Teachers as Mentors to encourage both power and simplicity in active mathematical learning

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Introduction

The teaching of mathematics is under stress around the world. Imposed targets in many countries press teachers to train their students to obtain higher marks on national tests. 'Teaching to the test' can produce higher marks on standard questions, but is often accompanied by a sense of disappointment that many students have not understood what they have learnt. Techniques taught sequentially may enable the individual to *do* mathematics, but not necessarily to *think* about it, making the mathematics grow increasingly complicated.

This presentation considers how flexible knowledge can be built by focusing on essential detail that leads to rich concepts that are both powerful and yet simple to use. This arises through a process that the Fields' Medallist Thurston (1990) called 'compression of knowledge'. I theorise that focusing on essential ideas can lead to 'thinkable concepts' that are easier to manipulate mentally and to link together in a more powerful conceptual structure. Over the long term, students who form rich compressed ideas are more likely to be able to build on them in a simpler way than students who learn procedures just to pass tests. As a consequence, this suggests that teachers need to act as mentors to encourage their students to build thinkable concepts that link together in coherent ways.

Compression of knowledge and powerful mathematical thinking

For many individuals, mathematics is *complicated* and it gets more complicated as new ideas are encountered. For others, by focusing on the essential ideas, it becomes possible to see mathematics in a more focused way that makes many ideas essentially more simple. As my colleague, and founding Chairman of the Mathematics Institute at Warwick, Christopher Zeeman (1977), said:

> "Technical skill is mastery of complexity, while creativity is mastery of simplicity."

The way in which complicated ideas can become simple to those who conceive them in a focused way is explained beautifully by Fields medallist, William Thurston:

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics. (Thurston, 1990, p. 847.)

Examples of compression occur throughout the learning of mathematics, starting with the child learning the highly complicated action of counting the items in a collection, which is steadily compressed into the concept of number. Symbols in arithmetic, algebra, trigonometry, calculus, and so on, also usually have two complementary meanings:

- as a *process* to be carried out by some procedure of calculation or manipulation,
- as a *concept* in its own right, to be thought of as a mental object that can be manipulated, as symbols are manipulated in arithmetic or algebra.

As a *process*, it allows us to *do* mathematics, as a *concept* we are able to *think about it* and make connections to other thinkable concepts in flexible ways.
A partial list of symbols in mathematics that represent both process and concept is given in table 1.

symbol	***process***	***concept***
4	counting	number
3+2	addition	sum
–3	subtract 3 (3 steps left)	negative 3
3/4	sharing/division	fraction
$3+2x$	evaluation	expression
$v=s/t$	ratio	rate
$y=f(x)$	assignment	function
dy/dx	differentiation	derivative
$\int f(x)\,dx$	integration	integral
$\left.\begin{array}{c}\lim\limits_{x\to 2}\left(\dfrac{x^2-4}{x-2}\right)\\ \sum\limits_{n=1}^{\infty} 1/n^2\end{array}\right\}$	tending to limit	value of limit
$(x_1, x_2, \ldots, x_n)$	vector shift	point in n-space
$\sigma \in S_n$	permuting $\{1, 2, \ldots, n\}$	element of S_n

Table 1: Symbols as process and concept (taken from Tall *et al.*, 2001)

Symbols as procepts

A symbol acting dually as a process (to *do* mathematics) and a concept (to *think about*) is called an *elementary procept* (Gray & Tall, 1994). However, different symbols, such as 6, or 4+2 or 1+5 or 5+1, or 2×3 all represent essentially the same thing. This is called a *procept.* A procept can be broken into its constituents and reassembled in many flexible ways. For example, to calculate 8+6, one may know that 8+2 is 10, and breaking the 6 into 2+4 allowing the 2 to be added to the 8 to get 10, then the 4 is added to 10 to get 14. It is for this reason, that we allow the procept to be represented by different symbols, such as 4+2, 2+4, 7–1, provided that each symbol is just a different way of writing the same underlying concept.
Gray and Tall found that young children learning arithmetic developed in a spectrum of different ways. The more successful could think of numbers as flexible procepts, deriving new number facts from old (‘I know 7 and 5 is 12’ so ‘12 take away 5 is 7’) while others

remained in the safety of counting which becomes time-consuming ('12 take away 5 is … 11, 10, 9, 8, *7*').

