EMBODIMENT, SYMBOLISM AND FORMALISM IN UNDERGRADUATE MATHEMATICS EDUCATION

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*In recent years I have been working on a theoretical framework of long-term learning that presents three ways in which mathematical thinking develops that operate so differently as to present essentially three distinct 'worlds of mathematics'—**conceptual embodiment**, **proceptual symbolism** and **axiomatic formalism**. Long-term human learning is seen to begin with facilities **set-before** birth in the genes and builds on successive constructions based on conceptions **met-before** in development. Thinking becomes increasingly sophisticated through **compression of knowledge** in which important aspects of a (possibly complicated) situation are named and built into rich **thinkable concepts** that are both powerful and simple in use. At the same time concepts that were met-before may enhance or impede new thinking where the latter requires explicit focus on re-thinking old ideas to develop new sophistication. This leads to a wide range of success from those who focus on the essential elements that compress into thinkable concepts and those who focus, if at all, more on incidental elements that lead to a more diffuse cognitive structure.*

The framework will be exemplified in three important areas—algebra, calculus and proof—to reveal how difficulties of algebra relate to the shift from embodiment to symbolism which underpins arithmetic but causes difficulty in algebra, how the embodied notion of local straightness can give a wider conceptual meaning to the calculus complementing the symbolic meaning of local linearity, and how proof develops in different ways in each world, with generic examples and thought experiments in conceptual embodiment, specific calculation and generic manipulation in proceptual symbolism and deduction from concept definitions in axiomatic formalism. The paper concludes by considering how formal proof often leads to structure theorems that link axiomatic systems back to more sophisticated forms of conceptual embodiment and proceptual symbolism.

INTRODUCTION

The development in interest in Research in Undergraduate Mathematics Education takes us a step further in looking at the whole framework of mathematical development from the young child to the research mathematician. It lies at the crossroads between school mathematics studying space and number and the formal mathematical theories and more sophisticated applications at college and university.

In recent years I have begun to build a simple framework that starts from the genetic inheritance of the newborn child and is broad enough to cover the spectrum of development of different individuals as they mature over the longer term. At the root of this increasing sophistication is the use of language to *compress* a complex phenomenon into a *thinkable concept* whose meaning can refined by experience and

Keynote presented at the *10th Conference on Research in Undergraduate Mathematics Education*, Feb 22–27, 2007, San Diego, California, USA.

discussion and connected to other thinkable concepts in rich cognitive schemas. This occurs both in the van Hiele development of generic conceptions in geometry and the symbolic process-object compression in arithmetic and algebra, leading eventually to the major compression of set-theoretic definitions into single axiomatic concepts such as infinite cardinal number or complete ordered field.

Three worlds of mathematics

The framework is based on three different but intertwined worlds of development, two of which dominate elementary mathematics, and the third develops in advanced mathematical thinking:

- the **conceptual-embodied** (based on perception of and reflection on properties of objects);
- the **proceptual-symbolic** that grows out of the embodied world through action (such as counting) and symbolization into thinkable concepts such as number, developing symbols that function both as processes to *do* and concepts to *think about* (called procepts);
- the **axiomatic-formal** (based on formal definitions and proof) which reverses the sequence of construction of meaning from definitions based on known concepts to formal concepts based on set-theoretic definitions.

(Tall, 2004, quoted from Mejia & Tall, 2006)

Terms such as 'embodied', 'symbolic', 'formal' have all been used in a range of different ways. Here I use a technique that arose from my friend and supervisor, the late Richard Skemp, in putting two familiar words together in a new way to signal the need to establish a new meaning (such as 'instrumental understanding' and 'relational understanding' or 'concept image' and 'concept definition').

'Conceptual embodiment' refers not to the broader claims of Lakoff that *all* thinking is embodied, but to the more specific idea of embodiment conceptualised through thought experiment based on perception and reflection on the properties of objects. We conceptually embody a geometric figure, such as a triangle consisting of three straight line-segments; we imagine a triangle as such a figure and allow a specific triangle to act as a prototype to represent the whole class of triangles. We 'see' an image of a specific graph as representing a specific or generic function.

'Proceptual symbolism' refers to the use of symbols that arise from performing an action schema, such as counting, where the symbols used become thinkable concepts, such as number. A symbol such as 3+2 or $\int \sin x \, dx$ represents both a process to be carried out or the thinkable concept produced by that process. Such a combination of symbol, process, and concept constructed from the process is called an elementary procept; a collection of elementary procepts with the same output concept is called a *procept* (Gray & Tall, 1994). I theorize that it is the flexible use of symbols as procepts in arithmetic, algebra, trigonometry, symbolic calculus, and so on, that enables the human mind to manipulate such symbols with great power and precision.

‘Axiomatic formalism’ refers broadly to the formalism of Hilbert that takes us beyond the formal operations of Piaget. In the famous lecture announcing his twenty-three problems that dominated the twentieth century, Hilbert remarked:

> To new concepts correspond, necessarily, new signs. These we choose in such a way that they remind us of the phenomena which were the occasion for the formation of the new concepts. So the geometrical figures are signs or mnemonic symbols of space intuition and are used as such by all mathematicians. Who does not always use along with the double inequality $a > b > c$ the picture of three points following one another on a straight line as the geometrical picture of the idea “between”? Hilbert 1900 ICM lecture

The formal axiomatic world of mathematicians is predicated on giving formal definitions to concepts and proving theorems by mathematical proof, but it is also underpinned by the experiences of mathematicians that suggests what theorems may be worth proving and how the proof might be carried out, which in turn builds on the mathematicians’ embodied and symbolic experience.

