Technology and Calculus

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Of all the areas in mathematics, calculus has received the most interest and investment in the use of Technology. Initiatives around the world have introduced a range of innovative approaches from programming numerical algorithms in various languages, to use of graphic software to explore calculus concepts, to fully featured computer algebra systems such as *Mathematica* (Wolfram Research, 2005), *Maple* (Maplesoft, 2005), *Derive* (Texas Instruments, 2005), *Theorist* (no longer available, replaced by Livemath, 2005) and *Mathcad* (Mathsoft, 2005). The innovations arose for a wide range of reasons—some because a traditional approach to calculus was considered fundamentally unsatisfactory for many students, others because ”technology is available, so we should use it.” Most had a pragmatic approach, trying out new ideas to see if they worked. Some began with a theory that formulated how the enterprise should work, others formulated their theories in the light of successive years of experience.

Technology brought with it new market-driven factors in which large companies cooperated with educators to develop new tools. The first round of materials were in a competitive situation, often with the main objective to get the materials adopted. The early years of using technology in calculus were characterized by hopeful enthusiasm and based on little documentation of the true success of the new ideas. The system was complex, and the wider effects of the changes would take several years to become apparent. Opinions were many, informed observations few. Over recent years, evaluations of reforms and research into learning of calculus have begun to provide some answers about the effects of teaching and learning calculus—effects that may be positive, negative or neutral (Ganter, 2001), Hurley, Kohn & Ganter, 1999).

Our main aim is to focus on the research on the use of technology in teaching calculus and to report what it has to say to the community of mathematicians, educators, curriculum builders, and administrators. We include an analysis of the conceptual learning of calculus to put the research results in perspective. Our chapter addresses the wide range of students with different needs and aspirations who take calculus, the views of mathematicians, and the needs of society in this changing technological age.

In an article addressed to the mathematical community, Schoenfeld (2000) formulated some broad principles about the mathematics education research enterprise. He emphasized that there are no “theorems” in mathematics education that can be used to build up a theory in the way that is familiar to mathematicians, but there are issues of replicability, explanative power, and predictive power that can be of value in reflecting on teaching and learning mathematics. One must keep these issues in mind when considering the results from research.

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THE CULTURAL CONTEXTS OF THE CALCULUS

Calculus is a rich subject with a varied cultural history. It serves not only as a basis of mathematical modeling and problem solving in applications, but also as a natural pinnacle of the beauty and power of mathematics for the vast majority of calculus students who take it as their final mathematics course.

Calculus is only 350 years old. But in those three and one-half centuries it has been enriched by new perceptions and belief structures of successive generations, starting with the original conceptions of Leibniz and Newton on infinitesimals and limits. Currently we have a range of modern formal methods, from intuitive dynamic approaches, through numerical, symbolic, and graphic approaches, culminating in theories ranging from formal epsilon-delta analysis, which banishes infinitesimals, to nonstandard analysis, which fully endorses them. The result is a wide range of viewpoints as to how the calculus should be conceived and taught. We therefore begin by considering some of these differing views to place our analysis in perspective.

The Views of Mathematicians

During the 1980s there arose amongst many mathematicians a growing concern for the quality of student learning in the calculus (Douglas, 1986). This led to the Calculus Reform movement in the United States proposing the integration of technology to make the subject more meaningful to the broad range of students. Other countries, each in its own way, worked to integrate technology into its learning programs and its culture. For example, periodic reviews of the curriculum in France turned attention to the use of technology in the transition to university mathematics (Artigue 1990) and, in Britain, the Mathematics Association (1992) considered the calculus as part of its review of the use of computers in the classroom.

The range of aspirations for a calculus reform course are exemplified by the following list of desirable characteristics formulated by the MAA Subcommittee on Calculus Reform and the First Two Years (CRAFTY) (Roberts, 1996):

- Students should leave the course with a “sense of the role that calculus has played in developing a modern world view, the place it holds in intellectual as well as scientific history, and the role it continues to play in science” (p.1).
- Students and instructors alike should find the applications real and compelling.
- Instructors should have high expectations for all students and should employ pedagogical strategies (e.g., cooperative learning, laboratory experiences, etc.) that engage students’ interest and enable most to succeed.
- Students should learn to read and write carefully reasoned arguments at a level appropriate for their stage of development—intuitive at first, with rigor coming later for those who need it.

The CRAFTY committee observed:

> A calculus course cannot be modernized simply by finding a way to make use of graphing calculators or computers. Neither should a modern course omit these tools where their use contributes to the goals of the course. Spelling checkers … will not make a good writer out of a poor writer, and computer algebra systems will not make a good mathematician out of a poor one, but efficient practitioners of any art will make intelligent use of all the tools that are available (p.2).

The Calculus Reform provoked a vigorous debate among mathematicians. Some (e.g., MacLane, 1997; Wu, 1996) put the case for rigor and precision in mathematical thought, while others (e.g., Mumford, 1997) downplayed the emphasis on formal proof and advocated meaningful experiences to give insight into essential ideas. A vigorous correspondence ensued in the *Notices of the American Mathematical Society*, advocating a range of opinions, often focusing on different mathematical approaches to the subject:

- "In praise of epsilon/delta" Norwood (1997) suggested that the students' problems with epsilon-delta analysis lies not in the use of Greek letters but in the students' problem to understand subtraction and absolute value.
- "Use uniform continuity to teach limits." Lax (1997) put forward the view that it was necessary to reduce the emphasis on quantification and keep the key concept of function closer to the students' experience with functions.
- "Defining uniform continuity first does not help." Briggs (1998) responded with the comment that the problem lies not in building a foundation for rigorous arguments, but in the need for sense-making.
- "Use convergence to teach continuity." Abian (1998) suggested that the notion of convergence of of sequences of function values is more intuitive than any epsilon-delta argument.
- "Teach calculus with big O." Knuth (1998) proposed the use of more intuitive order-of-magnitude notions, starting with the A-notation for "absolutely at most."

Pragmatic Issues

As the debate continued, students were voting with their feet and moving away from mathematics. Between 1994 and 1996, student applications to mathematics departments in the USA declined by 32% (Maxwell & Loftsgaarden, 1997). Similar crises were building in other countries. In Germany, between 1990 and 1999 there was a fall of 20% in students registering to study mathematics and a fall of 35% signing on for first semester mathematics courses (Jackson, 2000).

There is evidence that even those who follow a major course in mathematics do not maintain all their knowledge as time passes. Anderson, Austin, Barnard and Jagger (1998) gave a questionnaire focused on what were considered essential simple first year concepts to a selection of final (third) year mathematics majors from 15 British universities. Only about 20% of the responses were "substantially correct" and almost 50% did not contain anything "credit-worthy". These data challenge the belief that an undergraduate mathematics course builds up a broad conceptual understanding of the full range of mathematics in the course. It is however, consonant with the experience of mathematics professionals who have a powerful knowledge of the mathematics that they are currently researching, but may be less facile with other areas which, nevertheless, they may be able to reconstruct given a little time. (Burton, 2004.)