In the mathematics classroom we spend a great deal of time teaching students procedures: column addition, long multiplication, procedures to solve equations by 'doing the same thing to both sides' leading to rules such as 'change side, change sign'. In calculus we meet the rules of differentiation where '$d(uv) = vdu+udv$' is used to differentiate a product by replacing u and v by formulae and working out the symbolic derivative; then we move on to develop procedures for integration.

I suggest that the idea of using symbols flexibly as 'do-able' process or 'thinkable' concept is what makes mathematics essentially simpler at successive levels of sophistication.

Different ways of interpreting symbols

Let me ask a fundamental question. Do you personally see the following formulae being 'the same' or 'different'?

$$x(x+2),\ x^2+2x.$$

Think about it for a while.

As procedures they are *different*. The first adds x and 2, then calculates x times the result; the second squares x, also multiplies 2 times x, then adds the two results together. As procedures they involve *totally different* sequences of operations.

As algebraic expressions, we often say that they are *equivalent*. There are several reasons for this. First, as a procedure of evaluation, for a given numerical value of x, the two *always* give the same numerical output. Secondly the equation $x(x+2) = x^2+2x$ is true for all x and is often referred to as an *identity*. Thirdly if one shifts one's focus of attention from the successive steps of a procedure to the *effect* of the procedure, then the two procedures of evaluation *have the same effect*.

Finally, when a function is defined to be the set of ordered pairs

$$\{(x,y)\in\mathbb{R}^2 \mid y = f(x),\quad x\in D\}$$

then the functions

$$f(x) = x(x+2),\ g(x) = x^2+2x$$

are, *by definition*, exactly the same function on the domain $D=\mathbb{R}$.

Compression from Procedure to Process to Procept

In the mathematics education literature, a distinction is often made between a *procedure* and a *process*, by defining a procedure to be a specific set of steps and a process to be a transformation as a whole (Davis, 1983, p. 257). This means that a particular process may be carried out by several different procedures. With this interpretation, an algebraic expression may be considered as a specific *procedure* of evaluation, or, if we allow ourselves to think of equivalent expressions—such as $x(x+2)$ and x^2+2x—as being 'the same', then they are just different ways of writing the same *process*. At another level, by thinking of the expressions as functions that are both processes to evaluate and concepts to be mentally manipulated, the expressions now act as *procepts*. As procepts, two functions u and v can be operated on to obtain their sum $u + v$, difference $u - v$, product uv, quotient u/v and composite $u\circ v$ (where $(u\circ v)(x) = u(v(x))$.

This gives us four different stages which become progressively more sophisticated as:

1. *procedure*: carried out by a single step-by-step sequence of actions;
2. *multi-procedure*: several different procedures are available that give the same result;
3. *process*: a function with given input-output relationship;
4. *procept*: where functions act dually as a process that can be carried out by various procedures or as mental objects that can be operated upon.

As the meaning shifts from one stage to the next, the mathematical techniques become more efficient, more flexible and more interconnected to allow richer forms of mathematical thinking. (Figure 1.)

