EMBODIMENT, SYMBOLISM, ARGUMENTATION AND PROOF

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What is proof in mathematics? For a mathematician it is a formal argument to lead to the confirmation of a theorem without any possible room for doubt. For a student it may be an argument that convinces them that a statement is true. For all of us, our human knowledge builds on our genetic inheritance and our subsequent experiences that we have met before in our earlier life. Therefore our concept of proof grows as we build on increasingly sophisticated experiences.

In this presentation, I will look at the development of proof through the cognitive development of individuals from birth to adulthood, as different individuals develop different ways of argumentation to lead to their own conceptions of proof. Adults have conceptions that may make sense ***to them****, the question is, how to encourage growing students to conceptualise their own argumentation to lead to a mathematical form of proof. I will show that mathematical concepts have several different aspects that contribute to proof, beginning with the concept of number, which may be* ***embodied*** *as a number line traced with a finger,* ***symbolised*** *as decimal expansions useful for computation, and* ***formalised*** *as a complete ordered field. Each of these represents a distinct way of conceptualising number that work together as a* ***conceptual blend*** *to give various aspects of the number concept. Each of them involves a different form of proof appropriate in different contexts as the learner grows in sophistication. It is our purpose as mathematical teachers to understand this development and to encourage students to move through argumentation using embodiment and symbolism on to the formal meanings of mathematical proof.*

INTRODUCTION

This paper is a contribution to the conference on *Reading, Writing and Argumentation* at Changhua University, Taiwan, May 2007. It focuses on the underlying meaning of language and symbolism in mathematics and how this relates to the growth of argumentation and proof as different individuals mature, It is based on a theory of long-term development of three distinct worlds of mathematical thinking in the individual (Tall, 2004) through perception and action (the world of embodiment) the compression of actions such as counting into concepts such as number (the world of symbolism) and the construction of coherent theories based on set-theoretic definition and mathematical proof (the world of formalism).

We will consider how each development gives distinct forms of argumentation and proof which all depend on compressing ideas into thinkable concepts and blending conceptual structures into even more potent mental structures.

Language is the tool which gives us the power to build sophisticated conceptions. It enables us to *name* phenomena, with terms like mamma, dadda, smile, cold, dark, triangle, two, infinity. The verbal utterance is related to a person, an action, a

Keynote presented at National Changhua Normal University, Taiwan, May 2007.

sensation, a property, a situation, or any other phenomenon that we are able to focus upon. Language then enables us to talk about this phenomenon and give it more sophisticated meaning as a thinkable concept. Language allows us to *compress* experiences into small enough chunks to enable us to relate them in our minds. Compression can occur in many ways, in time, where events that happen over hours, days, or centuries can be spoken of in a single context, in space, where different events are considered in relation to one another, in conceptual structure where an action such as counting is compressed into a concept such as number.

Concepts are connected within particular contexts, which are termed *frames* in cognitive science, such as 'family', which includes not only parents and children, brothers and sisters, iterations such as grandparents and grandchildren, emotional relationships such as parents caring for children, with combinations of love and discipline. Such frames may operate at a generic level, such as 'family' or at a specific level, such as 'Joe, Mary and their son John'.

Existing frames may be blended together in new ways to give creative ways of thinking putting together previously constructed ideas in new imaginative ways, so that conceptual structures become more sophisticated.

Human development involves:

- **Compression** of complex situations into thinkable concepts using language;
- **Connectio**n between thinkable concepts in coherent **frames of reference**;
- **Blending frames of reference in new ways**, to create new conceptions and solve novel problems.

An example: Mrs Thatcher as President

Fauconnier & Turner (2002) give many examples of conceptual blends, for instance, the democratic systems in the USA and Britain have an elected leader, voters, and various pressure groups among the voters, allowing a generic frame with elected leaders, voters, pressure groups that can be blended to form the argument:

> If Mrs Thatcher stood for President, then she would not get elected because the unions would oppose her.

This argument creates a new imaginative blend that includes Mrs Thatcher, American Voters, and American Unions that does not exist in reality, but allows an innovative argument to be put forward. The two specific frames are not parallel in every aspect; for instance, the President is elected by the people, but the Prime Minister is elected by the party in power. It has *emergent concepts* that are part of the new blend but not of the original frames, such as the idea of Mrs Thatcher campaigning for votes in Michigan. In reality, Mrs Thatcher could not even stand as President because she is not a natural born American citizen. Nevertheless the argument imagining the scenario of Mrs Thatcher as President brings forward a valuable comment on the differences between the two electoral systems. (Figure 1.)

