known facts that may be used flexibly to derive new facts. Symbolic arithmetic benefits from blending with embodied conceptions, through a parallel development in embodiment and symbolism. For instance, the sum of the first 4 whole numbers can be seen as a succession of counters in rows of length 1, 2, 3, 4 and extended in each row successively by 4, 3, 2, 1, to give the sum 1+2+3+4 as a half of 1+4, 2+3, 3+2, 4+1, which is half of 4 lots of 5. This *specific* picture may be seen as a *generic* picture that works for any number of rows whatsoever, so that the sum of the first 100 numbers is $\frac{1}{2}\times 100\times 101$.

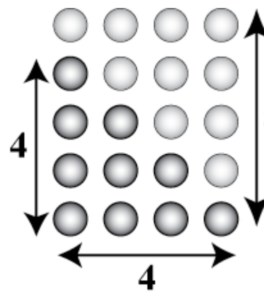

Figure 2

The specific and generic sum of the first few numbers may be generalized as an algebraic proof by writing the sequence 1, 2, …, n above the sequence in reverse as n, …, 2, 1, and adding the corresponding pairs to get n times $n+1$, to obtain a *general* algebraic proof that the sum of the first n whole numbers is $\frac{1}{2}n(n+1)$.

Symbolic operations develop from *specific* calculations to *generic* calculations, to *general* calculations represented algebraically. As this happens, the meanings of the ‘rules of arithmetic’ also develop in sophistication. Initially it may not be evident to the young child that addition and multiplication are commutative. For instance, calculating 8+2 by counting on 2 starting after 8 is much easier than calculating 2+8 by counting on 8 after 2. However, both calculations can be embodied by specific examples (for instance, seeing that a line of eight black objects and 2 white objects (●●●●●●●●○○) can be counted in either direction to get the same answer, 8+2 is 2+8 is 10). Such *specific* pictures can again be seen to be *generic* in the sense that the numbers of objects can be changed without affecting the general argument.

A significant shift of meaning occurs when observed regularities such as the shape of a figure in geometry or the property of an operation in arithmetic are formulated in terms of a *definition*. For instance, a figure that has four equal sides with opposite sides parallel and all angles are right angles is called a ‘square’. However, when it is *defined* to be a figure with four equal sides and (at least) *one* angle a right angle, a problem occurs. The young child may *see* that such a figure has four equal right angles, but the definition only insists on *one*. An embodied proof that a square, as defined, must have *four* equal angles can be performed by practical experiment in which four equal lengths are hinged to form a four-sided figure that can be placed on a flat table. Changing one angle automatically changes the others and it can be seen that if the one angle moves into a right angle, then the others do so as a physical consequence. Now the need for the implication is established. Such a embodied actions can also be carried out using appropriate dynamic geometry software such as *Cabri*, or *SketchPad*.

Embodied proofs continue to be of value to the learner as they mature, for instance it may be possible to translate them in terms of Euclidean proofs using congruent triangles. Embodied proofs can also be used to prove quite sophisticated statements, such as the fact that there are precisely five Platonic solids with faces given by particular regular polygon. Beginning with equilateral triangles, and considering how many can be placed at a vertex reveals that two are insufficient, three, four or five are possible, to give tetrahedron, octahedron and icosahedron, while six equilateral triangles fit to give a flat surface, so six or more is not possible. A similar argument with squares and pentagons reveals just one possibility in each case (the cube and dodecahedron). Hexagons and above do not fit to give a corner at all. Hence there are precisely five Platonic solids.

A second fundamental transition occurs in the shift from embodied and symbolic mathematics to the axiomatic formal world of set-theoretic definitions and formal deductions. Here, instead of a definition arising as a result of experiences with known objects, a definition is now given in set-theoretic terms, and the formal concept is constructed by proving theorems based on the definition. This leads us to the formal world introduced by Hilbert, as used today by most research mathematicians.

An example of the shift from symbolic formalism to axiomatic formalism can be seen in the nature of a proof by induction. Symbolically, it begins by establishing the truth of a proposition $P(n)$ for $n = 1$, then the general step that 'if $P(n)$ is true, then $P(n+1)$ is true' which is then repeated as often as desired for $n = 1, 2, 3, \ldots$ to reach any *specific* value of n. For instance, to reach $n = 101$, start the general step with $n = 1$ to get the case $n = 2$, and repeat the general step 100 times to reach $n = 101$. This is a *potentially infinite* proof. However, the formal proof using the Peano postulates is a *finite* proof in just three steps: first establish the proof for $n = 1$, then establish the general step, and then quote the induction axiom, 'if a subset S of $\mathbb{N}$ contains 1 and, when it contains n it must contain $n+1$, then S is the whole of $\mathbb{N}$'. Applying this to the set S of n for which $P(n)$ is true establishes the proof in a single step.

As students pass from elementary mathematics of embodiment and symbolism to the axiomatic formal world of mathematics, they must build on their set-befores and met-befores. Two different paths can be successful. One is to *give meaning* to the definition by using images or diagrams or dynamic change, building on met-befores to construct a *natural* route to formal proof. The other is to *extract meaning* from the definition by focusing on the logic of the proof, to become familiar with the definition and the various deductions that can be made, to build a *formal* knowledge structure that does not depend intrinsically on embodiment (Pinto, 1998).

Formal mathematical proof can then lead to what are termed *structure theorems*, which give rise to new meanings for embodiment and symbolism. For instance, a vector space is defined by formal axioms yet there is a structure theorem that proves that a finite dimensional vector space over a field F is isomorphic to F^n, thus the formal axiomatic system can be embodied by a coordinate system that can also be used for symbolic manipulation. In this way, formal mathematics returns to its origins in embodiment and symbolism.

The individual cognitive development of proof, which relates directly to the long-term social development of proof, builds on the three set-befores of *recognition*, *repetition* and *language*, which give concept construction through *categorization*, *encapsulation* and *definition*, giving rise to three mental worlds of mathematics based on *embodiment*, *symbolism* and *formalism*. Each world develops proof in different ways: embodiment begins with experiment to test predictions, and shifts through description then definition to verbal euclidean proof in geometry. Symbolism develops proof through specific, then generic, then general calculations and manipulations, leading to proof based on rules derived from regularities in symbol manipulation. Formalism is based on set-theoretic definitions and deductive proof.

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