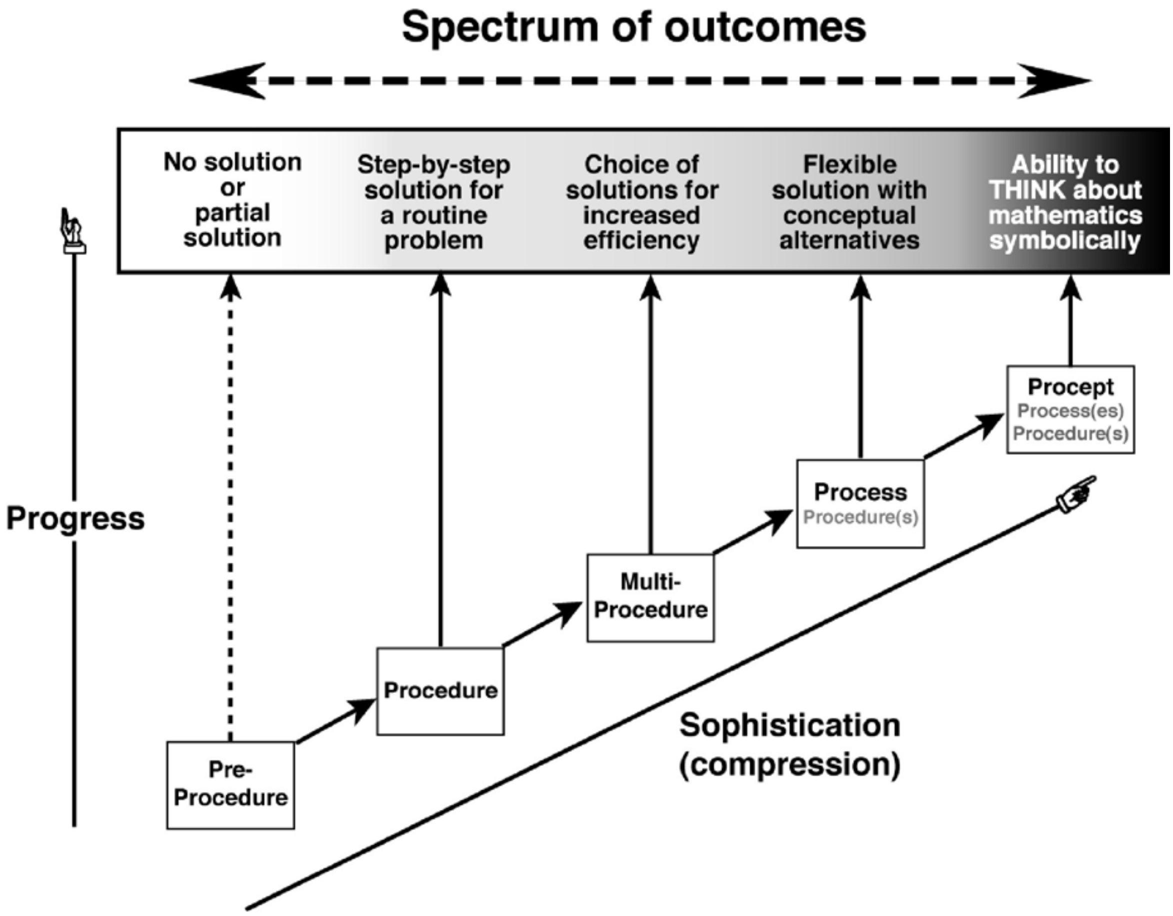

Figure 1: Spectrum of outcomes as symbolic compression becomes more sophisticated (expanded from Gray, Pitta, Pinto & Tall, 1991, p.121).

For example, the factorization of the quadratic to solve the equation $ax^2 + bx + c = 0$, can be performed by three related procedures: factorizing the quadratic, completing the square, or using the quadratic formula (which is derived from completing the square). A group of Brazilian teachers, conscious of their students growing difficulty with algebra and the need for them to gain success on tests, decided to focus mainly on the quadratic formula which could could essentially be used to solve any quadratic. Their students were asked to respond to the following:

> To solve the equation $(x–2)(x–3) = 0$, John answered in one line “$x = 3$ or $x = 2$”. Is the answer correct? Analyze and comment on it. (Lima & Tall, 2006).

Only six out of 77 students gave a satisfactory answer. Three substituted the values into the equation, and of the remainder, the most common response, begun by 16 students was to solve the equation from scratch: 14 attempted to multiply out the brackets, 6 did it correctly, of whom 4 then used the quadratic formula and just 3 found the correct roots. *No* student

mentioned the idea that ‘if the product of two numbers is zero, then one of them must be zero’. Using the analysis in figure 1, these students remained at the level of a single procedure, lacking the efficiency of having different procedures appropriate for particular contexts and the flexibility to think about mathematics symbolically.
As another example, the following calculus problem can be solved in several different ways:

