THE HISTORICAL & INDIVIDUAL DEVELOPMENT OF MATHEMATICAL THINKING: IDEAS THAT ARE SET-BEFORE AND MET-BEFORE

David Tall

University of Warwick – UK

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[DRAFT]

Historical development is sometimes used as a format for teaching mathematics in school. However, there is a significant difference between the historical development of sophisticated adults in successive societies and the development of the child in today's society. This talk will formulate a theoretical framework based on genetic facilities we all share that are the basis of mathematical thinking and operate in history and the experiences that we have in life that lead to the personal development of mathematical thinking. The genetic facilities that are ***set-before*** *our birth include three major facilities:* ***language****,* ***recognition*** *of similarities and differences and* ***repetition*** *of a sequence of actions until they can be performed routinely. Language enables us to categorise things, as we do in geometry, and to name actions such as counting to be symbolised as number and algebra. We repeat actions again and again to get the potential infinity of whole numbers and we categorise the whole numbers as a set leading to actual infinity. The three set-befores of recognition, repetition and language lead to* ***three distinct worlds of mathematical thinking****:* ***embodied*** *(based on our perceptions and actions),* ***symbolic*** *(based on the symbolisation of actions as thinkable concepts) and* ***formal*** *(based on set-theoretic definitions and formal proofs).*

As children learn, they build on their previous experiences, ***met-before*** *in their life to build individual meanings for mathematics. These met-befores act in part to build new concepts, but can also be inappropriate when used in new contexts. The notions of set-before and met-before within three distinct worlds of mathematics have vital consequences in the interpretation of history and in the teaching of mathematics.*

INTRODUCTION

In the first meeting of this conference in February 2002, I was privileged to be able to give the first public presentation of a theory of three distinct mental worlds of mathematics (Tall 2003). These I termed 'embodied' (based on human perception and action, building through thought experiment to Platonic thought), 'symbolic' (based on human actions such as counting being symbolised as operations that became thinkable mental concepts) and 'formal' (the formalism of Hilbert based on set-theoretic axioms and mathematical proof). Since that time, the theory has progressed to reach a point of simplification where it fits together as an inevitable framework building from the natural development of our species.

In this presentation I begin with the simplest of all possible ideas that are present in all of us: the ability to *recognise* similarities and differences, the ability to *repeat* a sequence of actions until they become routine and the human facility for *language* that enables us to talk about what we recognise and how we repeat actions to develop the whole of mathematical thinking. I claim that these three facilities, *recognition*, *repetition* and *language* are the basis of mathematical thinking that leads to the three mental worlds of mathematics that I introduced earlier. They have implications for understanding the historical development of mathematics and the cognitive development of children in today's societies. We must remember that every mathematician, however eminent, began his (or her) life as a tiny child, unable to speak, relying on his or her mother for sustenance and on adults and others for guidance in their early years. Thus even the mathematicians we read about in history all had personal cognitive developments from childhood to fully fledged mathematical minds.

Set-before

I will use the term 'set-before' to describe a mental ability that we are all born with, which make take a little time to mature as our brains make connections in early life. Set-befores are essentially genetic and shared by us all, even though they may take initial social interchange to bring them into action. Take for example, the act of pointing. This is an important action to communicate to others, for instance, pointing to a child's toe and saying the word 'toe' to establish the link between the spoken word and the object to which it refers. Pointing is a natural human activity that we all share, but it is not clear as to whether it is purely genetic or partly social. Chimpanzees in the wild do not naturally point to things. However, chimpanzees interrelating with humans soon pick up the gesture as a pointing of all the fingers on one hand to a desired object, such as a piece of food that is out of reach. Thus pointing is a natural act for humans, but it may be a combination of genetic inheritance and early socialisation.

More generally, a set-before may be laid down in our genes, but it also requires exposure to socialization to cause it to operate to its fullest extent. This happens with language, a natural propensity in humans that develops in very different ways in different societies. It happens with recognition that requires a social context to develop the language to describe the things recognised and also with repetition of sequences of actions, such as counting, that become part of the repertoire of human thought and action.