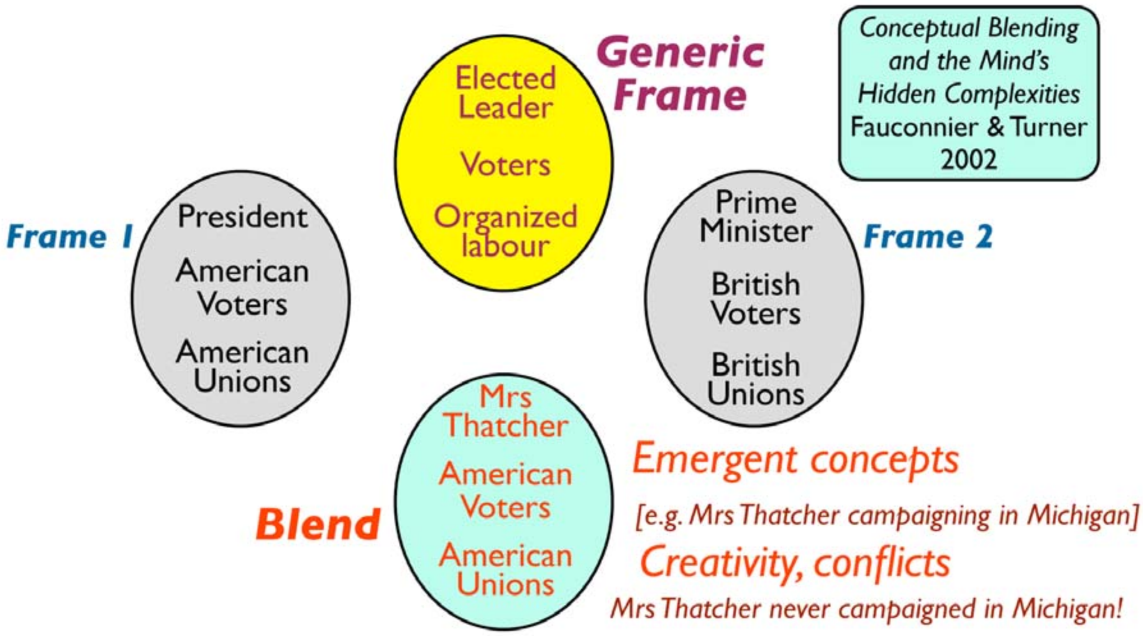

Figure 1: a conceptual blend of frames (based on Fauconnier & Turner, 2002)

The blending of frames is an essential aspect of human thinking. Although we build on concepts already in our minds, this need not be purely the mechanical use of ideas previously taught; blending enables the human mind to build new thoughts in original and creative ways, not only in general thought, but throughout mathematical thinking

Complex numbers as a conceptual blend

Another example cited by Fauconnier and Turner is the concept of complex number. (Figure 2).

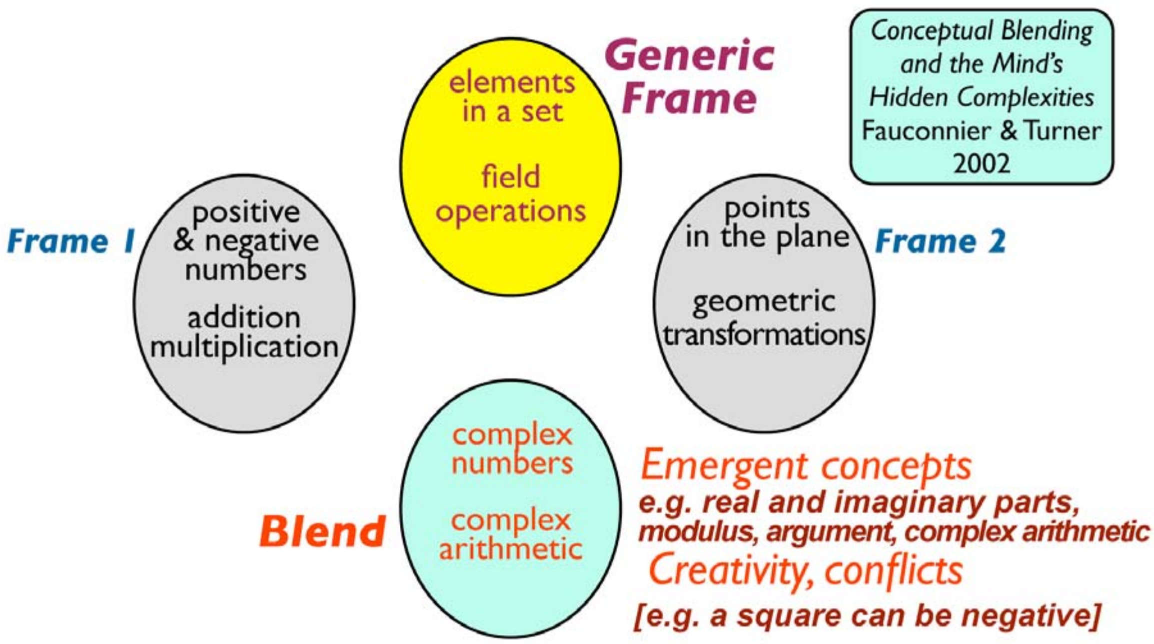

Figure 2: The complex numbers as a conceptual blend

The analysis is an intellectual one: seeing the arithmetic of positive and negative numbers being blended with the geometric transformations of points in the plane within a set-theoretic generic frame of operations on a field. The blend producing the complex numbers has many emergent concepts, such as the idea of real and imaginary parts of a complex number, its modulus and argument and the complex arithmetic which arises, adding the Cartesian coordinates and multiplying using the polar coordinates. The blend has new emergent properties that can create conflict with old ideas, such as the fact that the square of a complex number can be a negative number.