Most students studying calculus are not mathematics majors. Kenelly and Harvey (1994) reported that 700,000 students enrolled in calculus in the USA, including about 100,000 in Advanced Placement programs in high school. In 1997, 12,820 students graduated with bachelor degrees in mathematics (NCES, 2001), less than 2% of the calculus cohort. In addition to provisions for this small—but vitally important—group, it is therefore essential to take account of the needs of the other 98%. Some of these move on to science and engineering programs, using calculus in very different ways from their peers in mathematics; for many others, calculus is their last experience of mathematics in their formal education.

THE CHANGING NATURE OF THE CALCULUS IN A TECHNOLOGICAL WORLD

Successive waves of new hardware and software have made the prediction of the future of calculus notoriously difficult. In the early 1970s, when computers were beginning to be appear on the horizon, the Mathematics Association in the United Kingdom wrote "It is unlikely that the majority of pupils in this age range will find [a computer] so efficient, useful and convenient a calculating aid as a slide rule or book of tables" (Mathematical Association, 1974).

Such illusions were soon shattered, and slide rules and books of tables lingered for only a short time before they became obsolete. It is important therefore in analyzing calculus teaching to be aware of the changing landscape.

Numerical algorithms

The first microcomputers had the BASIC programming language built in, so the first wave of enthusiasm was to encourage students to program their own numerical methods. At that time there were too few computers available in the classroom to allow programming to become universal. Nevertheless, there was widespread belief that programming would encourage students to formulate mathematical ideas—in particular, they might program algorithms for limit, rate of change, Riemann sums, and solutions of differential equations. Working with highly able mathematics undergraduates at Cambridge, Harding and Johnson (1979) found very positive effects on conceptual understanding and mathematical problem solving through programming mathematical algorithms. For the broader range of students, however, there is the possibility that simultaneously programming algorithms and conceptualizing mathematics may impose a great cognitive strain.

Using True BASIC to program algorithms, Cowell and Prosser (1991) reported:

> The students largely agreed that the computer assignments were well integrated with the rest of the course, and that learning the necessary programming was easy, but they disagreed that the computer enhanced their interest in the course material, they disagreed that the computer should be dropped and they were divided on whether the computer assignments were a valuable part of the course. (pp.152, 153.)

In England, the Mathematics Association Committee reporting on the use of computers (1991) saw the use of short programs as a definite way ahead, producing *132 Short Programs for the Mathematics Classroom*. But such moves failed to take root as other languages such as Logo came and went, and more powerful software environments were introduced. By the new millennium, the use of programming in mathematics had waned (Johnson, 2000).

New languages designed to use explicit mathematical constructions in the syntax, such as ISETL (Interactive SET Language), have been developed to introduce concepts of (finite) sets, functions, and quantifiers. through programming. Dubinsky and his colleagues (Asiala, Cottrill, Dubinsky and Schwingendorf, 1997, Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas and Vidakovic 1996) have reported learning gains in conceptualizing calculus using a carefully constructed learning sequence which will be considered in detail in the following section: "Understanding of the calculus—A spectrum of approaches."

Graphic visualizations

In the early eighties, high-resolution graphics brought new graphical approaches to the teaching of calculus that were designed to help students visualize mathematical ideas. There was soon considerable evidence that a visual approach to graphs helped students to gain a wider conceptual understanding without necessarily affecting their ability to cope with the corresponding symbolization (e.g., Heid 1988, Palmiter 1991). But on the debit side there was also evidence that the drawing of graphs involved quite subtle techniques in choosing an appropriate domain and range to give a suitable picture. It proved possible for serious misinterpretations of the meaning of the graph on the screen (Goldenberg, 1988).

Enactive control

In 1984 the "mouse" was introduced to give the computer an enactive interface. Instead of having to type in a line of symbols, the user could now select and control the display by intuitive hand-movements. This allowed a completely different approach to learning that encouraged active exploration rather than writing procedural computations. For instance, *Function Probe* (Confrey 1992, Confrey & Maloney 2008, Smith & Confrey 1994) allows graphs to be manipulated enactively, using the mouse to transform graphs by translating, stretching, reflecting. Such an approach treats the graph as a single object to be transformed and has the potential to relate physical movement to algebraic translations.

More generally, an enactive interface allows the user to conceptualise mathematics based on underlying human perception, for instance, through using software to build up a solution of a first order differential equation from line-segments by "pointing and clicking". Such an interface can have profound benefits for students requiring intuitive insight into concepts to support work in applications.

Computer algebra systems

The next stage introduced the power of computer algebra systems, bringing with it the possibility of removing the "drudgery" of symbol manipulation to allow students concentrate on formulating solutions that could be carried out by computer algorithms (Davis et al, 1992). The research literature will be considered later to seek evidence for this viewpoint. Essentially, we will find that well-designed approaches using computer algebra systems can produce considerable gains. On the other hand, students learn from what they *do*; if they press buttons in a computer algebra package without focusing on underlying conceptual ideas, they may simply learn which buttons to press (Monaghan et al, 1994).

Newer technologies

Technology continues to develop at an astonishing pace, migrating from large mainframes to desktop calculators to portable calculators and portable computers that can be carried around and used anywhere, anytime. The World Wide Web is now a reality, allowing information and software to be passed around the globe. Programs in Java can be ported between different hardware environments, allowing a single program to run on a wide range of machines. Computer algebra systems such as *Mathematica* and *Maple* have the facility to write text explanations with graphical displays that can be modified and investigated experimentally. Multimedia interactive software grows ever more sophisticated, offering the learner a variety of facilities including explanatory text, spoken words, video and interactive software to explore mathematical processes and concepts. Assessing the nature of learning with technology continues to focus on a moving target.

III UNDERSTANDINGS IN THE CALCULUS — A SPECTRUM OF APPROACHES

Traditional calculus, prior to the arrival of technology, focused on building symbolic techniques for differentiation, integration and the solution of differential equations, complementing them where appropriate by static pictures of graphs to illustrate the phenomena involved. Technology provides dynamic pictures under user control that can give new insights into concepts. For instance, by highly magnifying a graph, the resulting picture may reveal that the graph is "locally straight." Then, by looking along the graph to inspect the changing slope, or by plotting a moving line through two close points along the graph, the learner may visualize the changing slope as a global function.

How does this affect the learning of calculus concepts? Those mathematicians who wish to build mathematics as a formal system may distrust visualisation as containing hidden deceptions, and hence consider a visual approach lacking in rigor and precision. The formal approach, however, involves manipulation of a highly complex epsilon-delta definition with several nested quantifiers. Anecdotal evidence from mathematicians and empirical research by mathematics educators both show that the formal limit is a grievously difficult concept to use as a foundation for teaching the calculus (Davis & Vinner, 1986, Williams, 1991).

On the other hand, a dynamic visual approach to the limit concept also has built-in conceptual difficulties. Research has shown that students imagine not only a *process* of tending to a limit, but a *concept* of an "arbitrarily small" quantity (Cornu, 1991). Monaghan (1986) notes that, if a sequence, such as $(1/n)$, consists of terms that tend to zero, then the mind is likely to imagine a "generic limit," which is a limiting object having the same properties as all the terms. The generic limit of $(1/n)$, which might be written by the student as $1/\infty$, may be conceived as a quantity that is infinitesimally small, but not zero. In the same way, every term of the sequence 0.9, 0.99, 0.999, … is strictly less than 1, so the limit 0.999… is considered strictly less than one.

