Abstraction as a Natural Process of Mental Compression

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This paper considers mathematical abstraction arising through a natural mechanism of the biological brain in which complicated phenomena are compressed into thinkable concepts. The neurons in the brain continually fire in parallel and the brain copes with the saturation of information by the simple expedient of suppressing irrelevant data and focusing only on a few important aspects at any given time. Language enables important phenomena to be named as thinkable concepts that can then be refined in meaning and connected together into coherent frameworks. Gray & Tall (1994) noted how this happened with the symbols of arithmetic, yielding a spectrum of performance between the more successful who used the symbols as thinkable concepts operating dually as process and concept (procept) and those who focused more on the step-by-step procedures who could perform simple arithmetic but failed to cope with more sophisticated problems. In this paper we broaden the discussion to the full range of mathematics from the young child to the mature mathematician and support our analysis by reviewing a range of recent research studies carried out internationally by research students at the University of Warwick.

Introduction

The term ‘abstract’ has its origins in the Latin *ab* (from) *trahere* (to drag) as:

- a verb: to ab*stract*, (a process),
- an adjective: to be *ab*stract, (a property),
- and a noun: an *ab*stract, for instance, an image in painting (a concept).

The corresponding word ‘abstraction’ is dually a process of ‘drawing from’ a situation and also the concept (the abstraction) output by that process. It has a multi-modal meaning as *process*, *property* or *concept*.

Gray and Tall (2001) envisaged at least three distinct types of mathematical concept: one based on the perception of objects, a second based on processes that are symbolised and conceived dually as process or object (procept) and a third based on a list of properties that acts as a concept definition for the construction of axiomatic systems in advanced mathematical thinking. Each of these is an abstraction: a mental image of a perceived *object* (such as a triangle), a mental *process* becoming a concept (such as counting becoming number) and a formal system (such as a permutation group) based on its *properties*, with the concept constructed by logical deduction.

Our purpose in this paper is to unite these various different ways of abstracting concepts in mathematics into a single construct by seeking an underlying mechanism in human thinking that gives rise to them all and then to review how this mechanism works successfully in some cases but not in others.

How do Humans do Mathematics?

We begin with a much more fundamental question: How does a biological creature like *Homo Sapiens* do mathematics? First, many individuals develop and build on each other’s work to construct a body of mathematical knowledge that is recorded in books and other products of human culture and shared with the community. Every individual develops from being a child who knows no mathematics into an adult who may learn to share the mathematical culture. Even mathematicians that created that culture also went through such a development from being a baby dependent on mother's milk to becoming a sophisticated adult such as Plato, Newton or Einstein. This has profound implications when we analyse mathematical thinking in general and abstraction in particular. By gaining insight into the way that mathematical thinking develops from child to adult, we also gain insight into the mathematical thinking of adults and into the nature of mathematics itself.

Homo Sapiens thinks using the biological structure of the human brain; hence mathematical abstraction in particular and mathematical thinking in general is built from the biological operations of the brain. The evidence shows that human brains, though exceedingly complex, are only able to concentrate consciously on a few things at once, requiring a mechanism to cope with the complication:

> The basic idea is that early processing is largely parallel – a lot of different activities proceed simultaneously. Then there appear to be one or more stages where there is a bottleneck in information processing. Only one (or a few) 'object(s)' can be dealt with at a time. This is done by temporarily filtering out the information coming from the unattended objects. The attentional system then moves fairly rapidly to the next object, and so on, so that attention is largely serial (i.e., attending to one object after another) not highly parallel (as it would be if the system attended to many things at once). (Crick, 1994, p. 61)

In addition to filtering out information, there must also be a mechanism to enable the essential information to be held in the brain in an economical manner.

Compression

The mechanism by which information is held in an economical manner relies on a phenomenon that we term *compression* (after Thurston 1990):

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics (Thurston 1990, p. 847).

Compression involves taking complicated phenomena, focusing on essential aspects of interest to conceive of them as whole to make them available as an entity to think about.

Although other species have such mechanisms to function in their own context, *Homo sapiens* has a tool that enables it to grasp a complex situation, reflect upon it at various different levels of sophistication and to communicate with others: language. The essential feature of this tool is to *name* a phenomenon as a word or phrase, to allow the name to be spoken when referring to that phenomenon and then to use language to discuss its various aspects and to focus on its various properties and its relationships with other phenomena. We use the term 'thinkable concept' to refer to some phenomenon that has been named so that we can talk and think about it. This can be any part of speech, and may refer to any phenomenon, such as number, food, warmth, rain, mountain, triangle, brother, fear, black, love, mathematics, category theory. The phrase 'thinkable concept' is, of course, a tautology, for a named phenomenon is a concept and is therefore thinkable. However, given the many meanings of the term 'concept', we choose to use the term 'thinkable concept' here to emphasise its particular use in this theoretical framework.