This is an intellectual top-down analysis performed by individuals who already have the requisite concepts in their minds and is very different from the development over time both in the history of complex numbers and in the understanding of complex numbers by individuals. For instance, in historical development, the generic frame of sets and field operations is implicit rather than explicit when the complex numbers were being constructed. Our purpose here is to use a similar analysis to focus on how the individual may develop cognitively using conceptual blends in argumentation and mathematical proof.

COGNITIVE DEVELOPMENT OF ARGUMENTATION

Development in Geometry

The long-term development of geometrical thinking and proof, as described by van Hiele, broadly goes through a sequence of stages:

1. Playing with objects to sense their properties and connect together perception, touch, actions etc.;
2. Naming objects and describing their properties to produce rich generic concepts:

   e.g. triangle, square, circle, corner, side, round,
3. Using properties to define objects, performing constructions to draw objects;
4. Arguing that if certain properties hold, then others follow;
5. Constructing a coherent framework of Euclidean Geometry;
6. Developing alternative geometries on the sphere, projective geometry, hyperbolic, elliptic etc.

It is not appropriate to go into further detail here. However, the broad development involves the development of language in geometric argumentation and proof. Initially the child uses language to *describe* properties, then, as more precision in meaning develops, language is used to *define* properties that specify when a figure is of a specific type. Now many specific figures are categorised with a single generic name, such as ‘triangle’ which may be further distinguished into sub-categories, for instance, an isosceles triangle is precisely a triangle with two equal sides. By building on the concept of two triangles being congruent if they have certain common properties then it is possible to produce arguments that show that if certain properties occur then others follow. For instance, by joining the apex to the mid-point of the base to divide the triangle into two congruent triangles (3 corresponding sides), it is possible to show that an isosceles triangle has equal base angles. This leads to the construction of a coherent theory of Euclidean geometry. Subsequently (two thousand years in history) more subtle geometric viewpoints can occur through the study of geometry on a sphere (where straight lines are great circles) or the projective geometry of drawing a scene on a transparent surface, or more subtle forms of geometry with new systems of formal definitions that give rise to non-Euclidean geometries.

Development in Arithmetic and Algebra

The long-term development of number concepts and arithmetic on into algebra begins with counting as an action taking place in time being compressed into the concept of number, and successive counting of sets being compressed into the concept of sum. All the operations of whole number arithmetic arise from embodied actions: putting sets together to give addition, taking a set away to give subtraction, taking several sets the same size to give multiplication and sharing equally to give division (perhaps with a remainder). Over time this leads to the broad conceptual frame of whole number arithmetic.

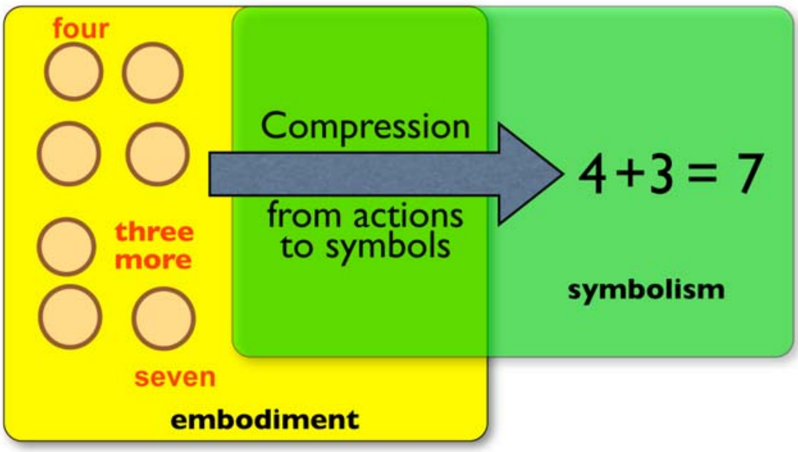

Figure 3: Compression of counting actions into number concepts

The repeated action of ‘one more’ builds up the potential infinity of whole numbers and their properties can be argued using both embodiment and symbolism. For instance, to show that the sum of the first 100 numbers is 5050 can be done ‘the long way, by starting with 1, counting on 2, then counting on 3, and so on up to counting on 100, but this can be compressed creatively by breaking up the sum

$$1+2+3+....+98+99+100$$

into the terms up to 50 plus the terms from 51 to 100, reversing the second sequence and adding them term by term to get each pair of terms adding to 101 and, with 50 pairs, the total is then 50×101 which is 5050:

$$\begin{array}{r} 1+2\ +\ 3+\ldots.+50 \\ +100+98+97+\ldots.+51 \\ \hline \underbrace{101+101+101+\ldots.+101}_{\text{50 terms}} \end{array} = 50 \times 101 = 5050$$