This view of a limiting process has a long history going back at least to the "law of continuity of Kepler and Cusa, which Leibniz re-expressed in a letter to Bayle of January 1687:

> In any supposed transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included.

Lakoff and Nunez (2000, p. 158) repackaged the same idea in a linguistic form as "the Basic Metaphor of Infinity" in which "processes that go on indefinitely are conceptualized as having an end and an ultimate result." The principle also has a biological basis in which a finite brain considering an ongoing process linking always to a particular brain structure will continue to have a link to the result. Hence the cognitive construction of variables that are "abitrarily small, but not zero" or functions that get "arbitrarily close to, but not equal to" a given limiting value. This natural product of human thought processes led to serious difficulties in the history of mathematics, requiring very subtle definitions to give satisfactory formal proofs.

Even when care is taken over the formal definitions, teaching mathematics can be subverted by the natural underlying human belief structure. Wood (1992) found that a significant minority of mathematics majors in their second year of analysis were able to simultaneously hold the two beliefs that "there is no *least* positive real number" but "there is a *first* positive number." Closer analysis revealed students using not one, but (at least) *two* models of the real numbers. One uses decimals, and reflects the student's dominant experience of numbers represented to a finite number of places that are naturally discrete. For instance, to four decimal places, after 0.0000 comes 0.0001 then 0.0002, and so on. A natural extension is to believe that after the "infinite decimal" 0.000…0 comes 0.000…1 (with an infinite number of zeros, followed by a 1, which might be written as 1 - 0.999), to give the "first" positive number. The other model involves dealing with numbers without writing them as decimals, so that, if a is positive, then $a/2$ is still positive and smaller, so there can be no "least" positive number.

Lakoff and Nunez (2000) formulate this beautifully by describing the real numbers as a "metaphorical blend" of two quite different metaphors, one a visual geometric metaphor of the "real line" and the other a numerical metaphor building up from counting and measuring number activities. These mathematically isomorphic systems are cognitively very different. Numerically a point has no size, but geometrically they fit together to give an interval of non-zero length.

Faced with a conflict between formalism and visual structure, the formal mathematician, in true Bourbakian style, distrusts the visualization as containing subtle aspects that conspire to deceive and relies totally on the formal theory. However, many of their students and their colleagues in other fields have a "metaphorical blend" view of numbers that includes such anomalies. There are two alternate ways out of this dilemma. One is to take a purely formal view and deal exclusively with mathematical symbols and quantified statements, which has proved to be notoriously difficult for the majority of students. The other is to educate visual intuition so that it is sound enough to build upon. Both strategies have been implemented using technology, leading to quite different approaches to the calculus.

A symbolic approach to calculus using a mathematical programming language

An approach to constructing mathematical concepts with both symbolic and psychological underpinnings has been designed by Dubinsky, a professional mathematician and a dedicated student of the theory of Jean Piaget. This uses the mathematical programming language ISETL in a carefully designed sequence of activities based on Piaget's theory of reflective abstraction. He and his colleagues formulate a theory of cognitive development with the acronym APOS (Action-Process-Object-Schema):

> An action is any physical or mental transformation of objects to obtain other objects. It occurs as a reaction to stimuli which the individual perceives as external. It may be a single step response, such as a physical reflex, or an act of recalling some fact from memory. It may also be a multi-step response, by then it has the characteristic that at each step, the next step is triggered by what has come before, rather than by the individual's conscious control of the transformation. … When the individual reflects upon an action, he or she may begin to establish conscious control over it. We would then say that the action is interiorized, and it becomes a process …. [Then] actions, processes and objects ... are organized into structures, which we refer to as schemas.
>
> (Cottrill, *et al*., 1996, p. 171)

Using this theory, Cottrill *et al* (1996) formulated a "preliminary genetic decomposition" of the limit of a function f as follows:

1. The action of evaluating the function f at a few points, each successive point closer to a than the previous point.

2. Interiorization of the action of Step 1 to a single process in which $f(x)$ approaches L as x approaches a. [This step was modified later in the paper in the light of empirical research to refer to the coordination of *two* processes, as "$x \to a$", and "$f(x) \to L$".]
3. Encapsulate the process of Step 2 so that, for example, in talking about combination properties of limits, the limit process becomes an object to which actions (e.g., determine if a certain property holds) can be applied.
4. Reconstruct the process of Step 2 in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, $0 < |x-a| < \delta$ and $|f(x)-L| < \varepsilon$.
5. Apply a quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of limit. […] applying this definition is a process in which one imagines iterating through all positive numbers, and, for each one called ε, visiting every positive number, called δ this time, considering each value, called x in the appropriate interval, and checking the inequalities. The implication and the quantification lead to a decision as to whether the definition is satisfied.
6. A completed ε-δ conception applied to specific situations. (Cottrill et al, (1996), p. ***)

This detailed analysis allowed the researchers to study the development in great detail, leading to the recognition of the need for an extra initial step in which the value of $f(x)$ is evaluated at a single point near, or equal to, a, before focusing on the process of calculating numeric values as x approaches a. More significantly the research revealed that the limiting process "as $x \to a$, so $f(x) \to L$" can be analyzed cognitively into two separate processes, one in which x approaches a, the other in which y approaches L, with these two processed being coordinated by the application of the function f.

This patient build-up of a concept through specific actions, routinized as a process, then encapsulated as an object, has a cost. In the first version of the development,

> … there were no students who progressed to the point where we could ask questions that indicated their thinking relevant to the last two steps of the preliminary genetic decomposition. We repeat them in the revised version, although they might be dropped for the present because there is no evidence for them. Cottrill et al (1996, p.187)

In other words, this approach may give a sound *process* conception of the limit, but it does not readily extend to an understanding of the epsilon-delta definition.

Other research has similarly underlined the difficulty of the formal definition. Knowing of the subtle problems using decimal numbers found by Wood (1992), mentioned above, Li & Tall (1993) designed an approach to limits of sequences through numerical programming for a class of pre-service school teachers. This involved gaining experiences intended to be a foundation for the limit concept. For instance, to get a better approximation to the limit of a sequence, say to 8 decimal places instead of 4, it is (usually) necessary to go further along the sequence to find the terms stabilizing to 8 decimal places. This was translated into the ε-N limit definition in the form "to get the limit within accuracy ε, you will need to find a suitable value of N for the limit to be given to that accuracy." However, such an approach had almost *no* effect on the students' belief that "nought point nine repeating" is strictly less than one. In addition, an attempt to study more general sequences had an unforseen side effect. The students were invited to consider the sum of the series whose nth term is "$1/n^2$ if n is prime, $1/n^3$ if it is not prime and even, and $1/n!$ if n is not prime and odd. The complexity of calculating the sum of this series to 1000 or 10000 terms was such that it took a considerable period of time. This phenomenon led half the class to believe that it would not tend to a limit even though they could accept that the sequence is bounded above using the comparison test. They reasoned that the sum would continue to increase slowly so that th any specified value might, after a long time, be exceeded. When introduced to the completeness axiom, these students refused to accept it *because they did not believe that it was true*.