Recognition and Repetition in Chimpanzees

Chimpanzees brought up in captivity have a great capacity for learning. For instance, they may be taught to recognise the digits, 1, 2, 3, 4, 5, 6, 7, 8, 9 and to point at each one in the correct order in turn. In a study at Kyoto University (Inoue & Matsuzawa, 2007), six chimps were compared with university students in various number-recognition tasks. The most successful, five-year old Ayumu, was more than the equal of the students. When five numbers were flashed for seven tenths of a second, Ayumu and the students were equally successful on 80% of the trials, but when the time was reduced to a fifth of a second, Ayumu was still accurate 80% of the time, while the students' success fell to 40%.

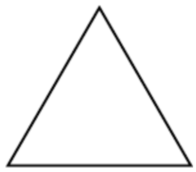
This shows that our close cousins in evolutionary development have both the capacity of recognition (of digits) and repetition (of the action of pointing to them in turn). Yet humans soon outstrip chimpanzees in development as they grow older. Homo Sapiens as a species has progressed beyond chimpanzees to develop the capacity for *language*, which has been with us for fifty thousand years and more. In the last ten thousand years, humans have evolved little genetically, however, there has been a vast increase in sophistication socially. Thus the historical development of mathematical thinking has built on the same genetic structure over the last ten thousand years, to address the needs of increasingly complex social structures, building on the same three set-befores of *recognition, repetition* and *language*, including the special mathematical symbolism of arithmetic, algebra and beyond.

BUILDING MATHEMATICAL THINKING

Building on set-befores

Mathematical thinking in every society builds on the three set-befores in specific ways. The *recognition* of similarities and differences enables us to use *language* to categorize things at varying levels of sophistication, such as 'cat', 'dog', 'triangle', 'duodecahedron', 'even number', 'area', 'function', 'finite-dimensional vector space', depending on the concepts available in the society at the time.

For instance, this figure may be called a 'triangle':



It happens to have three sides that look equal and three angles that look equal. Initially these properties are recognised as belonging to the figure itself. It is very sophisticated to realise that *if* the triangle has three equal sides *then* it will have three equal angles. Language is initially used to *describe* some of the properties of the figure and only later to use selected properties to *define* the figure and to *deduce* that other properties are a necessary consequence of the definition. Historically it was in the time of the Ancient Greeks that geometry developed a capacity for deductive proof.

The set-before of repetition enables us to learn to carry out sequences of actions to learn to operate procedurally without having to think consciously through every step.

Language enables us to *name* operations and to symbolise them to be able to think of them as mental concepts. The operation of counting gives the concept of number, the operation of addition gives the concept of sum, repeated addition gives multiplication, sharing gives fraction, evaluation of more complex arithmetic operations gives the concept of algebraic expression, and so on.

The counting procedure enables us to count one number, then another, then another, to continue as far as is desired to give the mental idea of *potential infinity.* The numbers 1, 2, 3, 4, … can be collected together and categorised as a set of all numbers, allowing us to speak of the set $\mathbb{N}$ of natural numbers, a set which has an *actual infinity* of elements. Philosophers may argue about the meanings of potential and actual infinity (and they certainly do), but today's mathematics undergraduates have no problem thinking about the whole set of natural numbers as a coherent mental concept (Tall, 1980).

Met-before

Human thought and action develop as a result of building on set-befores such as recognition and repetition. As children mature, successive experiences cause changes in the connections in the brain, building knowledge structures that are used to make sense of new situations. A 'met-before' is a personal mental structure in our brain *now* as a result of experiences met before. Many different met-befores are possible, depending on the experiences available in the child's previous development.

A child who is developing an acquaintance with arithmetic may learn that 2 and 2 makes 4, a met-before that holds good for the rest of the child's life, whatever situation that child is in. (Even in addition modulo 3 where 2 and 2 is 4 modulo 3 that gives the new meaning that 2 + 2 = 1 modulo 3, simply because 1 and 4 are equivalent modulo 3.)