Although this sum is a specific case of the first 100 numbers, the argument works for other whole numbers. This argument works more easily for *even* numbers that can be split into two equal parts. Another argument is to take a second copy of 1+…+100, turn it round and add 1+…+100 to 100+…+1 to get twice the sum as 101+…+101 (a hundred times) and divide by two to get $\frac{1}{2}\times 100\times 101$. This works for any whole number, so generically applies to *all* whole numbers. Algebraically, we can use this to say:

$$1+\ldots+n=\tfrac{1}{2}n(n+1).$$

The same proof may be embodied in a picture by putting elements in successive rows of length 1, 2, 3, … . This gives a triangular set of objects (yellow in figure 4) and when a second set of the same shape is rotated and placed with it (the blue set), the result is a rectangular array with sides n and n + 1, giving a visual argument for the sum $1+\ldots+n=\tfrac{1}{2}n(n+1)$.

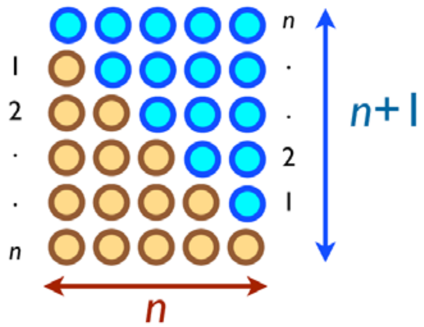

Figure 4: the sum 1+…+n is half of $n\times(n+1)$

Technically, it may be said that the pictorial proof is unsatisfactory because the picture shows not a general value of n but a specific value, in this figure the value is 5. However, if we look at figure 5, we see that the argument is visually satisfying even when the number of items is so large that it is not easy to count them.

The array is visibly rectangular. If there are n blue discs in the first row, the number of yellow discs in each row increases by 1 starting from 0 up to n, so there are n+1 rows. Most importantly, all of this can be *seen* without counting the actual number of discs involved.

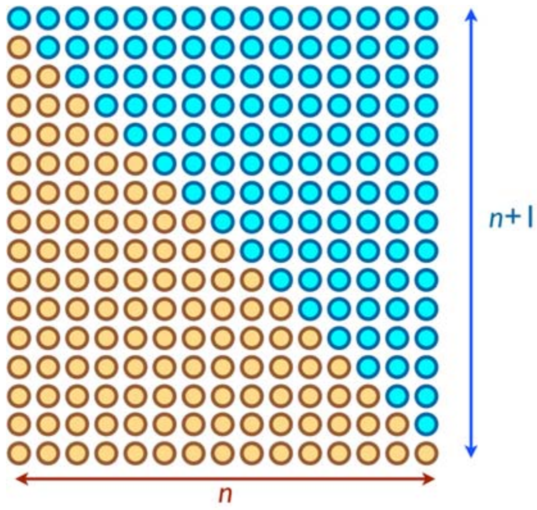

Figure 5: the generic pictorial proof

The formula can also be proved by induction in two similar-looking but subtly different ways.

One is as a potentially infinite proof:

Prove 1+...+n = n(n+1)/2 is true for n=1
in the form 1 = 1.(1+1)/2.
If true for n=k, use 1+...+k = k(k+1)/2
to deduce the truth for n=k+1,
and then use the potentially infinite proof:
It is true for n=1, hence for n=2, hence for n=3, and so on ....
(for any whole number n)

This can be recast as a *finite* proof, by basing it on the Peano Postulates in the following form:

Peano Postulates :
$\mathbb{N}$ is a set and s: $s:\mathbb{N}\to\mathbb{N}$ satisfies:
Axiom 1: s is one-one but not onto.
Let $\mathbf{1}\in\mathbb{N}$ where $s(n)\neq\mathbf{1}$ for any n, define n+**1** to be $s(n)$.
Axiom 2: If A is a subset of $\mathbb{N}$ with
$1\in A$,
and, for all $k\in\mathbb{N}$, $k\in A$ implies $s(k)\in A$
then $A=\mathbb{N}$.

Proof
Let $A=\{n\in\mathbb{N}\mid 1+\ldots+n=\tfrac{1}{2}n(n+1)\}$
1. show $\mathbf{1}\in A$,
2. show $k\in A$ implies $k+\mathbf{1}\in A$,
3. use axiom 2.

This is now a proof with just *three* steps (1, 2, 3 above). It reveals an *enormous* compression from a potentially infinite proof with the same step carried out time and time again and again, *ad infinitum*, to a proof with just three steps. The infinite part of the structure is now subsumed in the axioms themselves, because any set A with a map s satisfying axiom 1 *must be infinite*.