Given the metaphorical blend of numerical symbolism and geometric number line, we can now see that the visual experience of a variable tending to zero evokes infinitesimal concepts (Cornu, 1982, 1992, Tall et al, 2001). The students therefore may have an image of a number line with infinitesimals and such a number system is *not* complete.

A similar obstacle arises naturally in the second stage of Dubinsky's genetic decomposition where $f(x)$ approaches L as x approaches a. Calculating numerical values of $f(x)$ for x near a evokes discrete numerical values, while continuous motion of a variable x getting "arbitrarily close" to a evokes an image of arbitrarily small infinitesimal quantities.

While a considerable proportion of students have difficulties with infinitesimals on the one hand and quantifiers on the other, there is a small number of students who learn to handle the formal definition. Pinto (1988) found that able mathematics majors were able to handle the definition in at least two distinct ways (reported in Pinto and Tall, 1999, 2001). One is the evident *formal* way, working with the definition and deducing the properties and theorems from the definitions and previously proven theorems. However, there is an alternate route, reminiscent of the work of more visually motivated mathematicians, often including geometers and topologists. This she termed a *natural* approach building up meaning for the definition by working on one's own personal imagery. Some natural thinkers had imagery in which they believed theorems to be true as a result of their imagery. (For instance, if a_n gets close to a, and b_n gets close to b, then clearly a_n+b_n must get close to $a+b$ without any further need of proof). Such students saw little need for formal theory and built ideas largely on their imagery. For them the formal theory made no sense whatever.

However, natural thinking using visual imagery may also be successful in leading to formal proof. One natural thinker did not wish to learn the definition by rote. He wanted a meaningful interpretation and saw the limit of a sequence in a picture in which the terms a_1, a_2, … were plotted in the Cartesian plane over x-values 1, 2, …, with the limit L marked as a horizontal line and the prescribed interval $L–\varepsilon$ to $L+\varepsilon$ marked as lines at distance ε above and below the limit line. For this value of ε, he sought a value of N so that, from this point on, all the points of the sequence a_n for $n>N$ lay within the horizontal lines $L\pm\varepsilon$. Not only did he use the picture to "see" the definition, he also built up the definition sequentially from this imagery and used the imagery to guide his later development.

His natural approach caused him to be often in a state of excitation as he constantly wrestled with his imagery. However, the theory he developed caused him to continually update his relationships with his other ideas, giving him a rich schema of intuitions and deductions that he could use to predict and prove new theorems. He was therefore successful, first by developing visual imagery that supported the definition, then using this imagery to underpin formal proof. He therefore may be considered a *natural formalist*, using natural imagery for thought experiments then modifying them to underpin formal theory.

On the other hand, another student, following a formal route, remembered the definition through repetition and use. Once he was able to write it down from memory, he was able to begin to use it to deduce properties in a formal manner. His knowledge of analysis was quite separate from his informal knowledge, which he distrusted and did not regard as an appropriate way to build up mathematical theory.

Thus, students can learn formal mathematics successfully in different ways, a phenomenon of which all mathematicians are instinctively aware. No single approach is suitable even for those who are successful. What technology provides is a way of encouraging natural thinking using visualization that can complement and support formal theory for those students who prefer a natural approach.

An embodied visual approach using local straightness

At almost the opposite end of the spectrum from a symbolic/numeric programming approach is a "locally straight" approach, based fundamentally on the use of dynamic visual graphics. The quality of the approach depends on the total package. Whereas the ISETL approach based on APOS is designed to encapsulate a programmed process (of limit, derivative, or integral) as a mental object, the locally straight approach *begins* with an explicit visual image of a graph that "magnifies to look straight". This builds on natural perception in a manner that is consonant with the idea that so much thinking is *embodied* in human perception and action (in the sense of Lakoff and Nunez, 2000). Embodied foundations of the calculus include not only visual aspects, but other bodily sensations such as those that come from manipulating objects onscreen using a mouse. An embodied approach builds on fundamental human senses. We will see that it provides a foundational approach to the calculus that can be an end in itself, but ay also lead into a formal theoretic approach, either in the traditional epsilon-delta form or in the form of non-standard analysis using infinitesimals.

A "locally straight" approach has been followed in a range of curricula, for example, the School Mathematics 16–19 Project in Britain is designed to allow students with limited algebra resources to look at graphs and get the computer to sketch the graph of the changing slope of the curve. It is straightforward to guess that the derivative of x^2 is $2x$, of x^3 is $3x^2$, and to conjecture the general pattern. A similar experimental approach extends to "see" the derivative of $\sin x$, $\cos x$, and even to find the numerical value of k for which the derivative of k^x is again k^x. This introduces an embodied meaning to mathematics. For instance, the the derivative of $\cos x$ is *minus* $\sin x$ not because an algebraic manipulation magically produces this result, but because the shape of the gradient of $\cos x$ looks the same as the graph of $\sin x$ "upside down."

Empirical evidence shows that this enables students to visualize the changing slope of the graph as a function and to sketch it with far greater insight than a corresponding student who has used only a paper-and-pencil symbolic approach (Tall 1985).

In many calculus reform programs, the notion of "local straightness" is seen to be synonymous with "local linearity"(see e.g., Dick and Edwards, Chapter 18 in this volume; Hughes Hallett *et al*., 1994, pp. 132-135; Smith and Moore, 1996, pp. 80-84). Although the two ideas are mathematically equivalent, they are cognitively extremely different. Local straightness is an embodied visual conception that involves imagining the graph highly magnified to "see" how steep it is. Symbolism is not necessary at the outset; it can be introduced at an appropriate stage either with, or following, familiarization with the fundamental embodied concepts. Local linearity introduces symbolism at the outset to find a linear function that is "the best linear approximation" to the graph at a particular point. Local straightness carries with it complementary visual ideas of *non*-differentiability (in terms of "corners" on the graph at a point, or "wrinkles" over a range). Local linearity concerns itself only with examples of functions that have a local linear approximation at a point.

The "locally straight" approach is capable of far more sophisticated insights at an early stage. By magnifying wrinkled functions that remain wrinkled wherever they are magnified, it is possible to give a visual sense of a nowhere differentiable function, revealing the possibility of visualising not only differentiability (local straightness), but also *non-differentiability* (lack of local straightness).

It may also be used in a "natural formalist" manner to build on dynamic perceptions to inspire formal definitions. For instance, we may begin with the idea of "stretching" a graph horizontally in a computer window whilst keeping the vertical scale constant. If the value of $f(x_0)$ is in the middle of a pixel of height $f(x_0)\pm\varepsilon$, then it "pulls flat" if there is an interval $(x_0 - \delta, x_0 + \delta)$, over which the graph lies within the horizontal line of pixels. This gives the formal definition of continuity:

$$\text{Given } \varepsilon > 0, \text{ there exists } \delta > 0, \text{ such that } |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon .$$

Given the known difficulties of such formal ideas, it should not be imagined that such an approach will be universally applicable. However, it can be very beneficial not only to natural thinkers who go on to study formal mathematics, but also to give a sense of greater subtlety for those who are unlikely to be able to cope with the symbolism. Tall (1993) employed a computer-based visual approach to a group of trainee teachers who would be expected to struggle with formal analysis. They were able to discuss a wide range of concepts in a "natural" manner. These include the fact that the area under a continuous function is differentiable, that the derivative of the area function is visibly the original function, that the solution of a first order differential equation involves knowing the gradient of a curve and following the direction it specifies.