Thinkable concepts are noticed before they are named. First various properties and connections are perceived in a given phenomenon, but it is only when these are verbalised and the phenomenon is named that we can begin to acquire power over it to talk about it and refine its meaning in a more serious analytic way.

As an example of the development of a thinkable concept, consider the notion of *procept* itself (Gray & Tall, 1994). As we sat looking at data from children solving arithmetic problems, we saw how some children seemed to use number symbols both for counting procedures and also as thinkable 'things' to operate upon. Suddenly we realised that this phenomenon needed a name to talk about it and the word 'procept' was born. At this point it was just a word linked to a complicated phenomenon. But by naming that phenomenon, we acquired the power to think about it and talk about it to each other and to our colleagues.

Whereas others had talked about a process becoming encapsulated, or reified, as an object (Dubinsky, 1991; Sfard, 1991), they did not have a name to talk about the elusive underlying concept that was *both* process *and* concept. Though our invention built heavily on their work, it moved it to a more sophisticated level. We can now talk about different kinds of procepts, including *operational* procepts such as 2+3 in arithmetic, which always have a procedure to produce a result, *potential* procepts in algebra such as $2+3x$ which represent both a general process of evaluation that cannot be carried out until x is known and also a concept of an algebraic expression that can be manipulated, and *potentially infinite* procepts, including the concept of limit (Tall, Gray, *et al.*, 2000). We can go on to discuss how different kinds of procepts involve different kinds of cognitive advantages and difficulties and move the theory to a new level of sophistication.

This typifies the way in which a complicated phenomenon (here operating with symbols as process and concept) can be compressed into a thinkable concept (here 'procept') to allow us to think about the phenomenon in a more sophisticated way. We suggest that this is the underlying mechanism of abstraction to compress phenomena into thinkable concepts that enables human thought in general and mathematical thinking in particular to operate at successively higher levels of sophistication.

Making connections between thinkable concepts

Having considered the compression of knowledge into (thinkable) concepts, we now address the manner in which the brain connects them together. This is through a biological process called *long-term potentiation*, which is an electro-chemical modification of the links between neurons to favour those that are useful and build stronger neuronal connections (Hebb, 1949). All neurons have multiple inputs from other neurons and a single output (the axon) that passes electrochemical messages down its length and branches out to connect to other neurons. A particular neuron receives charges from other neurons and when it reaches a threshold, it fires down its axon. This occurs typically several times a second. Links that fire more often are changed chemically and are more easily fired for a time. This leads to the 'recency' effect, in which we continue to be conscious of more recent events and can sustain a continuous train of thought. If a link is repeated and put on a 'high', it may then be strengthened to such a level that it fires automatically, making the link essentially permanent. This process is 'long-term potentiation' that builds connections between thinkable concepts. It operates by a process akin to Darwinian selection in which successful links are enhanced and dominate others in the long-term (Edelman, 1992).

The necessary corollary of long-term potentiation is that the brain can only think using either built-in structures, such as vision, taste, smell and their respective connections, or mental constructions based on previous experience. The successive experiences that we have therefore deeply affect the ways in which we are able to think at later stages.

Homo sapiens shares a facility for learning through repetition that is common in many other species. Repetition can strengthen the connections in the brain to such an extent that repeated actions become routine and performable without conscious thought. Used properly, as part of a rich knowledge schema, such compression is a valuable and essential tool. Used on its own, to learn 'rules without reasons' without the subtleties of rich thinkable concepts, it is likely to lead to fragile knowledge that may fail as the situations become more complicated.

If a child compresses ideas into thinkable concepts, this will build the tools to work at a more sophisticated level. If not the ideas may simply become too complicated to cope. Krutetskii (1976) studied the success of four groups (gifted, capable, average, incapable) in terms of their compression of solutions procedures and found that the gifted were likely to curtail solutions to solve them in a small number of powerful steps, whilst the capable and average were more likely to learn to curtail solutions only after considerable practice, and the incapable were likely to fail. Gray and Tall (1994) report a spectrum of different performances in arithmetic that they described as 'the proceptual divide' between those who cling to the comfort of counting procedures that, at best, enables them to solve simple problems by counting and those who develop a more flexible form of arithmetic in which the symbols can be used dually as processes or as concepts to manipulate mentally. 'Proceptual thinking' occurs when counting procedures are compressed into number concepts with rich connections: knowing things like '4 and 2 makes 6' so '6 take away 4 must be 2' and using these 'things' to derive new knowledge, such as '26 take away 4 must be 22 because 26 is just 20 and 6'.