$$\frac{d}{dx}\left(\frac{x^2+1}{x}\right)$$

One is to write $u = x^2+1$, $v = x$ and to use the quotient rule, which leads to an expression that requires several steps to simplify it. Another is to rewrite the formula as $x + x^{-1}$, which can be differentiated term-by-term in a single step to give $1 - x^{-2}$. Another, more complex way is to write

$$y = \frac{x^2+1}{x}$$

so $\quad xy = x^2 + 1$

and to differentiate xy as a product to obtain

$$x\frac{dy}{dx} + y = 2x$$

which, after some simplification, gives the required derivative. This is less efficient than either of the other methods, but gives yet another procedure to carry out the same process of differentiation.
When 36 Malaysian students taking a calculus course taught mainly by procedural methods were asked how many solutions they could give to this problem, the 12 most successful students with grade A on the previous year’s examinations could offer significantly more procedures than the 12 who achieved grade C. (Ali & Tall, 1996). (Table 2.)

Student Grade	0 or 1 methods **[procedure]**	2 or 3 methods **[multi-procedure] [or process]**
A	3	9
B	7	5
C	9	3
Total	19	17

Table 2: Flexibility of student solution processes

This shows greater flexibility associated with higher examination grades.

Procedural and conceptual thinking

The two examples given so far are both solvable by procedural responses. The fluent and efficient use of procedures is part of the development of mathematical fluency, but it is not enough. Procedures *occur in time* and it is difficult to think about other things at the same time as carrying out a procedure. What has been noticed with some alarm is that there is a growing tendency for students to be able to solve 1-step problems requiring a single procedure, but a loss of ability in solving multi-step problems. (LMS, 1995).
Teaching procedures is *part* of the whole development, but is not sufficient for long-term development of mathematical sophistication.

To reach the higher levels of flexible operation requires the making of connections and I conjecture that making connections is easier when procedures are compressed into thinkable concepts after the manner indicated by Thurston. This involves a steady compression of meaning in the symbolism, and can also be seen by representing the concepts in other ways.

Embodiment and Symbolism

Mathematical concepts can be represented as physical actions (such as a transformation), as models, pictures, words and symbols. Bruner (1966) described three different modes of operation.

> What does it mean to translate experience into a model of the world. Let me suggest there are probably three ways in which human beings accomplish this feat. The first is through action. […] There is a second system of representation that depends upon visual or other sensory organization and upon the use of summarizing images. […] We have come to talk about the first form of representation as **enactive**, the second is **iconic**. […] Finally, there is a representation in words or language. Its hallmark is that it is **symbolic** in nature. (Bruner, 1966, pp. 10–11.)

The enactive mode through action and gesture is of particular value in mathematics. For instance, we may enact the slope of a graph by moving our hand along it to sense where it slopes up, where it is horizontal and where it slopes down (figure 2).

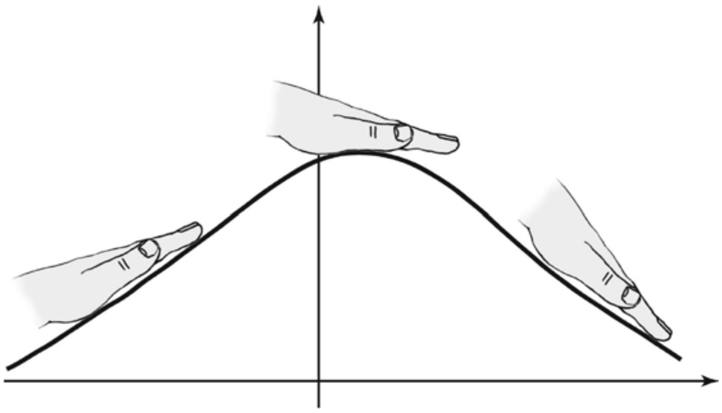

Figure 2: Dynamically tracing the changing slope of a curve with the hand

I term the combination of enactive and iconic, together with the mental thought experiments that accompany them, the *embodied* world of mathematical thinking. It complements the *symbolic* world of calculation and manipulation in arithmetic, algebra, symbolic calculus and so on. (For further details, see, for example, Tall 2004, Tall, 2006, obtainable from the website www.davidtall.com/papers.)