The child also knows that after 2 comes 3 and that 3 is the 'next' number after 2, with no other numbers in between. This met-before no longer holds in dealing with fractions, where there is no 'next' fraction after 2 and there are many fractions between 2 and 3. Other met-befores encountered in whole number arithmetic are 'addition makes bigger', 'take-away makes smaller', 'multiplication makes *much* bigger' which are all violated at a later stage when encountering fractions or negative numbers. The child also experiences the phenomenon that all expressions have *answers*, such as 2+3 is 5, 3.487×23 is 80.201, the area of a circle radius 3.76 is 44.41 to two significant figures. This can act negatively as a met-before when algebra is introduced, causing confusion because expressions like $2 + 3x$ no longer have an answer and cannot be 'worked out' as would be expected in arithmetic.

Three distinct forms of mathematical concept

The three set-befores of recognition, repetition and language lead to three distinct forms of mathematical concept. Recognition leads to *categorisation* of concepts using language. Repetition leads to operations that can be symbolised leading to *encapsulation* of processes as thinkable concepts. Language in eventual sophisticated forms can lead to *definition* of concepts.

Recognition, together with language, enables us to name perceptions such as 'point' and 'line' and figures such as 'triangle', 'square', 'circle' and ellipse. It allows us to *categorise* concepts such as these by recognising their properties and describing them so that they can be recognised when they are encountered again. The language used focuses on the more important elements of the concepts under consideration. For example a point is initially something marked on paper with some kind of writing implement, a pen or a pencil, but as we speak of it we focus attention that what matters is its position in space, so that we focus on a point as a location in space, not as a mark with a specific size. Likewise a line is drawn using a straight edge and we speak of it as having length but not breadth, and that it can be extended in either direction as far as is desired. In this way a physical mark on paper is imagined in the mind as having sophisticated meanings. One cannot actually *draw* a line that has no width, but one can *imagine* such a line in the mind's eye. Recognition and categorisation lead naturally to concepts that mathematician's regard as having a Platonic perfection outside what is achievable in the physical world. They have a natural perfection which arises from the set-befores of recognition and language acting together to imagine and describe such entities.

Repetition of a sequence of actions, such as counting, can be verbalised and named so that the operation of counting leads to the concept of number. More generally, operations are symbolised and encapsulated as mental objects where the symbol can operate dually as a process to be performed or a concept to be held in the mind and to be thought about. Such a symbol with dual meaning as process and concept is called a *procept*. Typical examples are 3+2, 5×4, 3^5, $2x+5$, $\sin A = opp / hyp$, ${}^{dy}\!/\!{}_{dx}$, $\int \sin x\, dx$, $\sum_{n=0}^{\infty} 1/n^2$. We refer the reader specifically to the more sophisticated notion of the notion of procept in which different symbolism is used for the same underlying concept, such as $\frac{3}{4}$, $\frac{6}{8}$, $\frac{1}{2}+\frac{1}{4}$, and so on. It is the flexible use of symbolism enabling it to be manipulated into different forms representing the same underlying concept that gives mathematics such power.

Definitions take on an initial role with objects that are categorised and number concepts that are formulated such as even number, odd number, prime number, factor, and so on. However, they take on a significantly different role when concepts are defined set-theoretically, such as group, vector space, topological space, and so on.

HISTORICAL DEVELOPMENT BASED ON SET-BEFORES

The set-befores of recognition, repetition and language have developed over the centuries depending on the needs of each emerging society. Practical measurement and geometry occurred in earlier civilisations, with the development of successive number systems, some with far more powerful properties than others. Number systems were initially developed, making marks to count tallies, counting fingers, body parts and so on. Different systems developed in different civilisations:

- hieroglyphic characters in Egypt that were fine for representing whole numbers but cumbersome for representing fractions.
- Babylonian numbers pressed on clay tablets that used a base sixty to represent place value that could cope both with whole numbers and approximate fractions.
- Greek numbers using nine letters for 1 to 9, nine letters for 10 to 90 and nine letters for 100 to 900 that could write numbers up to 999 with only 3 letter characters.
- Roman numerals, good for representing numbers but poor for calculation,
- Hindu-Arabic numbers, representing whole numbers of any size with just 10 digits, 0, …, 9,
- The decimal representation of Simon Stevin which in its modern form represents any number large or small with a combination of the 10 digits and a decimal point,
- Various number systems, such as rationals, irrationals, infinite decimals, complex numbers, and so on …

It is only as we look at the strengths and weaknesses of these systems that we can begin to realise the power of our modern number systems. Even these continue to betray their origins, with various pecularities from one language to another, with the French and its counting in twenties to have quatre-vingt dix neuf (4 twenties, 10 and 9) to represent 99, German with its reversal of the digits in numbers such as 137 spoken as ein hundert, sieben und dreisig (one hundred seven and thirty) and even English with its traces of older number names. For instance, eleven comes from the old English for 'ei lief on' (one left over), twelve from 'twe lief' (two left), indicating the number of fingers in excess of the ten on our two hands.