The compression of complex phenomena into thinkable concepts is a natural biological development. However, to trigger the compression requires a specific focus on relevant aspects of a situation: identifying salient features to sense an underlying phenomenon that is named and refined into a thinkable concept. Abstraction is the process of 'drawing from' the situation to focus on the thinkable concept (the abstraction) output by that process. This thinkable concept is then available to be used as in more sophisticated levels of thinking.

Abstraction may focus on the *properties* of perceived objects and give them names that are compressed by categorisation into different levels of sophistication (Rosch, 1978). A child may say, 'that is a "dog", that is a "cat", they are both "animals"', and a whole tree of classification becomes possible. It includes individuals such as 'Rover' or 'Tiddles' through generic 'cat' and 'dog', which are both 'mammals', as are also 'elephants' and 'rats', but not 'frogs', however all are 'animals', and so on. This kind of hierarchy occurs in geometry, studying categories of figures such as 'triangle', 'square', 'rectangle', 'circle' and classifying, for example, different kinds of triangle ('scalene', 'isosceles', 'equilateral', 'acute-angled', 'right-angled', 'obtuse'), seeing that a square is a special kind of rectangle and both are quadrilaterals which, along with triangles, pentagons, hexagons, and so on are all 'polygons'. Each category is a thinkable concept. The study of the properties of these objects and the actions upon them (geometric constructions) builds eventually into a coherent theory of Euclidean geometry.

Abstraction may focus on *actions* on objects, which leads through compression to the computable symbols in arithmetic, the manipulable symbols in algebra and symbolic calculus. The symbols are thinkable concepts that we can operate on with properties such as 'even', 'odd', 'prime'. They provide the basis for and extension to wider concepts such as 'fraction', 'decimal', 'rational', 'irrational', 'real', 'complex' and algebraic concepts as expressions we can 'factorise', 'simplify' and equations we can 'solve'.

Eventually the focus may turn to the properties of mental objects, compressed through several stages until it is possible to formulate a *concept definition* solely in terms of set-theoretic language and deduce further properties using mathematical proof.

All of these cases show abstraction in action: focusing on relevant aspects and naming or symbolising them as become thinkable concepts, be they mental images of objects (eg 'triangle'), symbolism for a process compressed into a concept (eg 3+2 as a process of addition and the concept of sum) or for structures defined by a list of set-theoretic axioms (such as the complete ordered field $\mathbb{R}$ or the infinite cardinal $\aleph_0$).

This formulates all forms of abstraction in neuronal terms as the compression of a coherent phenomenon into a thinkable concept. Within this wider framework, however, there remain significant long-term differences between abstraction based on perceived objects, abstraction based on actions and abstraction based on set-theoretic definitions. These differences are well represented in the research literature.

Earlier Theories

The distinctions between different forms of abstraction feature in a range of accepted theoretical frameworks. Piaget distinguished between construction of meaning through *empirical abstraction* (focusing on objects and their properties) and *pseudo-empirical abstraction* (focusing on actions on objects and the properties of the actions). Later *reflective abstraction* occurs through mental actions on mental concepts in which the mental operations themselves become new objects of thought (Piaget, 1972, p. 70). Here reflective abstraction is seen as an activity akin to pseudo-empirical abstraction, now applied to mental entities rather than physical objects.

Skemp (1971) considers the fundamental human activities to be perception, action and reflection. Perception involves input from the senses, action involves output through interaction with perceived phenomena, and reflection is the process whereby we think about relationships between perception and action. Skemp (1979) talked about two distinct systems, 'delta-one' which involves perception of and action on the actual world we live in, and a second internal system, 'delta-two', whereby our brains imagine internal perceptions and actions and reflect on them. Here we have perceptions of and actions on objects with reflection producing a developing mental framework.

Fischbein (1987) focused on three distinct aspects of mathematical thinking: fundamental *intuitions* that he saw as being widely shared, *algorithms* that give us power in computation and symbolic manipulation, and the *formal* aspect of axioms, definitions and formal proof.

Bruner (1966) focuses on three modes of operation: enactive, iconic and symbolic, which inhabit a similar theoretical discourse where embodiment relates to a combination of enactive and iconic while symbolic is subdivided into the symbolism of arithmetic and algebra the formalism of logical proof.

Biggs and Collis (1982) built on the stage theory of Piaget and the modes of Bruner to build a theory of Structure of Observed Learning Outcomes in assessing the progress of students through successive modes: sensori-motor, ikonic, concrete-symbolic, formal, and post-formal. Within each mode there were cycles of performance—unistructural, multi-structural, relational, extended abstract—which Pegg & Tall (2005) related to cycles of concept construction, dealing with one aspect, several separate aspects, related aspects, then the whole idea. They referred to this as a 'fundamental cycle of concept construction and noted this occurring in process-object encapsulation through a single procedure, several procedures (with the same effect), a process (producing the required effect possibly by several different procedures) and a procept.