An example: the derivative of cos*x*

Figure 3 shows the graph of cos x and the practical slope function which stabilizes to look like the graph of –sin x. (using Blokland et al, 2000). In this way it is possible to *see* the slope function *before* the need to compute it symbolically.

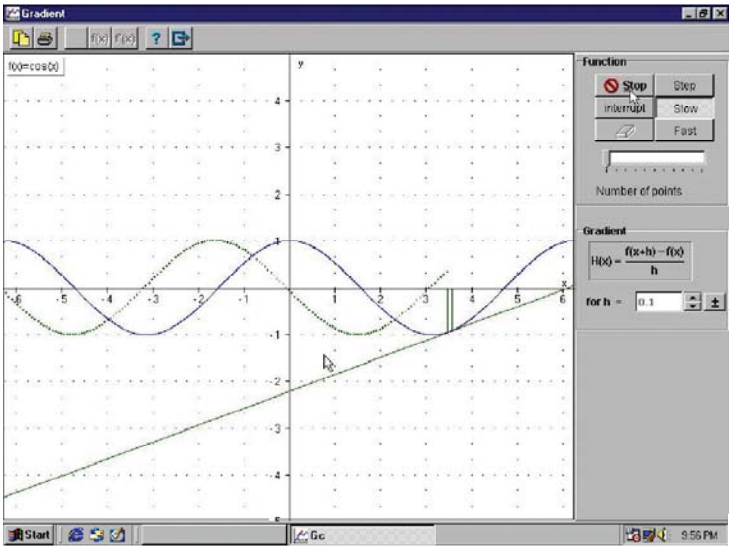

Figure 3: The gradient of cos$x$ looks like sin$x$ upside down

What is important here, is that the student makes conceptual links and *expects* the derivative of cos$x$ to be –sin$x$, rather than needing to use the limiting process to *prove* the limit exists.

An example: the case of vector

The concept of vector arises in many guises: as a displacement, a velocity, an acceleration, a force, expressed visually as arrows of given magnitude and direction or symbolically in terms of column vectors and matrices. Students can be taught *procedurally* to add two vectors together as arrows representing given magnitude and direction by following one after the other and using the triangle or parallelogram law. But this does not necessarily carry with it a conceptual meaning.

An embodied meaning may be given through considering a vector as a translation of a shape in space, say by translating a triangle on a horizontal plane. (Figure 4.)

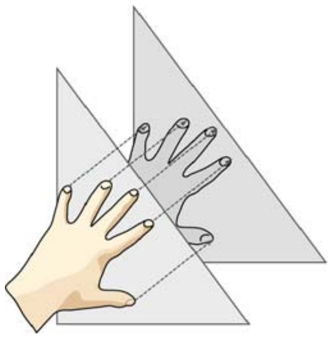

Figure 4: Translating an object

The effect of the translation can be seen by the relative shift of any point on the triangle, or on the hand that moves it, represented by an arrow from starting point to finishing point. One may regard all these arrows as ‘equivalent’ with the same magnitude and direction. A more powerful alternative is to imagine a *single arrow* of given magnitude and direction that can be shifted to start at any point whose endpoint then gives the finishing point. This gives the notion of *free vector*. Addition of free vectors is then accomplished by placing one after the other to obtain the unique free vector that has *the same effect* as the two free vectors traced sequentially. (Figure 5.)

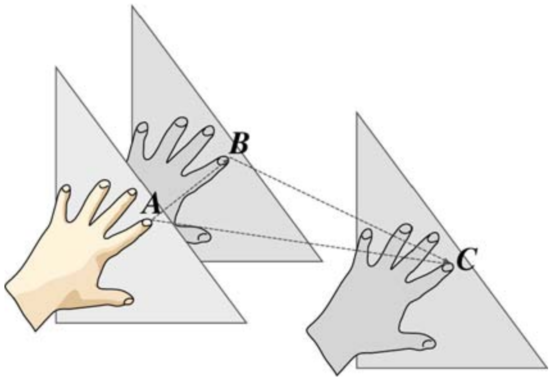

Figure 5: The sum of two vectors as the total effect of two translations.