If we look at the formal development of mathematics, we find the development of proof in the ancient Greeks and a form of mathematics that continued to specify clear definitions and deductions in subjects such as Newtonian mechanics into the nineteenth century.

The set-theoretic structures of modern formalism is a recent development from the reforms of Hilbert in his famous 1900 lecture, which not only announced 23 problems which kept mathematicians busy for the next century, but also a formal approach to mathematics based on axioms, definitions and set-theoretic proof.

TEACHING MATHEMATICS TODAY

The teaching of mathematics today relates to the body of mathematics that has been built up over the centuries. Our goal is to teach our children the ideas that we consider will be helpful to them in today's society. In doing this we can now see that this does not mean teaching a finished version of mathematics that exists outside of us independent of our activities as human beings. Mathematics is a product of the natural development of the human mind based on the three set-befores of recognition, repetition and language.

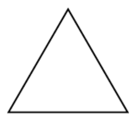
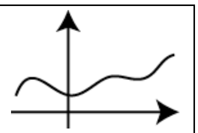
Every mathematician has entered the world as a child and passed through a cognitive development to become an intellectual mathematician. This includes Plato, who believed in the ideas of a pure Platonic world only after years of maturation in which he learned first to talk and then to learn new ideas of mathematics and philosophy. Even the great Hilbert was once a child and developed his mathematical ideas from his own experiences.

History does not focus on the cognitive development of mathematicians. Apart from a few anecdotes, the development of individual mathematicians in history as children is not part of the normal study of the history of mathematics. But it is there, nevertheless, silently underpinning the development of every professional mathematician.

Our job as teachers of mathematics is surely to be sensitive to the needs of children as they develop in sophistication. In doing so, we need to take account of their previous experience, and specifically we need an insight into their *met-befores*, the cognitive structures they have developed according to earlier experiences in their lives.

Cognitive development through three worlds of mathematics

In today's culture we have a blend of powerful ideas developed over the centuries that we teach to our children. We live in a society whose mathematics includes three distinct ways of building mathematical concepts, relating to the three set-befores in human evolution. These three distinct developments give the three mental worlds of mathematical development formulated in Tall (2003). They are:

The mental world of **embodiment** in which *recognition* leads to *categorization* (in which we build knowledge structures about things we perceive and think about);		
The mental world of **symbolism** in which *repetition* of actions (such as counting) leads through *encapsulation* to thinkable concepts such as number, developing symbols that act both as processes to do and concepts to think about (procepts);	$3+4$	$\int \sin x \, dx$
The mental world of **formalism** that uses (set-theoretic) *language* for *definition* of axiomatic systems and deduction by formal proof; this reverses the sequence of construction of meaning from definitions based on known concepts to formal concepts based on set-theoretic definitions.	$\mathbb{R}$	$\aleph_0$

In each of these worlds, the human brain compresses ideas into thinkable concepts that enable them to be manipulated in the mind in coherent ways. In the world of embodiments such thinkable concepts include geometric figures and graphs. In the world of symbolism, they include the procepts of arithmetic, algebra, functions, symbolic trigonometry, and symbolic calculus. In the world of formalism they include axiomatic structures where lists of axioms give rise to such formal concepts as the complete ordered field $\mathbb{R}$ and the infinite cardinal $\aleph_0$. In each case, complex ideas are symbolised into forms that can be manipulated on paper and in the mind.