A visual approach was used to "see" that the area under a continuous function has a derivative equal to the original function. Pre-service teachers working together in a group suggested that the area under a continuous non-differentiable function is a function that is differentiable once but not twice. This indicates the manner in which students with limited formal background may gain an intuitive grasp of highly subtle formal ideas, totally out of reach of most students following a traditional course.

The use of a perceptual approach therefore offers not only the possibility of a deep-seated human meaning of subtle formal concepts, but also has the additional possibility of extending the natural ideas into a natural formalist approach that moves from a visual basis to formal ideas.

A pragmatic locally linear approach using graphic, numeric, and symbolic software

Most of the approaches taken in the calculus reform (such as Hughes Hallett *et al.* (1994) or Smith and Moore (1996)) use the idea of local linearity, with a pragmatic choice of numeric, symbolic and graphic representations using computers and/or graphic calculators and/or computer algebra systems to approach ideas from a range of different viewpoints. As the reforms progressed, the role of verbal representation has been raised to an explicit level, formulated in the "rule of four" that encourages the use of verbal, numeric, symbolic and graphic representations to give a fuller sense of the concepts involved. New ideas are often introduced using specific contextual problems to situate the concepts in a practical context, using technology. Some approaches, such as *Calculus & Mathematica* (see Davis *et al*, 1994), use the power of a dynamic, computer-based text to explain concepts associated to numeric and graphic representations. On the other hand, *Calculus Using Mathematica* (Stroyan, **[REF]**). is an approach based on student programming in *Mathematica*. A major factor in many of these reforms is a new atmosphere of cooperation using technology to encourage active student exploration, discussion and commitment to personal construction of meaning.

IV THE ROLES OF TEACHER AND LEARNER

The role of the teacher

In approaches using technology, the teacher can play a significant role. Keller and Hirsch (1998) noted that the students' preference for numeric graphic or symbolic representations in part reflects the instructional preference. Kendal and Stacey (1999) report the effect of teachers privileging one more than others. In their research, three teachers agreed to teach the same syllabus with a TI-92 calculator but Teacher A enthusiastically used the computer algebra system at every opportunity, Teacher B was more reserved and underpinned the work with paper-and-pencil calculations. while Teacher C was enthusiastic about the calculator, using it more often for graphical insight. Their characteristics as observed by the researcher are given in Table 1.

Teacher Characteristics	Teacher A	Teacher B	Teacher C
Teaching style	Direct instruction	Guided discovery	Guided discovery
Direction of Lesson	Followed lesson plan	Controlled exploration	Open exploration
Attitude to using CAS	Enthusiastic	Reserved	Enthusiastic
Structured lesson around calculator use	Mostly	Sometimes	Mostly
Used algebraic explanations	Very often	Very often	Often
Used graphical explanations	Sometimes	Sometimes	Often
Used both algebraic and graphical explanations	Rarely	Sometimes	Often

Table 1: Researcher's categorization of teaching characteristics

Table 2 shows the teachers' predictions of their expectations of the students' work before the course began. These reveal links to the attitudes of the individual teachers and their teaching characteristics.

Teacher predictions	Teacher A	Teacher B	Teacher C
Algebraic competence	Moderate	Higher	Moderate
Graphical competence	Moderate	Moderate	Higher
With new technology	Likely to succeed	Will probably succeed	Very likely to succeed

Table 2: Teacher predictions of student success

An item-by-item analysis of student responses to the written test focusing on calculator usage and student success is given in table 3.

Student behavior	Class A	Class B	Class C
Use of calculator	Most frequent	Least frequent	Frequent
Decision to use calculator	Too frequent	Discriminating	Discriminating
Preferred approach	Algebra by calculator	Algebra by hand	Graphical with calculator
Algebraic proficiency	Moderate by hand	Higher by hand	Lower by hand

Graphical skills	Lower	Moderate	Higher
Procedural competence	Good	Good	Good
Conceptual understanding	Lower	Moderate	Moderate-higher

Table 3: Summary of the behavior and success on the written test

Although the classes had similar means on the test (24.1, 26.7, 27.9 respectively), students from the different classes got different questions correct. Class A students were most successful on questions that were procedural in nature while class C students were more successful in questions requiring conceptual understanding. It is interesting to note that although a number of procedural questions could have been performed (or checked) on the TI-92, there were many students in classes B and C who did not use the TI-92 to check them.

The role of the student

Students learning of the calculus encounter not only specific difficulties (such as the limit concept) but also a range of factors that are common to many environments. The Committee on Developments in the Science of Learning of the (U. S.) National Research Council (NRC) summarized research on learning in *How People Learn* (Bransford *et al*, 1999) and in a companion handbook for bridging the divide between research and practice (Donovan *et al*, 1999). In brief, in order for learning to take place, the handbook suggests:

- Students' initial understandings must be engaged.
- To develop competence, students must
  (a) have a deep knowledge base,
  (b) understand in a conceptual framework, and
  (c) organize for retrieval and application.
- Students must monitor progress toward goals.

Research on student *approaches* to learning (Entwistle and Ramsden 1983, Entwistle 1987, Ramsden 1992, Bowden and Marton 1998) tells us that deep learning approaches are quite different from surface learning approaches, and a given student – whatever his or her "learning style" – may exhibit different approaches simultaneously in different courses. These student-selected "coping strategies" are often determined, at least in part, by expectations set by the instructor, consciously or unconsciously. In particular, surface learning is encouraged by

- excessive amounts of material to be covered,
- lack of opportunity to pursue subjects in depth,
- lack of choice over subjects and/or method of study, and
- a threatening assessment system.

On the other hand, deep learning – the organized and conceptual learning described in the NRC study – is encouraged by

- interaction – peers working in groups,
- a well-structured knowledge base – connecting new concepts to prior experience and knowledge,
- a strong motivational context, with a choice of control and a sense of ownership, and
- learner activity followed by faculty connecting the activity to the abstract concept.

These are especially important messages for those whose goals include teaching mathematics to a much broader audience than just those who intend to become mathematicians. Calculus reformers and others interested in using technology, whether aware of the learning research or not, found that their task was as much about pedagogy as about choice of tools. As will be seen in our review of empirical research, the projects that successfully demonstrated learning gains are, for the most part, the ones in which use of technology is embedded in a rich learning environment that often looks quite different from the traditional lecture-plus-homework-plus-test environment of the typical college classroom.

V ANALYSING EMPIRICAL RESEARCH

In the preceding two sections we considered a range of pedagogical approaches to the calculus using technology, from an embodied visual viewpoint to a programming approach in which students are encouraged to write algorithms for mathematical processes to be encapsulated as mathematical concepts. Other pragmatic solutions use technology to utilise graphic, numeric and symbolic representations for student exploration and active construction of knowledge. We have emphasised the role of the teacher and also the participation of the learner in the enterprise. Wherever appropriate we have cited explicit empirical evidence to support any theories proposed. Now it is time to turn our attention to a broad review of research to draw out common themes that arise in the literature.