If students conceive vectors as actual *journeys*, they may add them naïvely by doing one journey after the other, jumping in the middle as necessary, rather than the mathematical addition as free vectors. (Figure 6.)

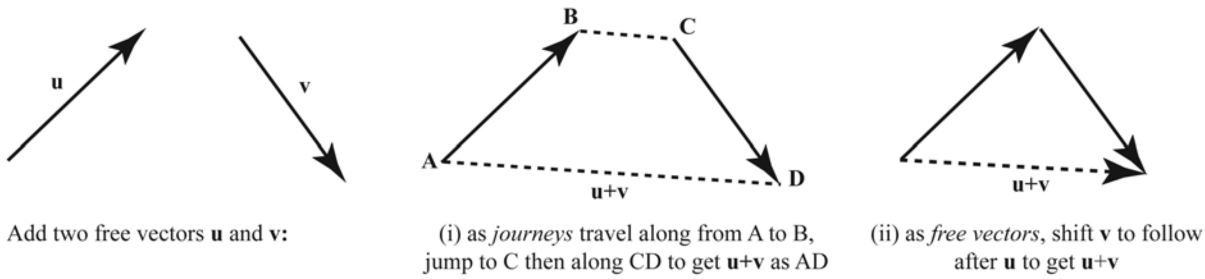

Figure 6. Adding together vectors

The error adding vectors as journeys arises through seeing them as actions, rather than as free vectors conceived as the effect of the actions. Taking this embodied approach to teaching vectors, focusing on the effect of a transformation, Poynter (2004) was able to significantly improve students' understanding of the mathematical notion of free vector.

Linking embodiment and symbolism

Mathematical symbolism arises from embodied actions such as counting, sharing, ordering, measuring, and so on. In this way, embodiment can give initial meaning and equivalence of symbols can be seen as actions *having the same effect.* For instance, equivalence of fractions (Figure 7.)

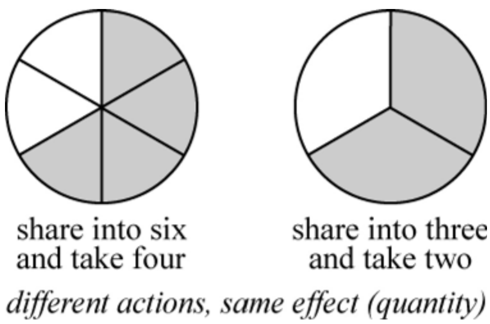

Figure 7: equivalence in terms of actions with the same effect

If the results of the actions are seen as objects, then picking up the four sixths and placing them over the two thirds shows that they are the same (in terms of quantity, if not in terms of the number of pieces).

Algebraic identities such as $a^2 - b^2 = (a-b)(a+b)$ can be given an embodied meaning as in the following picture (figure 8).

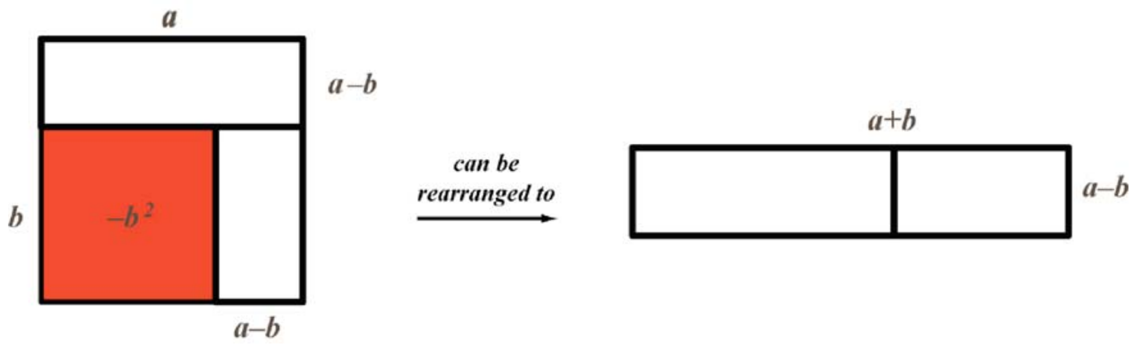

Figure 8: $a^2 - b^2 = (a-b)(a+b)$

This embodiment is straightforward when a and b are positive and $a > b$. But it soon gets complicated when $b > a$ or when the values a and b are positive or negative. This requires meaning given to negative lengths (by reversal) and negative areas (by turning over). Figure 9 shows the picture when $a < 0$, $b > 0$ and $|b| < a$. Can you see its meaning?