Blending different knowledge structures

Mathematical ideas are often a blend of distinct knowledge structures. For instance, the concept of number is a blend of *counting* and *measuring*. These are quite distinct actions. Counting starts at the number 1 and successively counts 1, 2, 3, 4, 5, … Each number word is followed by a next (with nothing in between). Measuring starts from zero and measures unit shifts along the line, to 1, then 2, 3, 4, 5 … Between any two units there are many more measuring numbers, as fractions (and later as irrational numbers such as √2).

Successive number systems have different properties that conflict.

Counting numbers start counting at 1, then 2, 3, … where each number has a next with none between. Addition makes bigger, take-away makes smaller, multiplication is defined as repeated addition so that 4×3 is $4+4+4$ and the result of multiplication is usually *much* bigger. Children counting sense these properties that act as met-befores when they encounter a new number system, such as fractions and integers.

Fractions have new properties that do not occur with counting numbers. A fraction has many names: 1/3 , 4/12 , 7/21 ... There is no 'next' fraction after any given fraction, indeed there are many fractions between any two distinct fractions. The operations of addition and multiplication involve new techniques. Addition continues to give a bigger result and take-away a smaller result, but the process of addition and subtraction requires the new idea of 'putting the fractions over a common denominator'. Multiplication of two fractions may not now be seen simply as repeated addition, although there are ways of representing multiplication of fractions as area. Multiplication may now give a smaller result, conflicting with the experience with counting numbers.

Positive and negative integers are also spaced out with no numbers in between, but now numbers can be positive or negative. Adding a negative number now gives a smaller result and subtracting a negative number gives a larger result. Multiplication in terms of repeated addition is more complicated. While $(-4) \times 3$ might be seen as 3 lots of –4 giving a total debt of –12, the definition of $4 \times (-3)$ as '–3 lots of 4' needs to be interpreted as 'taking away' 3 lots of 4, which is –12. Furthermore the product of two negative numbers is a positive number, which may seem very strange indeed.

Rote Learning as a response to conflict

The transition at each stage involves old ideas that act as met-befores that no longer work in the larger system. Such conflicts cause anxiety and fear of mathematics. When they are encountered, children often respond by learning to cope with the procedures by rote without understanding. This can have serious consequences, for learning procedures involves doing sequences of actions in time which may not have any compressed meaning that can be manipulated at the next stage. If the child learns only procedures for whole number arithmetic, fractions are difficult to cope with. If a a child sees a fraction such as $3/5$ as a procedure 'divide into five equal pieces and take three', how can that child make sense of $3/5 + 1/2$? More usually, a child who struggles does not see $3/5$ as 'divide into five equal pieces and take three'. It may mean 'divide 3 cakes between 5 children'. A practical solution is to divide each cake in half, to have six halves, then give each child a half-cake and divide the final half roughly into five small pieces. Each child now has half a cake and a fifth of half a cake in 'fair shares'. But the half and a fifth of a half is unlikely to be seen as $3/5$! (Kerslake, 1986).

The child who has a flexible knowledge of arithmetic may see $3/5$ is 'three fifths' or 'six tenths', while $1/2$ is 'five tenths'. So the sum is 'six tenths' and 'five tenths' which is 'eleven tenths'. So flexible knowledge of whole number arithmetic can lead to relating the complexity of fractional addition to familiar whole number arithmetic. The child who lacks this flexible knowledge has no such chance. For such a child, fractional arithmetic becomes *more complicated*, involving rules like 'put over a common denominator' which may make little sense. Thus the flexible thinker finds fractions easier while the procedural thinker finds fractions more difficult.

This phenomenon can be repeated at every transition. The child who is struggling with met-befores that fail to make sense may learn to operate 'rules without reason' which is more difficult to use coherently in any context which is more than a straight application to a particular rule.

These transitions occur throughout the curriculum. Those that involve unhelpful met-befores include:

- From counting to the whole number concept
- From whole numbers to fractions
- From whole numbers to signed numbers
- From arithmetic to algebra
- From powers to fractional and negative powers
- From finite arithmetic to the limit concept
- From description to deductive definition
- At many other transitions, such as teaching the function concept in stages (linear, quadratic, trigonometric, logarithm, exponential, etc) builds limitations at each stage that stunt long-term growth.

Research in many of these areas still needs to be done, so I invite you to do research into the effects of met-befores in transitions in the mathematical curriculum.