Calculus research at college level is of more recent vintage than research into teaching and learning mathematics in school. We therefore began with an analysis of the abstracts of some 40 dissertations completed in the last ten years, studying the original documents in detail where interesting ideas arose. After eliminating those that shed no light on the subject at hand, 29 studies remained, from which we draw the following observations.

Evidence that technology used inappropriately makes no significant difference

- Rasmussen (1997) studied six students in a traditional differential equations course (not calculus) with *Mathematica* exercises grafted on but not discussed in class. Interviews explored concept images with tasks adapted from reform curricula. One interpretation: Technology alone is not enough to overcome all the deficiencies of traditional pedagogy.
- Barton (1996) studied teachers overcoming skepticism about graphing calculators. There was little change in instruction, and the technology was grafted onto their usual classroom practice. Conclusion: Not much happens to teachers if there is no real professional development.
- Fredenberg (1994). Semi-weekly add-on computer labs (with unspecified software) in first-quarter calculus produced no change in achievement and little change in attitude or anxiety.
- Melin-Conejeros (1992) compared first-semester calculus students in a traditional course with experimental students whose only advantage was to use *Derive* for homework only, No technology was allowed on tests. Conceptual understanding was measured by a subset of questions on in-class tests; interviews were used to assess whether students used different problem-solving approaches. No differences in test performance or attitude were observed (although both groups went down in attitude). There was some interview evidence of slight conceptual gains by *Derive* group.

Partial evidence for technology producing changes

- Castillo (1998) introduced 10 supplemental TI-85 exercises in a multivariable calculus course. Relative to the control group, the calculator students improved their overall performance, but there was no change in processing patterns or attitudes.
- Parks (1995) compared two classes of first-semester calculus both with graphing calculators available where the experimental group had a *Mathematica* lab add-on (teacher demontrations and student explorations) to traditional instruction. Both classes were taught by the researcher based on the same traditional syllabus and were given the same tests. The lab promoted cooperation, exploration, use of alternate strategies, and better understanding of limit (the only concept studied).

Evidence that technology integrated intelligently with curriculum and pedagogy produces measurable learning gains

- Schrock (1990) compared first-semester calculus students in a *Maple*-supported, conceptually focused course with peers in a traditional course. The experimental group was allowed to use the computer on course tests, but no technology was permitted on pre- and post-tests or on a conceptual test constructed by the researcher. The results appear to confirm earlier results of Heid (1988, based on her 1984 dissertation), Palmiter (1986, 1991), Judson (1988, 1990), and others that students can learn concepts better and hand calculations just as well (with less time devoted to the latter) in a course designed to use CAS.
- Cooley (1996) compared a *Mathematica*-enhanced course with a traditional course and found significant gains in both concepts (as measured by a researcher-constructed test and by interviews) and calculation. This appears to improve on the results of Heid, Palmiter, and Judson.
- Porzio (1995) compared first-quarter students in *Calculus & Mathematica*, graphing calculator (with traditional content), and traditionally taught groups. Technology was allowed on quizzes and tests, and one interview per student focused on preference for certain representations. The C&M students were better able to use and connect multiple representations, but there was no difference between the other two groups.
- Park (reported in Park and Travers 1996) compared second-semester *Calculus & Mathematica* students with peers in a traditional course, using pre- and post-tests of achievement, an attitude survey, concept maps, and interviews. The findings from all instruments favored the *C&M* group.
- Hare (1997) studied students working collaboratively on five graphical function-derivative problems set in *MathWright* and found the software "invaluable" as a testing ground and in assisting students to generate cognitive understanding in cooperation with each other. This provides supportive evidence for combining cooperative learning with a well-designed computer environment.
- Connors (1995) compared results of a university calculus course before and after introducing a technology-integrated course with *TrueBasic* programs that students would use and modify. Connors measured gains for women relative to men in the "after" group and for both groups relative to "before." Females in the computer-integrated course benefited more than any other group; they also took fewer later courses but got significantly higher marks in them.
- Estes (1990) compared combined hands-on use of graphing calculators and computer demonstrations by the instructor with a traditionally taught control group. There were significant gains by the experimental group in conceptual understanding, but no difference in procedural ability. The students preferred what they could do hands-on to what they saw the instructor doing.

- Fitzsimmons (1995) studied combined use of a graphing calculator and *GyroGraphics* (dynamic 3D software) vs. a traditional class in solving 3D problems in calculus with pairs working together. There were positive effects of technology. Measured in terms of the five van Hiele levels they found that students at least two van Hiele levels below their partners made significant gains and recommended that assigned pairs should be either at the same van Hiele level or separated by at least two levels.
- Ramey (1997) studied why some high-achieving high school students choose project-based rather than traditional calculus. The positive factors included the challenge, hands-on applications, and extensive use of technology.
- Rich (1996) studied three groups of high school students: a control (traditional) group, a multiple representations group, and a dynamically linked multiple representations group. There were positive effects for the second and third groups in learning derivative concepts, but not much difference between these groups – i.e., dynamic linking appeared not to be the key characteristic.
- Ellison (1994) studied two sections of first-semester calculus students, a traditional class taught by another teacher and an experimental group that she taught herself. The experimental group used TI-81's and Tall's *Graphic Approach* software. Both groups used the traditional Leithold text, *Calculus with Analytic Geometry*, 5th ed. The researcher developed detailed case studies on 10 students, half from each section, chosen to represent a range of backgrounds and spatial abilities. Data included pre-tests, post-tests, unit exams, homework, lab reports, exit surveys, and three hour-long task-based interviews (one using computers). She found definite gains in understanding of derivative concepts, some of which were attributable to the use of technology. The experimental group had more positive attitudes than the control group.

Partial counter-evidence to the preceding category

- Roddick (1998) compared the performance of *Calculus & Mathematica* and traditional students in subsequent courses and found that the traditional students had an advantage in differential equations, whereas the C&M students had an advantage in physics and engineering. The C&M students were more likely to use a conceptual approach, the traditional students more likely to use a procedural approach. These results are similar to Bookman-Friedman study (1999) of Project CALC students in later courses. In particular, subsequent teachers did not know how to assess and reward the skills learned in non-traditional course.
- Soto-Johnson (1998, based on 1996 dissertation), using her own pre- and post-tests, compared three groups of second-semester calculus students – Project CALC (*Mathematica* version), modified *Calculus & Mathematica*, and traditionally taught – with a focus on conceptual understanding of series. She found no differences on that score. However, traditional students were better at series computations than the Project CALC students (who spent half as much time on series). The Project CALC students were better able than the other groups to make connections among derivatives, integrals, and series. The C&M students were better able to relate rate of decrease of terms to convergence.
- Meel (1996) compared third-semester honors students in *Calculus & Mathematica* and traditional courses. Students were given a 10-item instructor-constructed test (no technology), plus interviews on problem-solving and conceptual understanding in which technology was available. He found no differences except the C&M students were better problem solvers, and the traditional students were better on concepts (!) and on questions without figures.
- Crocker (1993) carried out a qualitative study of nine students using *Calculus & Mathematica* in second-quarter calculus, with a focus on derivative concept development and on problem solving strategies. She found that "stronger" students were less likely to

try multiple approaches – which may have something to do with how we identify "stronger" in the US: Students who have been successful with a single strategy (algebra) are unwilling to give it up or to consider alternatives.