The levels of success in achieving the compression of a complicated situation into a thinkable concept can be formulated in general using this fundamental cycle: focusing on an isolated aspect of a situation, encompassing several aspects without grasping their relationships, relating several aspects, grasping the situation as a whole.

In this way SOLO taxonomy encompasses the analysis of learning outcomes, both in terms of the long-term growth of different modes of operation and also the local cycles of abstraction in which complicated phenomena are compressed into thinkable concepts.

There are many differences between each of these theories and it would be highly inappropriate to attempt to combine them into a single theory. However, there are underlying themes that relate to the natural growth of mathematical thinking.

Gray and Tall (2001) reviewed a range of these theories and hypothesised that elementary mathematical thinking builds by focusing on objects (for instance, in geometry), and on operations on objects represented as symbols (in arithmetic, algebra, etc) that operate dually as process and concept. At a later stage, in advanced mathematical thinking, the focus changes to the properties (of objects and operations) formulated set-theoretically as fundamental axioms for mathematical theories.

Based on these ideas, Tall (2004) formulated three distinct ‘worlds of mathematics’:

> the *conceptual-embodied* (based on perception of and reflection on properties of objects);
>
> the *proceptual-symbolic* that grows out of the embodied world through action (such as counting) and symbolization into thinkable concepts such as number, developing symbols that function both as processes to do and concepts to think about (procepts);
>
> the *axiomatic-formal* (based on formal definitions and proof) which reverses the sequence of construction of meaning from definitions based on known concepts to formal concepts based on set-theoretic definitions. (Tall, 2004, quoted from Mejia & Tall, 2006)

This framework is consistent with the research we have performed over recent years in our studies of the development of the individual from the young child to the sophisticated adult. Gray and Pitta (1996) consider the way that more successful children focus on the subtle compression of knowledge of arithmetic while less successful children remain fixated on the more visible complication of the physical detail. Tall, Gray *et al.* (2001) look at the long-term growth of procepts through arithmetic, algebra, calculus, and on to formal definitions in terms of properties and proof. Tall (1999) considers the distinct forms of proof available as the child develops cognitively from physical interaction with the world, through thought experiments, the properties of procepts proved by calculation and manipulation and on to formal proof as a mathematical expert. Tall (2002) reviews the calculus in terms of an embodied enactive-iconic approach manipulating graphs, symbolic-proceptual representations (manipulating formulae) and formal proof (in analysis). Tall, Gray *et al.* (2000) focus on the development through the proceptual-symbolic world and the transition to the axiomatic formal world that reveals a bifurcation between conceptual and procedural thinking occurring because of different levels of success in compressing procedures into thinkable concepts. All of these papers are available on the web (www.davidtall.com.papers and www.eddiegray.co.uk).

Recent data from around the world

Recent work with our own research students collecting data from around the world in Britain, Malaysia, Turkey, Brazil has further studied concept formation at various stages of mathematical development of the individual. These reveal in many cases the desired abstraction of thinkable concepts often does not occur as required, with many students remaining at a fragile procedural level of operation. This has long-term consequences for the successful teaching of mathematics at all levels.

The development of early arithmetic

Early arithmetic evolves from actions of manipulating and counting collections of objects. It involves repeating and refining the action-schema of counting, until it becomes apparent that the effect of a counting procedure on a given set always gives the same number word leading to the thinkable concept of number. The addition of two numbers begins by putting two sets together and counting the combination by various counting procedures with a symbol such as 3+2 evoking either a process of addition or the concept of sum, 3+2, which is 5.

In any context that involves an action on objects, the individual has the possibility of attending to different aspects of the situation: a theme that Cobb, Yackel and Wood (1992) see as one of the great problems in learning mathematics. In the terms of our framework the essential question is whether the child focuses on the actions of counting leading to a procedural interpretation or is able to contemplate the effect of those actions in terms of thinkable number concepts.

The greater majority of young children count (and so do many adults). For a young child, counting can be seen as part of a stage in concept development. However, an older child's extensive reliance on counting may be the result of necessity. Counting procedures that work with small numbers—such as calculating 8+3 by counting-on three after 8—are no longer practicable in dealing with larger numbers such as 855+379. Learned routines may be used without meaning and—when they lead to error—confusion and alienation may ensue. Pitta (1998) obtained data illustrating that children who succeed naturally focus on compressed number concepts to perform arithmetic tasks, while children who fail often focus on other aspects of what they see. A child looking at the picture  in a mathematical way may see it representing the fraction one half (the black or white parts of the square). Another child may see it as the doors of a lift or even an open window at night with a white curtain. As children grow older, some focus on the powerful use of number symbols as compressed thinkable concepts, some can combine both embodiment and symbolism in flexible ways, but others remain trapped in an increasingly complicated world of fragile procedures.

