THE UNIVERSITY OF WARWICK

The Historical & Individual Development of Mathematical Thinking:

Ideas that are set-before and met-before

David Tall
Emeritus Professor in Mathematical Thinking
University of Warwick UK

Historical development is sometimes used as a format for teaching mathematics in school.

There is a difference between the historical development of sophisticated adults in successive societies and the development of the child in today's society.

This talk formulate a theoretical framework based on:

genetic facilities that we all share that are the basis of mathematical thinking and operate in history

SET-BEFORE

and

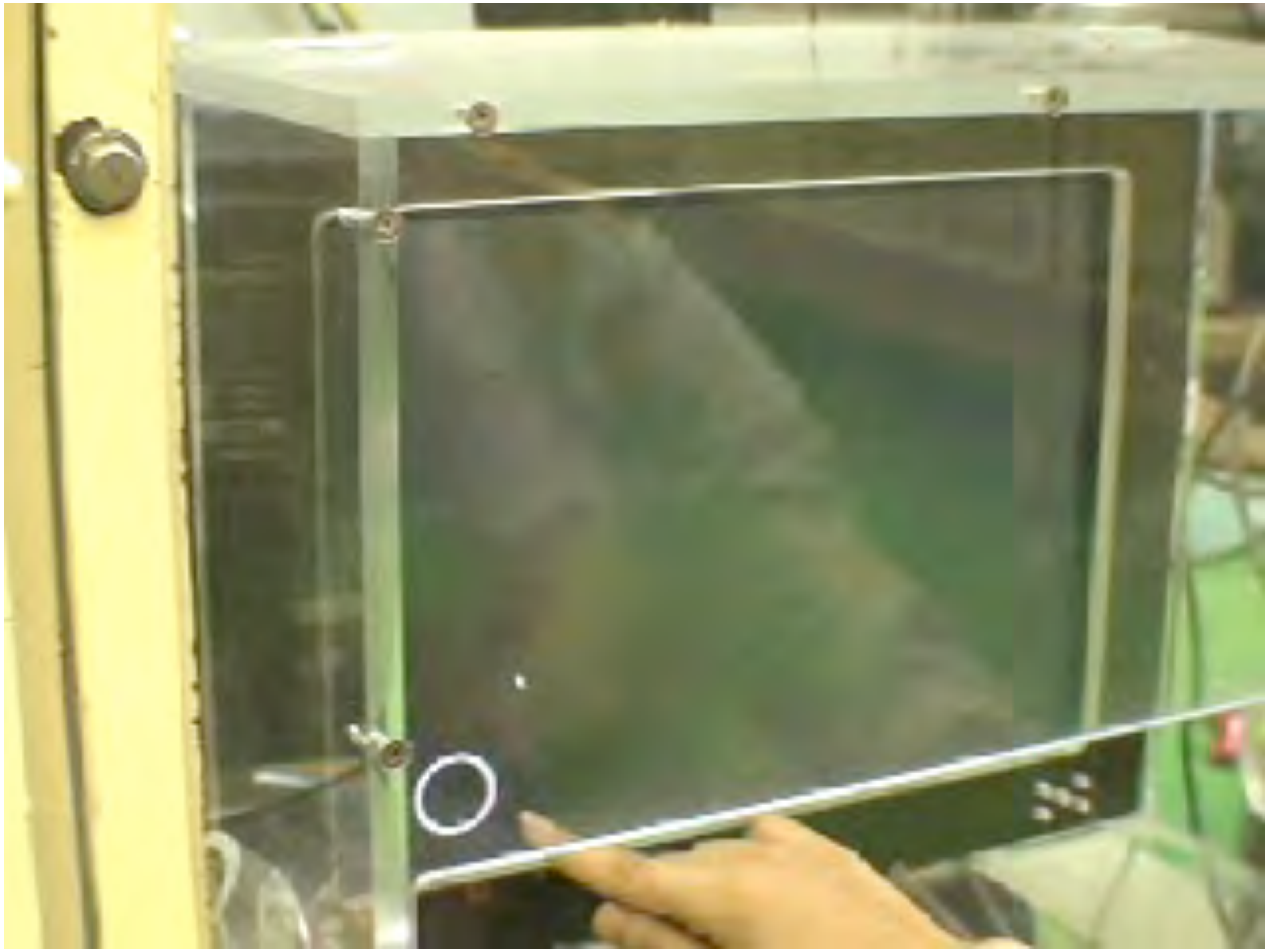
individual experiences we have in life that lead to the personal development of mathematical thinking.