Calculator-only effects

- Almeqdadi (1997) studied the attitudes of students and teachers to graphing calculators and found no effect on achievement. Students were generally positive in their attitudes, and gender differences favored males.
- Stroup (1997) studied 8th and 9th graders' learning of "rate" via "steepness" on a graphing calculator. There was an apparent paradox that the linear case was harder for students than nonlinear cases. This is not really surprising because of degeneracy (no change of steepness to observe) – the linear case is "easier" only in an algebraic sense, and algebra was in the background here.
- Williams (1996) studied relationships among learning styles, attitudes, achievement, and frequency of use of calculators. The conclusions are not entirely clear, but apparently attitude influenced achievement and graphing calculator use influenced attitude, so the author recommends integrated and maximized use of the graphing calculator, along with open-ended, long-term projects and investigations.
- See also Barton (1996), Castillo (1997)
- The research presents a range of issues, however, we did not see a clear pattern attributable to calculator use.

Computer-only effects

The dissertations above (Rasmussen, 1997; Fredenberg, 1994; Melin-Conejeros, 1992; Parks. 1995; Schrock, 1990; Cooley, 1996; Hare, 1997; Roddick, 1998; Soto-Johnson, 1998; Meel, 1996; Crocker, 1993) present a range of phenomena, but no pattern distinguishing the use of the computer over the calculator appeared, other than obvious differences between the facilities offered between the current generation of computers and calculators (eg the power of computer algebra systems on computers, and the ever-present portability of calculators compared with fixed computers in labs.)

Combined use of calculators and computers (either by students or researcher)

- Maldonado (1998) studied social construction of knowledge in an environment that included TI-85 and *Maple* (no control group). The abstract indicates that technology sometimes gets in the way of understanding, but no details were given.
- Galindo-Morales (1994, 1995) compared three groups of first-semester calculus students using, respectively, a graphing calculator, *Mathematica*, and no technology. Text materials also varied: The first group used the calculator-oriented Finney-Thomas-Demana-Waits **[REFERENCE]**, the second, *Calculus & Mathematica*, **[REF]** and the third, the traditional Finney & Thomas **[REF]**. Calculator students were allowed their technology at all times, but the second group was not allowed to use the computer on midterm or final exams. Conceptual understanding was measured by Presmeg's Mathematical Processing Instrument. Those with a visualizer learning style did better than non-visualizers with *C&M* and with no technology, but there was no difference for the graphing calculator group. He also found no gender differences in degree of visualization or in performance.
- See also Estes (1990), Fitzsimmons (1995), Ellison (1994) – there is no clear pattern in these studies.

Lack of evidence comparing different technologies

Most studies focus on a particular technology, rather than comparing technologies. Some of the things being done on computers can't be done at all on calculators (e.g., linked dynamic representations) or can't be done well (e.g., rotation of 3D graphics); otherwise there is little evidence that the choice of technology matters much. None of these studies looked at the possible effects of having a technological tool constantly at hand, as one can with a calculator, but not yet with a computer. What clearly does matter is how the tool is used and how well it is integrated with sound pedagogy.

Other studies using technology but focusing on other aspects

- Thomas (1996). The thesis is mainly about refining the genetic decomposition of learning the Fundamental Theorem in C^4L, a curriculum that depends on ISETL for delivery. (See our earlier discussion of ISETL as a conceptual language for programming mathematical ideas.)
- Fiske (1995) studied secondary students, with a focus on local linearity vs. secant/tangent definition of the derivative. Both groups used the computer extensively – each group was better at some aspects, but there was no difference overall. The instruction was teacher-centered, with some active investigation in both groups.
- Emese (***) compared three groups: graphing calculator + discovery, graphing calculator only, and traditional (Discovery without technology was not considered.) He found no difference in achievement, but concluded that discovery is a viable alternative.

Summarising Our Analysis of Doctoral Theses

Reflecting on the general trends found in the dissertations, some broad themes begin to surface.

- Technology used inappropriately usually makes no significant difference, although sometimes the mere access to better tools appears to alter the environment enough to allow observable benefits.
- Technology integrated intelligently with curriculum and pedagogy produces measurable learning gains.
- There is little evidence that the "brand" or type of technology makes any significant difference, beyond to obvious fact that some tasks require more powerful tools than others. Some, such as *Mathematica* and *Maple* give a broader environment for presentation using notebooks and exploration using built-in programs . The important thing is not *which* tools are used but *how* they are used.
- There is evidence that using tools such as *Mathematica, Maple* for conceptual exploration, to learn how to instruct the software to carry out symbolic calculations, leads to conceptual gains in solving problems that can transfer to later courses. In comparison, students following traditional courses tend to use more procedural solution processes.
- Technology enables some types of learning activities (e.g., discovery learning) and facilitates some others (e.g., cooperative learning) that are harder or impossible to achieve without technology.

Research Papers

The growing number of research papers in the use of technology in the calculus cover a wide range of considerations, to such an extent that attempting to summarize them would lose much of the subtle detail that makes them so distinctive. Most focus on implementations of a specific approach, often compared with a traditional course, a few focus on broader aspects, such of the effects of taking different calculus courses on subsequent study. Some papers make useful distinctions between various aspects of reform, in the *curriculum* (change in content of the syllabus), *pedagogy* (change in teaching strategies) (Lefton and Steinbart, 1995). In addition, we would place a high premium on the changes that occur in the *learner*. Here there are a wide spectrum of aspects, including the balance of conceptual and procedural understanding, in their ability to communicate mathematical ideas, to use technology productively, to develop problem-solving strategies, and their flexibility in coping with new situations. In this section we review a number of the broader aspects, referring the reader to the original paper for finer subtleties.

Two different studies concerning *Calculus and Mathematica* (Lefton and Steinbart, 1995, Park and Travers, 1996) yield similar conclusions. The philosophy of the course (using technology as a toolbox for measurement both exact and approximate) gives the opportunity for a new curriculum that emphasises the underlying meaning by building on prototypical examples, at the same time eliminating the epsilon-delta definition and de-emphasizing formal proof. The pedagogy eliminates lectures and introduces the exploration of concepts through Mathematica notebooks and discussion of ideas. Park and Travers (op. cit, p.175) suggest that "the C&M course allows the student to spend less time on computation and better direct their efforts to conceptual understanding". The comparison of students following these courses with those following a traditional approach consistently reveals a higher conceptual understanding and problem-solving ability, together with more positive attitudes towards technology and mathematics. The changes in the learner arise through a combination of both new curriculum and pedagogy. Lefton and Steinbart (op. cit., pp. 93-94) found that client departments were "overwhelmingly supportive", reflecting "the way our colleagues actually used the calculus"; the response of the mathematics department was "lukewarm", a view which may reflect the changing emphasis to inductive reasoning rather than proof, and the perceived effort that is involved in changing to the use of new technology.