Is it possible to help those who are struggling with counting procedures unable to cope with more sophisticated problems? Gray and Pitta (1997) worked with an eight-year-old child who had difficulty counting on her fingers and instead used mental images of counters in specific arrays to perform arithmetic calculations with small numbers. They provided her with a graphical calculator and tasks to perform such as 'find a sum whose answer is 9'. Trying 5+3 gave 8, but adding another 1 gives 5+3+1 is 9. The essential facility of the graphic calculator is that it shows both the arithmetic expression and the result without the need for counting, allowing the focus of attention to be shifted from long procedures of counting which are no longer required to the visible relationships in arithmetic displayed on the screen. She found combinations such as 4+5, 3+6, 4+4+1, 3+4+2 and as she did so, she began to see number patterns. Slowly her activities no longer depended totally on counting. As she worked through the programme, she became more adventurous, building sums such as $2+9+1-6=6$, $90-80-4=6$, $30-15-9=6$, $5+20-19=6$, $40-30-5=5$, $10+30-30-2=8$. Over ten weeks of exploration she became comfortable with larger numbers and using number patterns. At the end of the course she was asked what "4" meant to her, and she replied "a hundred take away ninety-six." Not only had she become familiar with number relations in a way highly unusual for a 'slow learner', she had developed a quality that is usually only shown by much more able children: a sense of humour with numbers.

However, such remediation has not proved successful in other cases. Howat (2005) found that there were failing students in arithmetic with a median age of 8.5 who were unable to cope with place value because they had not constructed the concept of 'ten' as a thinkable concept that could be ten ones or one ten. Without this they were overwhelmed by the problem of coping with the arithmetic of two digit numbers since their response to any problem was to attempt counting on in ones. Even when working closely with them, Howat found that some had cognitive difficulties that started far back in their development that were deeply ingrained and seemed no longer subject to her remedial action.

The Ambiguous Number Line

The shift to number as measurement is embodied using a number line. In the English National Curriculum, this is intended to give learners an overall picture of numbers in order on a line as part of a long-term development of successive abstractions, from a line drawn with pencil and ruler, to a mental image of an arbitrarily long line with no thickness that is subdivided to imagine fractions, finite decimals, then infinite decimals and eventually to the formal thinkable concept of the real numbers as a complete ordered field. It is a journey that is made by those who become mathematicians, but the initial path proves stony for many young children in school.

In English Primary Schools (for children of median ages 4.5 to 10.5), the number line is introduced as a key classroom resource within the Primary National Strategy (PNS) in the *Framework for Teaching Mathematics* (DfEE, 1999). It begins with a 'number track' consisting of blocks placed one after another to count in order, then moves on to a number-line in several different guises including a 'washing line' of numbers, table-top number lines, some marked with specific numbers, others left open to place the numbers in an appropriate place. The overall aim is to use these representations to promote the understanding of the number sequence and order of the whole numbers marked on the line, introducing addition and subtraction in terms of operations on lengths, then later expanding the children's knowledge to include fractions, decimals and negative numbers.

Within the documentation there is no reference to the conceptual differences between a discrete number track and a continuous number line, or to the subtle shifts in meaning involved in the introduction of broader number concepts.

This ambiguity is reflected in schools by the way in which teachers interchangeably use the terms number line and number track as if they are the same idea, when they are not. The number track consists of discrete numbers, one, two, three, … with each number followed by a next number and no numbers in between. The number line is a continuous line on which we may mark numbers as points, with fractions between whole numbers and the possibility to extend the line in either direction to include positive and negative numbers, rationals, decimals, and so on.

When Doritou (2006) interviewed a range of children with median ages between 6.5 and 10.5, most of them simply described the number line in terms of some perceptual features of a particular line or explained a particular line in the context of an action. Overall, the quality of the children's responses did not change significantly between children in Year 3 (aged 7.5) and those in Year 6 (aged 10.5). There was an over-riding preference to label calibrated lines with whole numbers and a limited acknowledgement that an interval could be subdivided, linking back to their experiences with the number track rather than the intended number line.

When the teachers came to use what they perceived as a number line to demonstrate the subdivision of intervals for fractions and decimals, most children carried an embodiment pre-loaded with prior active, linguistic and relational experience with whole number. There was no sense of the conceptual structure that underscores the use of the number line as an ideal representation for connecting whole number and fraction (Baturo & Cooper, 1999).