When such proofs are presented to students, the symbolic and pictorial proofs are usually seen as meaningful, but the proof by induction appears confusing to many.

Why? The symbolic and pictorial proofs show *why* the argument is meaningful. Superficially, the induction proof uses the result to prove the result. The subtlety of the quantifiers—that one wishes to prove it true *for all n*, while using the result *for any n so far*—is potentially confusing.

It leads to a new world of mathematics: the formal world built on set-theoretic definition and mathematical proof. While formal deduction builds in an increasingly sophisticated way throughout the worlds of embodiment and symbolism, there is a significant change in the formal axiomatic form of mathematical thinking in that it signals a change in focus from having definitions based on known objects in the earlier stages, shifting round to build proofs on axioms where the mathematical objects involved are based on formal definitions.

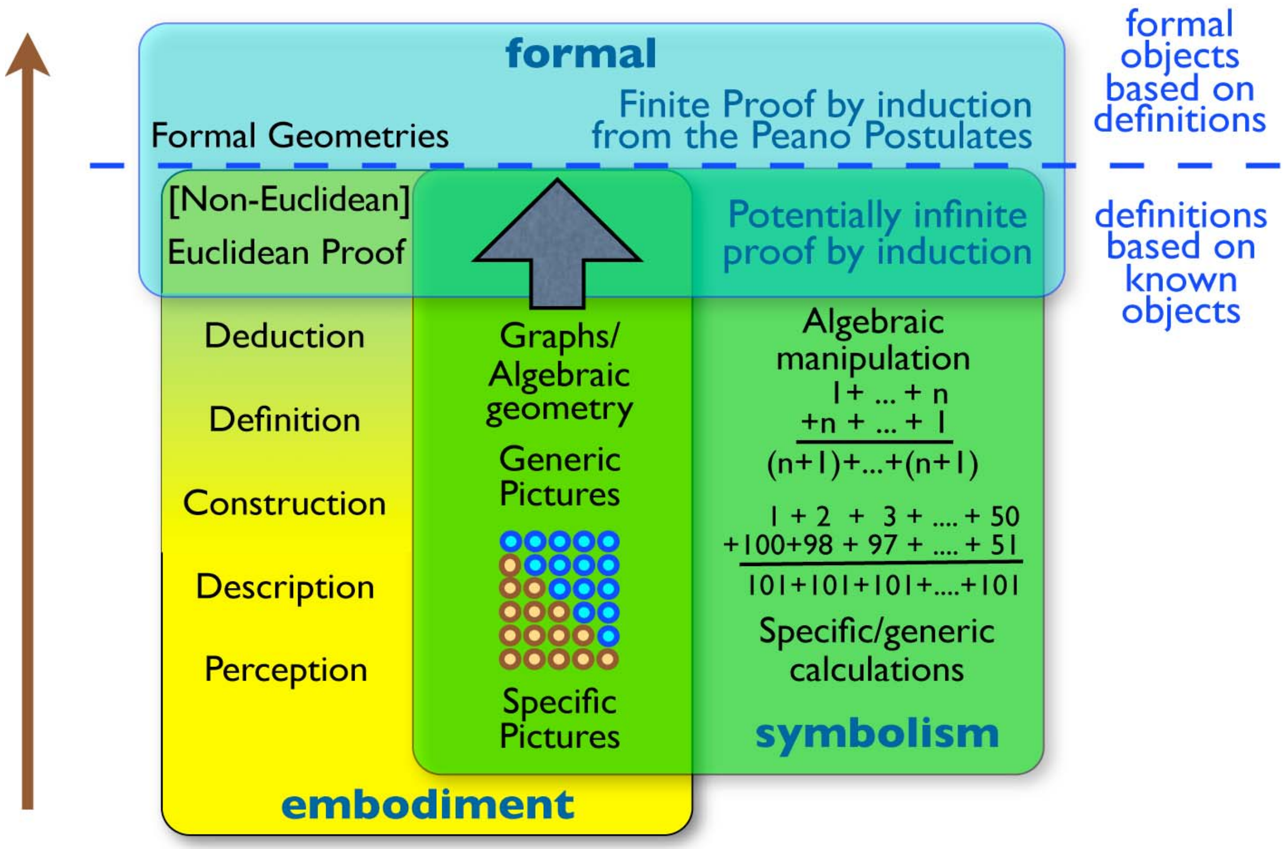

Figure 6: Argumentation and Proof in three different worlds of mathematics

This leads to increasingly sophisticated developments of argumentation and proof as each individual grows over their lifetime. Some will make the journey, but many may be confused by the conflicts which occur in conceptual blends which may not seem to make sense.

In the embodied world, there is an increasingly sophisticated use of language to define and deduce properties of categories of objects that are then built into a deductive theory. While Euclidean proof has a formal deductive framework, it still builds on mental imagery of known geometric figures to build its deductive framework. Even some forms of non-euclidean geometry now have embodiments that give them meaning in a new way, and it is only when we reach formal geometries defined set-theoretically in terms of axioms that we enter the formal world in its fullest form.