A modification of the course using the language *Maple* instead of *Mathematica* showed some differences in perceived outcome. Aldis, Sidhu and Joiner (1999) used *Calculus and Maple* in two classes studying differential equations at the Australian Defence Academy over a seven week period. Both classes experienced difficulty with Maple syntax in the time available but were appreciative of the group-work and small class aspects. The authors hypothesized that a longer exposure to Maple could reduce the difficulties with syntax. Based on their explicit experience, they suggested:

> … key concepts in a course should be *introduced* through a CAS. This would allow students to explore ideas without the hindrance of specific notation or technical terms and to develop a memorable, visual impression. Once introduced, the mathematical concepts could be followed up in small-group lectures and non-computer activities. (Aldis, et al, 1999, p. 186.)

Such a conclusion is consistent with an embodied approach to the calculus that was discussed earlier in this article, using multi-media tools to enable the students to begin by gaining insights into the concepts before tackling the detail of the more powerful symbolism. Indeed, the approach of *Calculus and Mathematica* has a deep underlying philosophy that builds concepts in a manner that is meaningful to the students in their own terms.

Similar conclusions arise with the Project CALC materials using …**[David - please fill in the software you used in the Bookman Friedman research - they don't say what it is!]**. Again there is a commitment to both new curriculum and new pedagogy using computer laboratories for cooperative learning. The Project CALC students performed significantly better than traditional students on problems that could be solved by pencil, paper and scientific calculator. Some of the test items, such as word problems that required translation into mathematical form, were more appropriate for the Project CALC students than for those following the traditional course. However, the manner in which the experimental students responded shows the ability of the reform course to raise awareness and success in achieving higher order learning goals.

Using MathCad as an environment for computer-aided learning, Blachos and Kehagias (2000) report positive effects of the reform course that led to their decision to implement the approach in all sections of their business calculus. MathCad offers a combination of text, graphics, numeric and symbolic facilities in a free-form environment that the authors found particularly easy to design their course The improvement in student performance showed by comparison with a standard approach on standard tests. They also report subjective experiences of improvement in attitudes, enthusiasm, systematic study, flexibility in using the representations, and personal "empowerment" in which weak students could "survive" and strong students have the opportunity to excel. They also counsel designers of such courses to be aware of negative factors, which they felt they had addressed in all cases, but needed to be listed for consideration by others. This included the possible "overhead" in teaching specific computer skills, the possibility of the study degenerating into "computer gaming", the possible effects on later courses with very different traditional content, and the disorientation that may occur for both students and instructors caused by the radical change in approach.

The effects of various approaches on later courses have been researched in a major study involving over two thousand students. Armstrong & Hendrix (1999) compared students on courses following three different approaches to calculus I and II: a traditional approach (Ellis & Gulik, 1991), *Calculus using Mathematica* (Stroyan, 1993) and the Harvard Calculus Consortium (Hughes-Hallett, et al, 1994). *Calculus Using Mathematica* is a computer-intensive, project-based course for honours students, quite different from the traditional courses which currently follow it and also involving more student programming than *Calculus & Mathematica*. It attracted students with marginally better ACT scores and its students performed marginally better in subsequent courses. Students who switched from this course in Calculus I to a different course in Calculus II performed significantly worse in later courses. The researchers attribute this in part to the fact that those students switching to other courses were already having difficulty and continued to have difficulty in their new courses. There was no significant difference in the later performance of students who switched in either direction from the traditional approach to the Harvard approach between calculus I and II. For the majority of students who stayed with the same approach in both calculus courses, *there was no significant statistical difference in performance on subsequent courses regardless of which approach they followed in Calculus I and II.*

The authors concluded that this justifies the teaching of traditional and reform courses concurrently as alternatives. In particular, they refute the opinion of "critics who blame reformed calculus when their students 'can't differentiate' or 'never learned to integrate'," (Armstrong & Hendrix, 1999, p. 98).

A word of caution should be expressed here. The research does not say there is no difference between the two approaches, only that there is no statistical difference in the marks attained on subsequent examinations. Wells (1995), in a broad review of calculus research, noted that procedural items can give the impression that "most" students can be successful in the material covered, whereas more conceptually based items will reveal a broader range of success in understanding. If there is a qualitative difference, this may not be apparent statistically in the marks awarded for performance on traditional examinations. Mathematicians, in their desire to set examinations on which students can perform satisfactorily after following traditional courses, may therefore set questions which set more store on procedural competence than on deeper mathematical understanding.

Although there is a broad consensus that students using technology do not score significantly differently from traditional students in pure mathematics exams, there is other evidence that those who go on to courses requiring problem solving skills have better conceptual tools to formulate and solve problems (Meel, 1996,1998; Roddick, 1998).

It is sensible to consider any empirical research with a healthy scepticism. Fortunately, authors are usually scrupulously fair in reporting their data. Keller and Russell (1997) describe a preliminary study investigating students' performances on three different type of problems, simple directive computations such as evaluating a specific integral, multi-step problems such as finding a maximum, and complex problems formulated in a manner that requires information to be translated into appropriate mathematical form. In the preliminary study with a small number of students, they found that students using TI-92 calculators in a supportive group-oriented classroom context were better at all three forms of problem than students who had access to other types of calculator. However, on a later large-scale study (Keller, Russell & Thomson, 1999), their results showed a significant difference ($p<0.001$) in one-step and multi-step problems, but not in complex problems.

VI. DRAWING CONCLUSIONS

As we stated at the outset, a review of the research does not, and cannot, give definitive answers to whether reform courses using technology are "better" in general. There are no "theorems" in mathematics education as there are in mathematics. However, our journey through the research literature has produced some consistent themes that may be of benefit for those designing curricula and more generally for those seeking a deeper understanding of the issues.

First, a single opinion based purely on a particular philosophical viewpoint, be it from formal mathematics, crusading reformers, practical users, or whoever, needs to be seen not only in its own context, where it may have considerable substance, but also in the wider framework where other views have their own validity. In this chapter we have explicitly looked at the following headings:

- *Cultural contexts of the calculus*. In this section, the purpose was to generate a sense of the wider range of viewpoints
- *The changing nature of the calculus in a technological world*. By reviewing the changes over the recent years, we give a perspective to show how the situation continues to change, so that our analysis continues to consider a moving target.
- *Understanding the calculus – a spectrum of approaches*. Different approaches are possible in the calculus and have differing cognitive issues arise. A spectrum of approaches to the calculus is formulated from a human embodied approach through to a formal mathematical viewpoint, with technology being used for dynamic enactive visualization, conceptual programming, and the pragmatic use of a variety of technological resources involving symbolic, numeric and graphic representations.

- *The Roles of Teacher and Learner*, a consideration of the way in which the teacher can affect the learning and
- *Analysing empirical research* - in which we made a broad review of research dissertations and research studies.

In this journey, a number of themes arise. The first is the role not just of the technology, but the manner in which the technology is used. Clearly the skills of using software such as Mathematica or Maple is a valuable resource in its own right. However, the way in which it is introduced is extremely important.

… To be concluded.

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