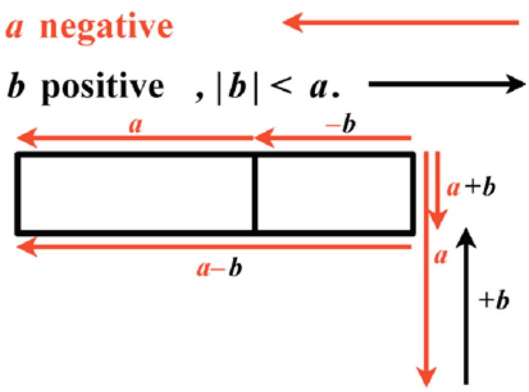

Figure 9: $a^2 - b^2 = (a-b)(a+b)$ for $a < 0$, $b > 0$, $b < |a|$

Matters become even more complicated with the embodiment of formulae for higher powers:

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$a^4 - b^4 = (a-b)(a+b)(a^2 + b^2)$$

The former can be 'seen' in three dimensions, but the embodiment of the latter in four dimensions is far more sophisticated for mortals living in three-dimensional space. The situation becomes far more complicated when a and b have a mixture of positive and negative signs. Fortunately the symbolic manipulation is the same in all cases. For instance, the difference of two fourth powers is no more than the difference of the squares of a^2 and b^2 that can be handled in two easy stages:

$$\begin{aligned} a^4 - b^4 &= (a^2 - b^2)(a^2 + b^2) \\ &= (a-b)(a+b)(a^2 + b^2). \end{aligned}$$

As mathematics gets more sophisticated, the power of symbolism carries the successful mathematical thinker further. However, to do this requires not only meaning through linking symbolism to embodiment, but also meaning *within symbolism itself*. I suggest that such meaning occurs through switching focus from the steps of a procedure to its effect and the flexible use of symbols as procepts to give sophisticated meaning to symbol manipulation.

Making connections in the classroom

The teaching and learning of mathematics may be approached in a range of possible ways, related to what is learnt, how it is learnt, and the responsibilities of the teachers and learners to themselves and to others. In a traditional *transmissio*n approach, the teacher provides a class of learners with an introductory explanation, followed by graded exercises for them to work on their own to build up specified concepts in an organised sequence. On the other hand, in a student-centred classroom the learners are seen as the architects of their own learning by *discovery*, with the teacher as a facilitator and organiser of resources. A more sophisticated approach is a *connectionist* classroom, in which the teacher as mentor orchestrates classroom activities, encouraging the learners to make connections by focusing on essential ideas, working in groups, sharing ideas and being challenged by problems.

To categorise individual teachers in a study of the effective teaching of numeracy, Askew et al., 1997 designed a battery of phrases of the type given in Table 3. (The ones in the figure are modified slightly and I take responsibility for this.) The brief phrases only tell part of the story, though taken as a whole they reveal broad trends.

As you look down the list to see which items you tend to agree with personally, you may find yourself choosing some items from one column and other items from others. To test myself, I decided to consider each line of three choices in turn and award a maximum of 3 points to the line in total, dividing the points according to my strength of feeling. I might give all 3 points to a single preference, or 2 points to a strong preference and 1 point to a weaker preference, or use just two points to give one each to two statements that I preferred equally over the third. My own scores were transmission 4, connectionist 12, discovery 4, showing a strong agreement with many of the connectionist items in the table, and with some agreement with elements of transmission and discovery.