Procedural conceptions of fraction

After the initial challenges of handling whole numbers, the shift to handling fractions is a problematic one for many children. Instead of a single name for a single number, like '3', a fraction has many names: 'two-thirds', 'four-sixths', 'sixteen twenty-fourths'. A fraction begins as an embodied activity of breaking an object or collection into equal parts and assembling the required number of parts. Whereas two-thirds, four-sixths and sixteen-twenty-fourths involve quite different procedures and produce parts of different sizes, the *quantity* in each case is the same. The major act of abstraction shifts attention from the sharing procedure as a sequence of steps to the *effect* of that procedure, namely the quantity produced in the result. Focusing on the effect, one-third is the same as three-sixths. If fractions are seen as procedures, then addition is almost too complicated to contemplate, but if fractions with the same effect are seen as 'the same' then adding one-third and one-half is the same as adding two sixths and three sixths, a process not unlike adding two apples and three apples. The child with proceptual flexibility of number is more likely to see the essential simplicity of adding fractions than the child with a procedural view that may involve learning complicated rules which may have little meaning, such as 'put the fractions over a least common denominator'.

As part of the Malaysian Vision for 2020 to develop the country's economy to the highest standards by that date, the Malaysian curriculum is designed to teach fractions in a caring and helpful way to include such things as seeing that multiplication can be done in different ways to give the same result. For instance, 'two-fifths of twenty-five' can be performed either by working out a fifth of twenty-five, which is five, then multiply by two, to get ten or by multiplying two times twenty-five to get fifty and divide by five, also to get ten.

To ensure that all children can accomplish these tasks, a teacher encourages the pupils to remember the procedures, reciting successive parts of the procedure and inviting the children to fill in missing words. For instance, the teacher might say (in Bahasa Malay), 'How do we work out two-fifths of twenty five?' and draw three circles on the board one above the other for numerator and denominator of the fraction, the other for the whole number. 'What do we put in the top circle? The nu…' to which the class gleefully says 'the numerator.' 'What do we put in the bottom circle? The de…' 'Denominator'. 'Of means mul…', 'multiply', and so the lesson continues, building up the ritual of the procedure of multiplication by a fraction.

In a study observing the classroom and interviewing the students, Md Ali (2006) found that children's achievement in fractions was indeed improving, but that although the teachers subscribed to the aspirations of Vision 2020 to help children 'really understand mathematics', they general consensus was that they felt constrained by the teaching schedule and the need for success in the National UPSR Examination. Success in examinations was achieved procedurally with some degree of flexibility in choosing which procedure to use—but the compression of knowledge into flexible thinkable concepts to solve unfamiliar fraction problems proved elusive.

'Magic' embodiments in algebra

The shift from arithmetic to algebra involves an abstraction from the computable operations of arithmetic to the use of expressions representing generalised arithmetic operations. Such a transition proves relatively easy for some who have a flexible proceptual approach to arithmetic, but it is far more difficult for those who continue to think of expressions purely in procedural terms. Working with a group of committed teachers in Brazil, Rosana Nogueira de Lima (de Lima & Tall, 2006) found the teachers concerned with teaching their students how to solve equations.

In linear equations, they elected to teach the students the principle of 'doing the same thing to both sides' to maintain the equality and manipulate the equation to give the solution. In practice the students focused not on the general principle but on the actions they performed. Subtracting 2 from both sides of the equation $3x + 2 = 8$ and simplifying to $3x = 8 - 2$ was soon seen as 'change sides, change signs' while the final simplification of $3x = 6$ by dividing both sides by three to get $x = 6/3$ soon became 'move the 3 over the other side and put it underneath'. The students interpreted these moves in an embodied way, 'picking up the terms and putting them somewhere else' with additional actions such as 'change signs' or 'put it underneath'. This 'procedural embodiment', carried out by mentally moving the terms with an additional meaningless piece of 'magic' to get the right answer was used successfully by some but proved fragile for others who made errors mixing up the rules such as shifting the 3 in $3x = 6$ with the additional magic of 'changing signs' to get $x = \frac{6}{-3}$. Such errors may increase the confusion of students who may then try alternatives to get the correct answer, producing what appear to be random errors.

In solving quadratics, the problems became worse as the teachers, knowing the difficulties with linear equations, focused only on teaching the formula because it gives a solution for any quadratic. However, this single procedure lacks flexibility and it is often necessary to manipulate the equation to get it into a form to use the formula. A problem arose when the students were asked to show that the equation $(x - 2)(x - 3) = 0$ had roots 2, 3. Most students failed to respond at all and those that did attempted to multiply out the brackets to solve the equation using the formula. Few succeeded.

Here we see lack of flexibility in abstracting thinkable concepts has a cumulative effect. Some students see the essential simplicity of an algebraic expression as a potential calculation that can be manipulated in its own right and develop an effortless mastery of algebra. Meanwhile others who focus on procedures become entrapped in more complicated activities that increase the cognitive strain and may become unmanageable.

Complications in the function concept

As we move through into the secondary curriculum we come to concepts like the notion of function, which the NCTM standards see as being an essential underpinning of a wide range of mathematics. In some countries, such as Turkey, the function concept is taught from its set-theoretic definition and seen as a fundamental foundational idea. It is quite simple. There are two sets A and B and for each element x in A, there is precisely one corresponding element y in B which is called $f(x)$ (eff of eks). That's it!