MET-BEFORE

A chimpanzee, Ayumu, **recognizing** and **repeating**

A chimpanzee, Ayumu, **recognizing** and **repeating**

A chimpanzee, Ayumu, **recognizing** and **repeating**



A human not doing so well!

A chimpanzee has genetic abilities for

recognition of similarities and differences
repetion of complex actions that become automatic

A human also has genetic abilities for the development of

language to name, describe and refine meaning

These abilities are the foundation of mathematical thinking.

They have been essentially the same in homo sapiens for tens of thousands of years.

What has changed is the increasing sophistication of cultures over the centuries.

Set-befores & Met-befores

A ‘**set-before**’ is a mental ability that we are all born with, which make take a little time to mature as our brains make connections in early life.

The three major set-befores in mathematical thinking are

Recognition, Repetition, Language

A ‘**met-before**’ is a personal mental structure in our brain *now* as a result of experiences met before.

Many different met-befores are possible, depending on experiences available in our society at the time.

2+2 is 4 **after 2 comes 3** **addition makes bigger**

take-away makes smaller **multiplication makes bigger**

all expressions (such as 2+3, 22/7, 3.48x23.4) have answers.

Building Mathematics on Set-befores

Mathematical thinking in every society builds on the three set-befores as follows:

Recognition of similarities and differences enables us to use language to ***categorize*** things such as 'cat', 'dog', 'triangle', 'duodecahedron', 'even number', 'area', 'function' ...

Repetition of operations enables us to learn to operate procedurally, and language enables us to *name* operations and think of them as mental concepts.
Counting gives number
addition gives sum
repeated addition gives multiplication
sharing gives fraction
evaluation gives algebraic expression, and so on.

Building on Set-befores

The counting procedure can itself be repeated, to count another number, and another, and another ...
to give ***potential infinity.***
The numbers themselves can be *categorised* to speak of the set of all numbers, to give ***actual infinity***.

Mathematical thinking grows in three distinct ways related to the three set-befores.

Recognition *together with language allows us to name 'point' and 'line', and figures, such as triangle, square, circle, ellipse.*
It allows us to **categorize.**

Building on Set-befores

The counting procedure can itself be repeated, to count another number, and another, and another ...
to give ***potential infinity.***
The numbers themselves can be *categorised* to speak of the set of all numbers, to give ***actual infinity***.

Mathematical thinking grows in three distinct ways related to the three set-befores.

Repetition *together with language allows us to name actions such as counting and the results of those actions such as number.*
It allows us to **encapsulate** *processes as mental objects.*

Building on Set-befores

The counting procedure can itself be repeated, to count another number, and another, and another ...
to give ***potential infinity.***
The numbers themselves can be *categorised* to speak of the set of all numbers, to give ***actual infinity***.

Mathematical thinking grows in three distinct ways related to the three set-befores.

Language *itself becomes more powerful and, as we build up sophisticated relationships it can be used to* ***define*** *mathematical concepts.*

Building on Set-befores

Recognition *together with language allows us to name 'point' and 'line', and figures, such as triangle, square, circle, ellipse.*
It allows us to **categorize.**

Repetition *together with language allows us to name actions such as counting and the results of those actions such as number.*
It allows us to **encapsulate** *processes as mental objects.*

Language *itself becomes more powerful and, as we build up sophisticated relationships it can be used to* **define** *mathematical concepts.*

Building on Set-befores

Historical Development

Recognition, repetition and language

have developed over the centuries, depending on the needs of the society.

Practical measurement and geometry, practical counting and number representations, some far more powerful than others.

Formal theories, such as Euclidean Geometry and modern set theory.