The cognitive development of symbolism is quite different from that found in geometry, involving an increasingly sophisticated compression of actions into manipulable symbols in arithmetic, algebra, symbolic calculus and other topics such as clock arithmetic modulo a given whole number.

In both embodiment and symbolism it is possible for arguments concerning specific cases to be conceived by the human mind as typical generic representations of a whole category of cases, which leads on to symbolic proofs using algebra.

There is an interface between embodiment and symbolism which enables both embodiment to give meaning to the symbolism (as in the number line, or graphs of functions in the Cartesian plane) or to provide symbolism to underlie the geometry (as in algebraic geometry).

At more advanced levels of embodiment and symbolism, definitions and deductions arise built on experience that form a transition stage before the switch around to formal arguments based on axiomatic systems that occur in mathematical proof. All of this development is built on the increasingly sophisticated use of language blending together frames of reference with underlying mathematical similarities. (Figure 7.)

Formal axiomatics is an ideal form of proof desired by mathematicians. The truth is that, as biological creatures, we all re-cycle what we know. For instance, our experience of arithmetic is not replaced by the formal definition of the real numbers, we continue to use the connections made in our brain when we learnt simple arithmetic as a basis for thinking about numbers. Therefore, despite the claim of mathematicians that the real numbers are simply a formal complete ordered field, they remain, in all of us, a subtle multi-blend of embodiment as a number line we trace with a finger, a decimal number system we use for calculation and a complete ordered field we use for formal proof. (Figure 8.)