The transition from arithmetic to algebra

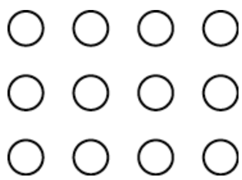
I close my presentation with reference to the PhD study of Rosana Nogueira de Lima, part of which has been published as Lima & Tall (2008). This investigates the met-befores that occur in the shift from arithmetic to algebra.

This is a study of teaching and learning in Brazil which began by investigating what difficulties students have in learning linear and quadratic equations and looks at the met-befores that act as obstacles to the learning of algebra.

For some students, the step from generalized arithmetic to algebra is trivial, for algebra only expresses in generality what happens in arithmetic in particular. Students who are flexible with arithmetic concepts may find algebra extremely easy.

But many students find algebra a total turn-off. It does not make sense at all! So what is the problem. Fundamentally there are a number of met-befores that the beginning algebra student has encountered which can act as obstacles to understanding algebra.

In arithmetic, the operations can be embodied by a physical representation. For instance, the operation 4×3 can be embodied as a row of 4 discs repeated three times. This can also be seen as a column of 3 discs repeated 4 times, revealing an embodiment that multiplication is independent of order.



All expressions in arithmetic have an answer, 2+3 is 5, 4×3 is 12, and so on. However, an algebraic expression $2 + 3x$ has no 'answer', unless the value of x is known. And if it is known, say $x = 4$, why not write it as an arithmetic expression $2 + 3 \times 4$ anyway? And even if it is written this way, what does it mean? Is it 2+3 (which is 5) times 4 to give 20, or 2 plus (3×4) which is 14? Some students who cannot make sense of the expression $2 + 3x$ work out the bit they *can* calculate $(2 + 3)$ which is 5, and write the answer erroneously as $5x$.

Sometimes students are introduced to algebra using letters to represent objects, for instance using $3a$ to represent 3 apples, $4b$ to represent 4 bananas, then simplifying the expression $3a + 4b + 2a$ by putting the apples together to get $5a + 4b$. This often helps students get a start in manipulating simple expression, but it soon fails. What does $3a - 2b$ mean? How can you take 2 bananas from 3 apples?

Letters are used to represent units, for instance 3m is '3 metres' and we write

$$3\text{m} = 300\text{cm}$$

to represent 3 metres is the same as 300 cms.

This is a powerful met-before to mislead students, as is shown in the student-professor problem:

> Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university." Use S for the number of students and P for the number of professors.

Approximately a third of the highly educated people who answered this question wrote the equation as $6S = P$ and two thirds wrote $S = 6P$ (or their equivalents). The equation $6S = P$ might be read as 'six students equal one professor' which treats S and P as students and professors. However, the correct interpretation of the task is $S = 6P$ which says that *the number* of students is *six times the number* of professors.

The earlier use of letters to represent objects or units is very strong. It creates met-befores that cause individuals to reverse the algebraic meaning of the students-professors problem. Subsequent attempts to teach students to recognise the true meaning of the task have met with surprisingly strong resistance which is consistent with them clinging to the interpretation of letters representing 'things' rather than 'the number' of things.

Overall, there are a number of met-befores that act as obstacles in the transition from arithmetic to algebra. They operate as a mine-field of misconceptions in algebra which makes it difficult for the students to make any real sense of algebra.

What is worse occurs when the students first meet linear equations such as $3x + 4 = 19$. Often the solution is to teach them that the two sides 'balance' and the balance can be maintained by 'doing the same thing to both sides'.

This 'balance' method has proved to be successful in the past, but Vlassis (2002) showed that it may work in simple cases where the equations do not involve negative values or subtraction, but in more general cases, the embodiment as a balance does not work. Indeed, it seems, given the many met-befores from arithmetic that cause problems, there seems to be no single way of embodying linear equations that works in general. For instance, the equation $2x - 3 = 5$ cannot be represented as a 'balance' because it is not possible to subtract the 3 from $2x$ on the left-hand side if x is not known. One possibility is to represent taking off 3 on the left hand side by putting 3 on the right hand side. Another is to imagine helium balloons that lift up the balance and so act as negative numbers. Such problems even arise in equations that have all positive-looking quantities in them such as $2x + 3 = 2 + x$, which seems at first sight to be amenable to a physical representation, yet reduces to $x + 1 = 0$, which seems impossible.