*Every culture begins from its human **perception** of the world and its **action** upon that world to survive and prosper, giving distinct mental worlds of mathematical thinking:*

*An **embodied** mental world of perception, recognition and categorization, typical in geometry and graphical representations.*

*A **symbolic** mental world of operations that are symbolised and the symbols operate dually as operations or as thinkable concepts.*

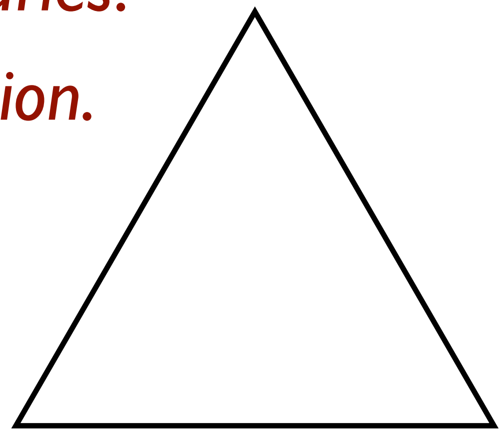
*A later development is a **formal** world of definition and proof.*

Building on Set-befores

Historical Development

The mathematics that we use today is a blend of successive developments over the centuries.

Practical recognition and action.



Here is a triangle. It has three sides and three angles. It looks as if the sides are all the same and the angles are all the same.

The Greeks realised that

IF the sides are equal THEN the angles are equal.

The development of Euclidean Geometry.

Set-theoretic proof is a late development in the nineteenth century.

The Development of Number

Historically, the number concept developed according to the needs of the given society.

Counting fingers, body parts, tallying, ...

Egyptian hieroglyphics, Babylonian base sixty on clay,

Greek letter numbers

Roman numerals

Hindu-arabic whole numbers

Simon Stevin decimals

rationals, irrationals, infinite decimals, complex numbers etc

Modern number systems build on old cultural elements

e.g. English numbers build on finger counting

eleven: old English for ei lief on (one left over)

twelve: twe lief (two left)

Building on Set-befores

Cognitive Development

Every mathematician enters the world as a child

and goes through a cognitive development.

Historical studies do not focus on this aspect, but it is essential to realise that mathematicians themselves develop mathematical thinking from learning in a particular society with particular beliefs.

All mathematicians build on their own personal ***met-befores****.*

Even Plato!

Even Hilbert!

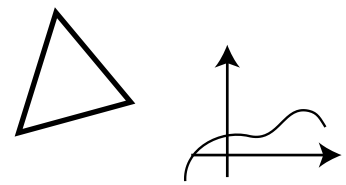
In teaching our goal may be to encourage thinking like mathematicians, but we *must* take into account the development of the learners' met-befores.

Building on Set-befores

Cognitive Development

In today's culture we have a blend of the most powerful ideas over the centuries which we teach to our children.

- ***recognition*** leads to **embodiment** (in which we categorize and build knowledge structures about things we perceive and think about);



- ***repetition*** leads to **symbolism** through action (such as counting) and symbolization into thinkable concepts such as number, developing symbols that function both as processes to do and concepts to think about (called procepts);

$3+4$

$\int \sin x \, dx$

- ***language*** leads eventually to **axiomatic-formalism** (based on formal definitions and proof) which reverses the sequence of construction of meaning from definitions based on known concepts to formal concepts based on set-theoretic definitions.

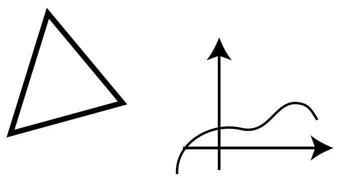
$\mathbb{R}$

$\aleph_0$

The Biological Brain

The brain needs to **compress** ideas into thinkable concepts that enable it to manipulate them in a simple way.