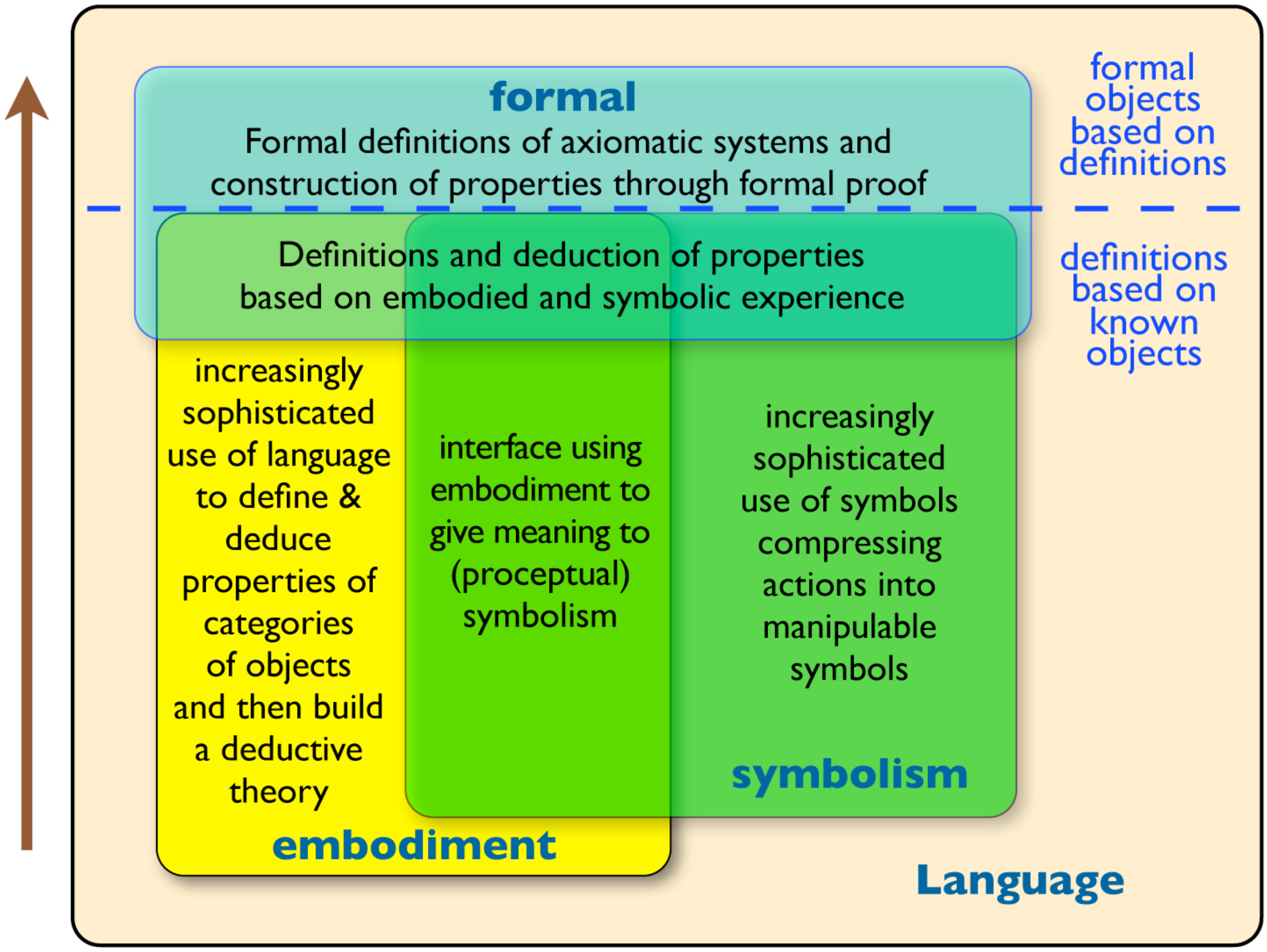

Figure 7: The cognitive development of argumentation

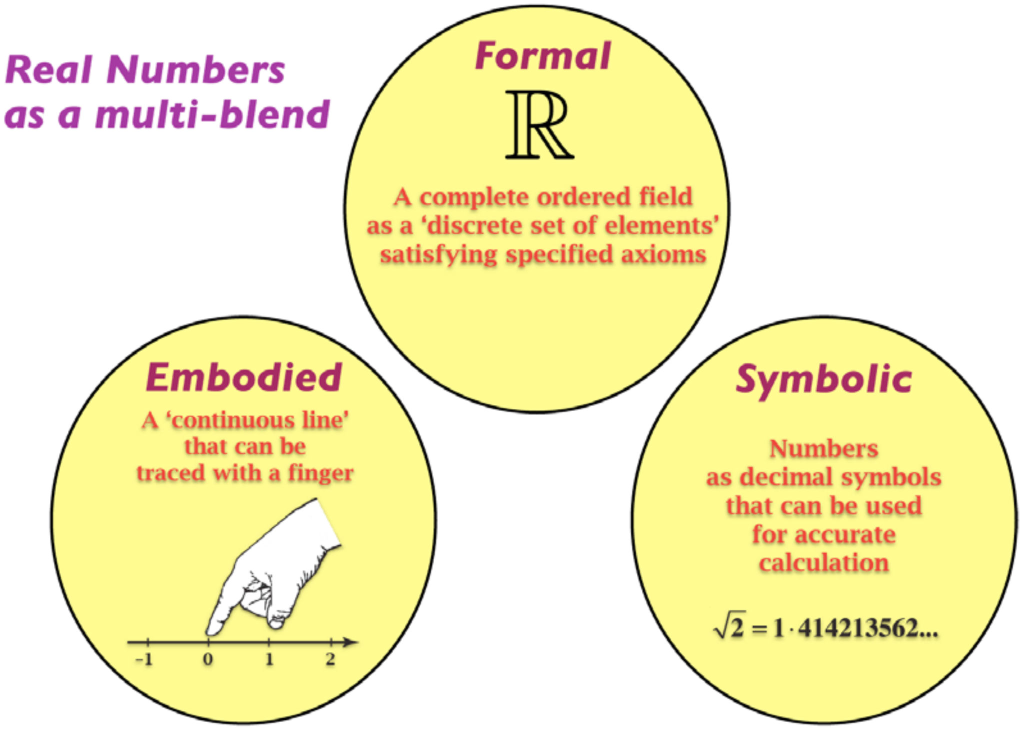

Figure 8: The real numbers as a multi-blend

A mature mathematician may handle this blend believing that the formal definition coincides with the symbolic and that the embodied definition gives a useful but limited approximation to the same underlying structure. Learners often sense conflict between the different aspects of the blend.

The mathematical curriculum is usually seen as a continual expansion of the number system from counting numbers, broadening to positive and negative integers, fractions, rationals, reals, then complex:

$$\mathbb{N} \begin{array}{c} \subset \mathbb{Q}_+ \subset \\ \subset \mathbb{Z} \subset \end{array} \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

For the developing student, however, the different structures involved can cause serious conflict. For example, if we ask the question:

How many numbers are there between 2 and 3?

the answer is different in different cases:

$\mathbb{Z}$ none;

$\mathbb{Q}$ lots (a countable infinity);

$\mathbb{R}$ even more (an uncountable infinity);

$\mathbb{C}$ none (because the complex numbers are not ordered).

Mathematically, of course, as *sets* we have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, but *as mathematical structures*, they are thoroughly different. $\mathbb{N}$ is the smallest mathematical structure that starts with an element and repeats continually with each element having a new successor, $\mathbb{Z}$ is the smallest ordered integral domain, $\mathbb{Q}$ is the smallest ordered field, $\mathbb{R}$ is the only complete ordered field.

Cognitively, successive number systems encountered in school are not contained one within the next; they are successive blends where one expands into the next, sharing some properties but having others that are not carried over from the smaller system to the expanded system. For instance, Natural numbers build from *counting* objects and this can be represented as a number *track* built with successive cubes set out in line, counting them one after another. Each number has a next number and there are no numbers in between. However, when the focus shifts to *measuring*, then the numbers can be written on the number line as they are on a ruler, with the numbers at the end of successive unit lengths and the space in between divisible into fractions. (Figure 9).