The teachers in the study of Lima (2007) decided to use the general approach of 'doing the same thing to both sides' to retain the balance of the equation. However, rather than use the general principle, many students saw the operation of taking the same number from both sides as 'move the number of the other side and change its sign', so that $3x + 2 = 8$ became $3x = 8 - 2 = 6$. Then when both sides were divided by 3, the students saw this as 'move the number on the other side and put it underneath', giving

$$x = \frac{6}{3} = 2.$$

Instead of learning the general principle, the students learned to 'shift the symbols around' with extra elements that were not fully understood, such as 'change the sign' or 'put it underneath'.

This procedural technique of mentally moving the terms with an additional meaningless piece of magic to get the right answer, was used successfully by some but proved fragile for others who made errors mixing up the rules such as shifting the 3 in $3x = 6$ with the additional magic of changing signs to get $x = \frac{6}{-3}$. Such errors may increase the confusion of students who may then try alternatives to get the correct answer, producing what appear to be random errors.

This is consistent with earlier research in Brazil where Cortés and Pfaff (2000) found that the principles used by 17-year old students to solve equations in their study were all based on the 'movement' of symbols from one side to the other of an equation, as if the symbols were physical entities that students 'pass' to the other side of the equation with additional changes of signs or a shift to place it below the number in the other side. Freitas (2002) found that procedures related to phrases such as 'change side, change sign' were usually meaningless to students and often resulted in mistakes. As procedures, they are likely to become more complicated as the mathematics gets more sophisticated, and the increasing complication will cause even greater problems at later stages.

Quadratic Equations

Faced with the difficulties with linear equations and knowing the struggles that the students were encountering, the teachers took the decision *to teach the formula for quadratic equations*. They decided that, at least, students could be expected to solve simple quadratics that would be given on the examination. The general feeling was that too much emphasis on completing the square, or factorising the quadratic were likely to make everything just too complicated.

However, this single procedure first requires the equation be manipulated into the form $ax^2 + bx + c = 0$ and so proves to be more complicated in cases where an alternative approach may be more insightful. For instance, a problem arose when the students were asked to show that the equation $(x - 2)(x - 3) = 0$ had roots 2, 3. Most students failed to respond at all and those that did attempted to multiply out the brackets to solve the equation using the formula. Few succeeded.

The consequences of teaching limited procedures

Here we see that the focus on limited procedures and lack of flexibility in the meaning of the symbols has a cumulative effect. A few students may see the essential simplicity of an algebraic expression as a potential calculation that can be manipulated in its own right and develop an effortless mastery of algebra. However, most focus only on procedures and become involved in more and more complicated activities that increase the cognitive strain and become unmanageable.

The message is clear. If students do not make sense of the mathematics and learn meaningless procedures, the problem of coping with mathematics gets more and more difficult with each new topic.

Now we can see that difficulties occur through met-befores: experiences that create ways of working that no longer apply in the new situation, even if they worked well in an earlier context.

This paper provides a framework for the development of mathematical thinking. It reveals the three set-befores of recognition, repetition and language which are fundamental for long-term development of mathematical sophistication. Progress requires the compression of complicated ideas into thinkable concepts through categorisation, encapsulation of process as concept and at university through definition of mathematical structures whose properties can be deduced by formal proof.

In school the main source of difficulty in the use of symbolism is the meaningful compression of procedures into symbolic representations that can be used in flexible ways for solving new problems. If a new context is not understood, the fall-back position is to learn the procedures by rote and the long-term prognosis may be increasing reliance on meaningful procedures.

A major focus in improving the curriculum is to address the difficulties that students encounter through met-befores that worked well before but are no longer appropriate. It is essential for researchers, teachers and students to become aware of these naturally occurring difficulties and to address them directly, by seeing that they worked successfully in one situation but may be no longer applicable in learning a different topic.

There are research projects out there to be performed by researchers and teachers in collaboration, to identify met-befores that cause problems and to address ways of making sense of new ideas that make sense in new contexts. It is a new area of research that I recommend to you.

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