embodied



symbolic

$3+4$

$\int \sin x \, dx$

formal

$\mathbb{R}$

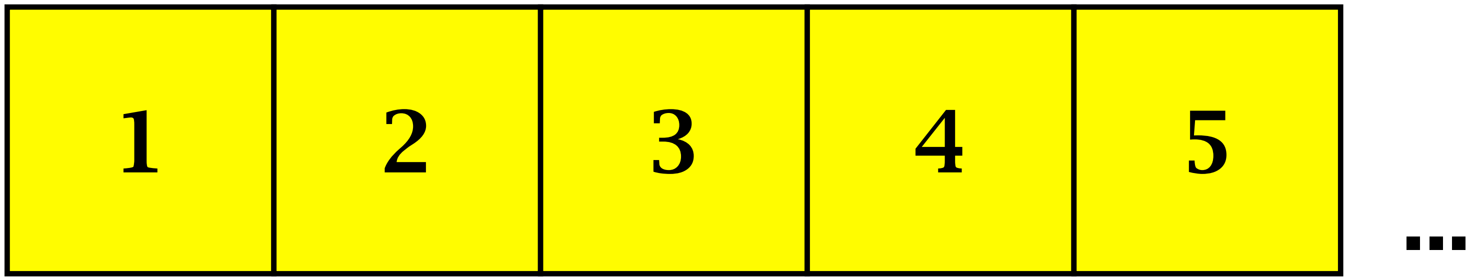
$\aleph_0$

The brain uses information it already has to build new concepts. It **blends** together old ideas that fit in useful ways.

Aspects that fit give *pleasure*,
aspects that clash give *challenge* and/or *anxiety* ...

Blending different conceptions of number

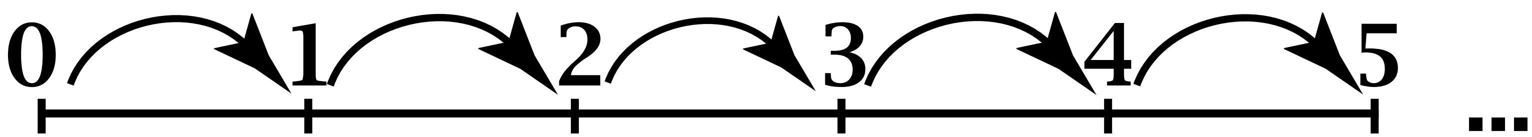
Number is a blend of ***counting*** and measuring



The ***number track*** ...

Counting: each number has a next with nothing between, starts counting at 1, then 2, 3, ...

The number line builds from *measuring*



The ***number line*** ...

Measuring: each interval can be subdivided, starts from 0 and measures unit shifts to the right.

Blending different conceptions of number

Successive number systems have properties that conflict

Counting Numbers

each number has a next with none between,
starts counting at 1, then 2, 3, ...
addition makes bigger, take-away smaller, multiplication bigger

Fractions

a fraction has many names: **$^1/_3$, $^4/_{12}$, $^7/_{21}$ ...**
there is no 'next' fraction
addition and multiplication involve new techniques
addition makes bigger, take-away smaller,
multiplication may be smaller

Integers

each number has a next with none between,
numbers can be positive or negative,
addition may get smaller, take-away may get larger,
multiplication of negatives gives a positive.

Increasing sophistication of Number Systems

Mathematicians usually view the number systems as an expanding system:

$$\mathbf{N} \subset \mathbf{F} \subset \mathbf{Q} \subset \mathbf{R} \subset \mathbf{C}$$

$$\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q}$$

In cognitive development, each new system is a blend of old ideas that still hold and new features that conflict with the old.

Different knowledge structures for numbers

The properties change as the number system expands.

Take away leaves less?

N	True
Q	Yes and no
R	Yes and no
C	No - complex numbers are not ordered
Infinite Sets	Not necessarily eg Z – N

Different knowledge structures for numbers

The properties change as the number system expands.

How many numbers between 2 and 3?

N	None
Q	Lots – a countable infinity
R	Lots more – an uncountable infinity
C	None (the complex numbers are not ordered)

A mathematician has all of these as met-befores

A learner has a succession of conflicting met-befores

Blending different conceptions of number

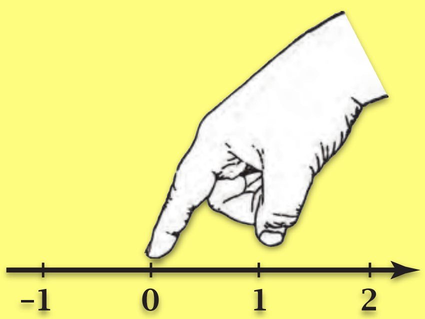
Formal

R

A complete ordered field
as a 'discrete set of elements'
satisfying specified axioms

Embodied

A 'continuous line'
that can be
traced with a finger



Symbolic

Numbers
as decimal symbols
that can be used
for accurate
calculation

$$\sqrt{2} = 1 \cdot 414213562...$$

Implications for teaching

Powerful long-term learning requires compression of procedures into flexible symbols as processes to do and concepts to think about. (procepts).