However, this is used in the curriculum to weave a huge web of knowledge: linear functions, quadratic functions, trigonometric functions, exponentials and logarithms, formulae, graphs, set diagrams, and so on. How does one help the student make sense of this complicated array of ideas? Two routes are possible. One is to focus on the simplicity of the definition and continually link back to it to make powerful connections. Another is to look at the individual difficulties and teach the students how to cope with them.

Bayazit & Gray (2006) reports a study of the teaching of two teachers with very different approaches. Ahmet saw his duty to mentor the students and help them make sense of the function concept. At ever opportunity, he emphasised the simple property that a function $f: A \to B$ mapped each element of the domain A to a specific element in the domain B. For example, in considering when a graph could be a function, he looked at the definition and related it to the fact that each x corresponded to only one y, and linked this to the 'vertical line' test. When he considered the constant function, he considered the definition and revealed the constant function $f(x) = c$ as the *simplest* of functions which maps *every* value of x in A onto the element c in B. Likewise, when he studied inverse functions, such as the square root, the inverse trigonometric functions and the relationship between logarithm and exponential, he patiently referred everything to the definition. With piece-wise functions, which were new to the students, he again used the definition to confirmed that these too satisfied the simple requirement that for every x there was a unique y.

The other teacher, Burak, was well aware of his students' potential difficulties and misconceptions. He considered that students rejected the constant function because 'it did not vary with x'. He interpreted the students' difficulties with the inverse function as an indicator of their inability to move back and forth between the elements of domain and co-domain. He knew that students had problems with the discontinuities of the graphs of piecewise-defined functions, predicting that they would draw lines to fill in any gaps.

However, he made no effort to eliminate these obstacles during his teaching. Instead he gave the students the details he considered that they needed to answer the examination questions. He taught the 'vertical line test' as a specific test for functions, practising examples to get it right. He introduced the inverse function with a simple case, finding the inverse of $y = 2x + 3$, by seeking to express x in terms of y, subtracting 3 from both sides and dividing by 2 to get $x = \frac{y-3}{2}$ then interchanging x and y to get $y = \frac{x-3}{2}$. He dealt with the problem of the constant function by affirming that a function does not need to involve x and that its graph is a horizontal line parallel to the x-axis. He dealt with piece-wise functions by showing students how to cope with particular examples.

He would often indicate that an examination or test required particular tactics:

> If you want to succeed in those exams you have to learn how to cope.
> Do not forget simplification. It is crucial, especially [in] a multiple-choice test.

Even though he was aware of student difficulties he did not attempt to address them meaningfully as teachers had done in other studies (Tirosh *et al.*, 1998; Escudero & Sanchez, 2002). His students scored significantly lower than those of Ahmet who had concentrated on building the notion of function as a thinkable concept by linking the definition to many different function contexts such as set diagram, formula, graph rather than focus separately on how to cope with specific difficulties in different cases.

Drawing Together the Threads

The theoretical framework for the development of mathematical thinking through the compression of phenomena into thinkable concepts encompasses learning over a life-time. It begins with young children learning to count, which requires a process of mental compression of the counting-schema into the concept of number. Arithmetic knowledge builds on operations of addition, subtraction, sharing and multiplication where the more successful student sees the symbols as flexible thinkable concepts that dually evoke processes 'to do' and concepts' to think about'. Such students use old knowledge to create new and have the flexibility to formulate and solve new problems. The less successful stay more with their perceptions of the real world and their inflexible counting procedures that cannot cope with the increasingly sophistication of situations that they encounter.

We have found the use of a graphic calculator representing both the arithmetic expression to be computed and the result of the computation can enable a child to focus on relationships rather than on the time-occupying procedures of counting, although serious cases of difficulty may not respond even to this remediation. Our framework suggests that abstraction of thinkable concepts by focusing on the proceptual relationships between numbers rather than the procedures of counting leads naturally to more successful operations with arithmetic.

The physical representation of number as a number track, then a number line has strong perceived properties that encourages many children to cling to an interpretation of the number line as a counting and computational tool for whole numbers, so that they fail to grasp its measurement characteristics to recognise it as a visual and dynamic image of numbers beyond whole numbers. However, this occurred in a situation where much activity was provided without an apparent focus on the change of meaning from counting on a discrete number track to measuring on a continuous number line. We hypothesise that children learn from their experience and if that experience does not focus on the thinkable concepts required for more powerful thinking then that thinking may not occur.

The procedural teaching of fractions, even though it improves performance on examinations nevertheless can leave the student inflexible and less able to cope with problems beyond the scope of the narrow framework taught so that success may be achieved within the examination. Again, the focus on different procedures in counting produces gains in accuracy and efficiency but does not extend so easily to more sophisticated problem solving.