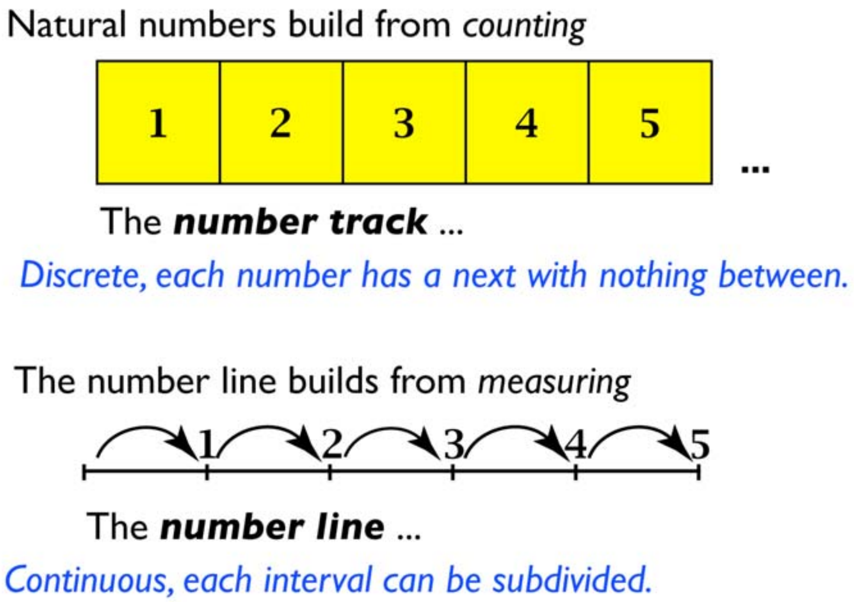

Figure 9: number track and number line as a blend of two frames

Does the difference between number track and number line matter? The English National Curriculum specifically starts with a number track and later uses various number-line representations to expand the number line in both directions and to mark positive and negative fractions and decimals. Doritou (2006, Warwick PhD) found that many children had an overwhelming preference to label calibrated lines with whole numbers, with limited ideas that an interval could be sub-divided. Their conception of the number line did not change significantly between children in Year 3 (aged 7.5) and those in Year 6 (aged 10.5). This suggests that the children's early experiences of counting and number track did not expand to give the broader properties of the number line. For some reason, the blend of number track and number line was not apparent to many children, with the number line being invested mainly with the earlier discrete properties of whole number. This applies also to the introduction of the fraction concept where the blend of whole number arithmetic is expanded as a blend with fractional arithmetic.

The transition from arithmetic to algebra is difficult for many. For example, the conceptual blend between a linear algebra equation and a physical balance works in simple cases for many children (Vlassis, 2002, Ed. Studies). However, the blend breaks down with negatives and subtraction (Lima & Tall 2007, Ed. Studies). I conjecture that there is no single embodiment that matches the flexibility of algebraic symbolism, causing another difficult transition in expansion of mathematical concepts.

Students conceiving algebra as generalised arithmetic may find algebra simple. However, those who remain with inappropriate blends may find it distressing and complicated.

This successive obstacle of blending as the number system expands with its combination of powerful connections that enhance understanding for some and conflicting factors that cause anxiety for others underlies the difference between success and failure in developing mathematical thinking.

SUMMARY

This survey of the development of meaning in mathematical thinking reveals an underlying development of argumentation and proof as the individual is faced with a succession of acts of increasing sophistication through:

- **Compression** of complex situations into thinkable concepts using language;
- **Connectio**n between thinkable concepts in coherent **frames of reference**;
- **Blending frames in new ways**, using *embodiment, symbolism* and *formal definition and proof* to build theories and solve novel problems.

In teaching students to develop methods of argumentation and proof, we need to be aware of their cognitive development and the way in which their previous experiences will build conceptual structures that can help or hinder their development. New ideas which mathematicians may see as logical extensions of simpler ideas may be subtle conceptual blends for students who have different experiences to build upon. There is need for much more detailed understanding of the strengths and conflicts of conceptual blends that give powerful insight on the one hand and confusion and anxiety on the other.

REFERENCES

Fauconnier & Turner (2002). *The way we think: Conceptual Blending and the Mind's Hidden Complexities.* New York: Basic Books.

Doritou, M. (2006). *Understanding the number line: Conception and Practice*. PhD Thesis, University of Warwick.

Lima, R. N. de & Tall, D. O. (2006). The concept of equation: what have students met before? *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education*, Prague, Czech Republic, vol. 4, 233–241.

Tall, D. O. (2004). Thinking through three worlds of mathematics, *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Bergen, Norway, 4, 281–288.

Vlassis, J.: 2002, The balance model: hindrance or support for the solving of linear equations with one unknown, *Educational Studies in Mathematics*, 49, 341–359.