When learning new ideas, if met-befores conflict with new knowledge, then the new ideas will not make sense.

The fall-back position is then **procedural learning**, without flexible links and successively less appropriate for later learning.

Procedural learning without understanding is limited in power and may fail in situations requiring greater flexibility.

Implications for teaching

Transitions that involve unhelpful met-befores:

from counting to whole numbers

from whole numbers to fractions

from whole numbers to signed numbers

from arithmetic to algebra

from powers to fractional and negative powers

From finite arithmetic to the limit concept

from description to deductive definition

at many other transitions in development of concepts such as the function concept. (linear, quadratic, trig., log., exponential ...)

In each case, conflict between old knowledge (met-before) and new knowledge, can lead to procedural learning.

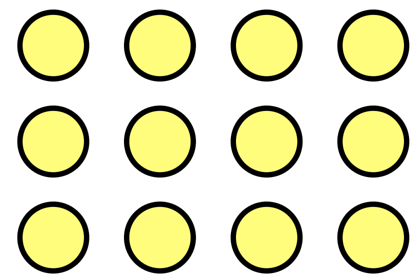
From then on, ***procedural learning*** *may be the only option!!!*

Implications for teaching

Example: the transition to algebra.

Met-befores:

Early arithmetic can be embodied: **4x3**



All expressions have an answer: 2+3 is 5, 3x4 is 12, etc

Letters represent units eg 3m is 3 metres, 3m = 300cm

For each professor there are six students.
p is number of professors, s number of students: **1*p* = 6*s***

letters sometimes used as representing objects

eg 3a is 3 apples, 4b is 4 bananas, 3a + 4b + 2a = 5a + 4b

*What is **3a – 2b** ?*

The transition to algebra has many conflicting met-befores!

Implications for teaching

A particular example from Brazil.

The PhD Thesis of Rosana Nogueira de Lima (2007)

Students have difficulty with linear and quadratic equations.

She notes that there is no natural embodiment that works for **all** linear equations.

The 'balance model' works OK for simple cases with positive terms and positive values (Vlassis, 2002, Ed. Studies).

The model fails with negatives and subtraction (Lima & Tall 2008, Ed. Studies).

Implications for teaching

A particular example from Brazil.

The PhD Thesis of Rosana Nogueira de Lima (2007)

The teachers decided to teach linear equations by 'doing the same thing to both sides to maintain the balance'.

e.g. take the same number from both sides and simplify.

Most students learned 'move the number over the other side and change its sign'. eg $3x+2 = 8$ becomes $3x = 8 - 2$.

The teachers taught 'divide both sides by the same number'.

Most students learned 'move the number over and put it beneath. e.g. $3x = 6$ becomes $x = \frac{6}{3}$

Implications for teaching

A particular example from Brazil.

The PhD Thesis of Rosana Nogueira de Lima (2007)

Overall, the teachers taught from general principles.

In general, many students learned to move the symbols around with extra 'magic' such as 'change sign', 'put it underneath' but with little meaning.

By the time they reached quadratic equations, algebra had become mainly procedural, so the teachers decided to concentrate on *the formula* to solve quadratics.

Implications for teaching

A particular example from Brazil.

The PhD Thesis of Rosana Nogueira de Lima (2007)

Students could use the formula to solve equations such as

$$3x^2 - 4x + 2 = 0$$

but they had difficulty with

$$(x-2)(x-3) = 0.$$

I leave the details to Rosana to report, or see:

Rosana Nogueira de Lima and David Tall (2008). Procedural embodiment and magic in linear equations. *Educational Studies in Mathematics*, 67 (1) 3-18.

Conclusions

Mathematical Thinking is based on three main SET-BEFOREs:

Recognition, Repetition, Language

where Recognition and Language give **categorization**

Repetition gives **procedural learning**

Repetition and Language give **encapsulation and flexible use of symbolism**

Language itself gives **definition** **and formal proof**.

Cognitive development builds on MET-BEFOREs

some supportive, some conflicting with new knowledge.

Conflict leads to challenge or to confusion.

Without making sense of new ideas, the only option is procedural learning, lacking flexibility for future development.

Meaningful learning requires understanding of MET-BEFOREs.