The shift to algebra requires a sense of the general operations of arithmetic, and we hypothesise that a child who is proceptual in arithmetic is more likely to have the flexibility of arithmetic symbols as thinkable concepts to carry over to the manipulation of algebraic expressions. Data to support this explicit hypothesis still needs to be collected. What we do have is data that a group of children taught by the principle of 'do the same thing to both sides' focus on the specific experience they see before them and solve equations by shifting symbols with a perceived 'magic' of changing signs or shifting the number 3 in $3x = 6$ over the other side and 'putting it underneath'.

The case of the research on the function concept shows the positive way in which a gifted teacher can assist in the abstraction of powerful thinkable concepts by relating the simplicity of the definition to all its representations and relationships. Meanwhile the desire to focus on a host of specific instructions to pass the examination may be counter-productive.

What does the available evidence tell us? The overwhelming message is that the pressure to 'teach to the test' leads to the teaching of specific procedures that compresses the routine procedure into an automatic way of solving specific problems without necessarily producing the conceptual flexibility to think in a more sophisticated way. The 'natural process' of the brain to focus on relevant data and to form thinkable concepts is here focusing on the *steps* of the procedure, not what the procedure is intended to *do*. For so many learners the focus is on the steps needed to perform column subtraction, long division or factorisation of quadratics. The procedure of factorisation turns $x^2 + x - 6$ into $(x-2)(x+3)$ which can be seen as converting one thing into something that (looks) different. These two expressions are different as procedures of evaluation, but they are the same in effect. Being able to 'see' 8+6 as 8+2+4 and then as 14, or to 'see' $\frac{1}{2}+\frac{1}{3}$ as $\frac{3}{6}+\frac{2}{6}$ and 'hear' 'three sixths and two sixths' to get 'five sixths' transforms arithmetic procedures into flexible thinkable concepts. In the same way, 'seeing' $x^2 + x - 6$ and $(x-2)(x+3)$ as different ways of writing *the same thing* is a significant simplification that turns algebraic expressions into thinkable concepts that can be handled fluently.

If at one stage a learner fails to focus on the relevant aspects to produce the subtle thinkable concepts and instead learns the steps of the procedure to carry out a specific task, then the human brain lacks the thinkable concepts to build on the sophistication required at the next stage and is more likely to resort to the primitive strategy of learning by rote. The effects are cumulative. As all of us go through the long-term development of learning mathematics, if compression of knowledge required for the next stage does not occur, then procedural learning becomes more likely, not only in the children we are teaching, but in those who are adults and have already been through their mathematics education. Thus, despite the widespread call for more meaningful conceptual learning, the perception that the way to learn mathematics procedurally proliferates and is held, not only by children, but also by many teachers, administrators and politicians.

To improve long-term conceptual learning, the framework formulated here suggests that the whole curriculum must be framed with an awareness of the abstraction process to produce thinkable concepts *at every stage*. This requires the teacher as mentor to encourage children to focus on the appropriate essential ideas in a way that enables them to compress the phenomena into thinkable concepts. This in turn requires mathematics educators to aid the development of such a vision by working to formulate how this can be attempted in ways that make sense both to the teacher and the many different learners.

This journey will not be easy. In the UK, the National Numeracy Strategy (DfEE, 1999) was initiated as a response to low attainment in mathematics in many schools and it informed teachers, through its annual objectives that built new ideas on those previously taught, what should be taught in mathematics during each school year. It also explained how the mathematics specified should be taught by presenting a recommended three–part lesson format: mental/oral phase, main phase and plenary phase and advocated that the first and last phase, together with part of the main phase, should be taught with the whole–class together. While this strategy was not enforced by law, the majority of schools responded to the initiative, partly due to pressure from government agencies such as OfSTED (Denvir & Askew, 2001).

In an evaluation of mathematics provision for 14 to 19 year-olds OfSTED (2006) reported that the majority of teaching preparing students for examinations by "teaching to the test" might ensure that students pass examinations but it would not ensure mathematical flexibility. The government report intimated that the problems arose from the inadequacies of teaching.

Our theoretical framework suggests differently. We believe that the natural process of abstraction through compression of knowledge into more sophisticated thinkable concepts is the key to developing increasingly powerful thinking. It occurs naturally with the most able and others can be helped by using techniques that encourage a focus on the essential elements to compress into thinkable concepts. But there is no evidence that it can work for *all* children. Until we grasp the nature of the required sophistication to compress complicated phenomena into thinkable concepts and are able to express it in a way that makes sense to teachers, students and, if possible, to politicians, mathematics will remain for many a world of overbearing difficulty relieved only partially by limited rote-learning.

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