Teacher's beliefs		
Transmission	**Connectionist**	**Discovery**
About pupils *being numerate*		
Primarily the ability **to perform calculations** by **standard procedures**	**Calculation methods** which are both **efficient and effective**	**Finding the answer** to a calculation **by any method**
Heavy reliance on **paper & pencil methods**	Confidence & ability in **mental methods**	Heavy reliance on **practical methods**
How pupils *learn* to become 'numerate'		
Individual activity based on **following instructions**	**Interpersonal activity** through **interaction with others**	**Individual activity** based on **actions on objects**
Pupils vary in their *ability* to become numerate	***Most* pupils are able to become numerate**	**Pupils vary in the *rate* at which their numeracy develops**
Pupils learn through being introduced to one mathematical routine at a time and remembering it	**Pupils learn through being challenged** and struggling to overcome difficulties	**Pupils need to be 'ready' before they can learn** certain mathematical ideas
How to *teach* pupils to become 'numerate'		
Teaching has priority over learning	**Teaching and learning are complementary**	**Learning has priority over teaching**
Teaching is based on ***verbal explanation*** so that pupils understand the teacher's methods	Teaching is based on ***dialogue*** between teacher and pupils to explore understandings	Teaching is based on ***practical activities*** to enable pupils to discover methods for themselves
Application is best approached through **'word problems'**	Application is best approached through **challenges** that need to be reasoned about	Application is best approached through using **practical equipment**

Table 3. Teacher orientation towards numeracy (selected from Askew et al., 1997)

In the original research, teachers were classified using these categories and the improvements of their students was traced over a period as measured using National Curriculum tests and classified as highly effective, effective or moderately effective. (See Table 4, selected from

Askew et al, 1997, pp. 31, 32). The data showed that those who were strongly connectionist were all highly effective, those who were strongly transmission or discovery oriented were only moderately effective, while those with no strong orientation were in a spectrum between the two extremes, with the majority being effective rather than highly or moderately effective.

	Highly Effective	Effective	Moderately Effective
Strongly Transmission			Beth Cath Elizabeth
Strongly Connectionist	Anne Alan Barbara Carole Faith		
Strongly Discovery			Brian David
No Strong Orientation	Alice	Danielle Dorothy Eva Fay	Erica

Table 4: The relation between teacher orientation and effectiveness.

Discussion

Even though the data from the previous section relates to a single study focusing on numeracy, its potential implications are clear. The teacher who acts as a mentor, balancing the act of teaching and learning to encourage students to make connections is likely to be more effective.

In many curricula, the emphasis is on giving meaning by relating the mathematics to real-world problems, suggesting the need to link symbolism to human embodiment. Embodiment works fine initially with simple cases but we have seen it may grow more complicated as the context becomes more sophisticated. Hence it is necessary to seek sense not only in embodiment but also within symbolism itself.

Here the symbolic procedures need to be conceived in ways that are easy for the human brain to comprehend. I suggest that this involves compression of knowledge, switching the focus of attention from step-by-step procedures, not just to more efficient step-by-step procedures, but to the *effect* of those procedures. By doing this we see equivalent procedures having the same effect as representing one and the same *process* which can then be manipulated symbolically as a thinkable *concept*, giving a flexible *procept*.

In whole number arithmetic, increased power arises from decomposing and recomposing numbers as procepts to give flexibility in arithmetic operations. In fractional arithmetic, increased power comes from shifting focus from fractions such as 2/3 and 4/6 as different sharing procedures to processes representing one and the same rational number. In algebra, different procedures of evaluation—such as $(x-1)(x+1)$ and x^2-1 —having the same effect can be conceived as a single *process* that becomes a manipulable *concept* as a function. The shift from *action* to *effect* and on to *thinkable concept* occurs both in embodiment and symbolism. It gives the underlying mechanism to shift attention from routine procedures to build increasingly sophisticated mathematical thinking. Some students remain with procedures that become increasingly complicated to use. Others are more successful at developing more flexible thinking. There is a major role for teachers to act as mentors encouraging a greater number of students to compress knowledge into thinkable concepts that are appropriate for connecting ideas together in sophisticated and simple ways.

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