The question often arises as to why the framework refers to three *worlds* of mathematics, as opposed to, say, three different *modes* of operation. The reason is because the modes of thinking used in different contexts become more sophisticated in each world as the individual matures. For instance, the conceptual-embodied world has a long-term development essentially formulated by van Hiele: objects are first seen as gestalts, then various properties are *described*; there is a shift of attention in which the objects are *defined* and new objects tested to see if they fit the definition; then these definitions are used in verbal ‘if … then …’ statements that lead to Euclidean geometry and beyond. Meanwhile the proceptual-symbolic world grows out of embodiment of counting procedures that are compressed into manipulable whole number concepts, with successive process–object encapsulations not only in whole number arithmetic, but also in broader number systems through fractions, negatives, integers, rationals, reals, complex numbers each expanding to the generalised arithmetic expressed in algebra and on to the potentially infinite limit processes in the calculus (Tall *et al.*, 2001). The fundamental shift to the axiomatic-formal world occurs through a shift in attention from the focus on properties that belong to *known* objects to properties formulated as concept definitions to *define* mathematical objects.

Having formulated the terms ‘conceptual-embodied’, ‘proceptual-symbolic’ and ‘axiomatic-formal’, I make a conceptual compression by using the shortened forms ‘embodied’, ‘symbolic’ and ‘formal’, with new meanings given to them in the theoretical framework of three worlds. This enables us to consider new combinations, such as ‘symbolic-embodied’ where symbolism is embodied, ‘embodied-formal’ where embodied ideas are translated into formal structures, and ‘symbolic-formal’ where symbolic ideas are translated into formalism (figure 1). (Not shown in the figure is the supporting language structure which operates in ways appropriate for each world and the underlying conscious and sub-conscious mental processing.)

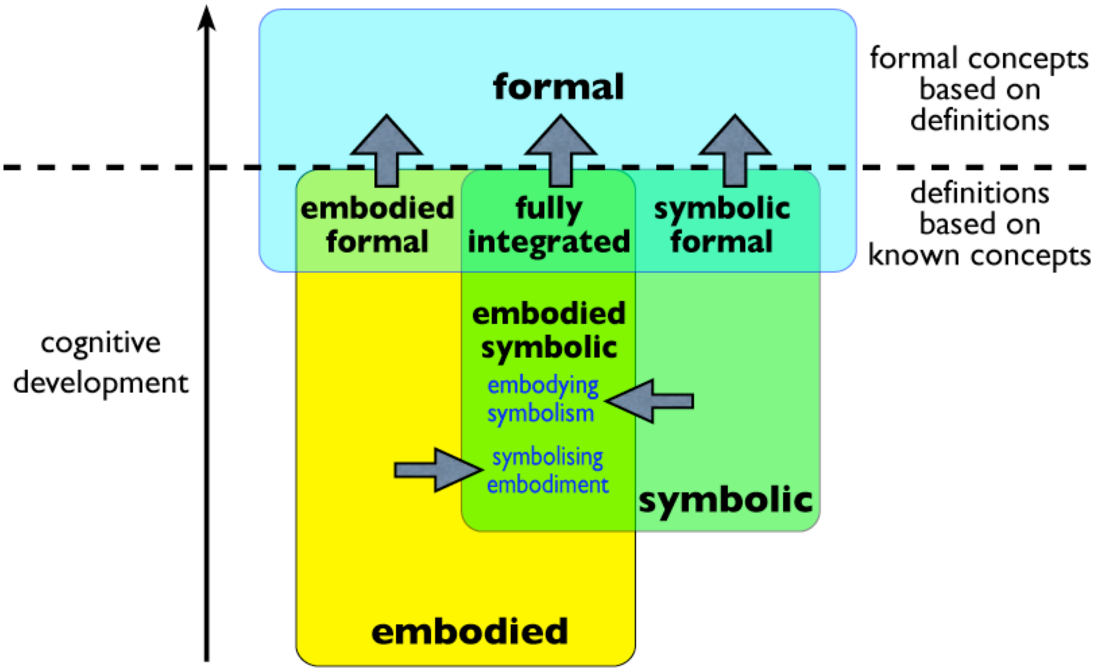

Figure 1: Cognitive development through three worlds of mathematics

Compression, Connection and Thinkable Concepts

Compression into thinkable concepts occurs in several different ways. One, discussed thoroughly by Lakoff (1987) in *Women Fire and Dangerous Things*, is through categorisation where concepts are connected in various ways in a category and the category itself becomes a thinkable concept. Another described by Dubinsky and his colleagues is APOS theory where an Action is internalised as a Process and is encapsulated into an Object, connected to other knowledge within a Schema; a schema may also be encapsulated as an object. Following Davis (1983, p. 257) who used the term *procedure* to mean a specific sequence of steps and a *process* as the overall input-output relationship which may be implemented by different procedures, Gray, Pitta, Pinto & Tall (1999) represented the successive compression from procedure through multi-procedure, process and procept, expanded in figure 2 in parallel to the SOLO taxonomy sequence: unistructural, multi-structural, relational, extended abstract (Pegg & Tall, 2005). This models the way in which a procedure which is thinkable sequence of steps *to do* in time is steadily enriched to give the efficiency of choosing the most suitable procedure to perform the task in a particular concept, condensed into a process and compressed as a procept *to think about* and to manipulate mentally in a flexible way.

Some students who may have difficulty with the procedure may become entrenched at the procedural level, perhaps reaching the multi-procedural stage that can lead to procedural efficiency. Others who focus on procedures as overall processes and then as flexible procepts can lead to a far more sophisticated proceptual level of operation.

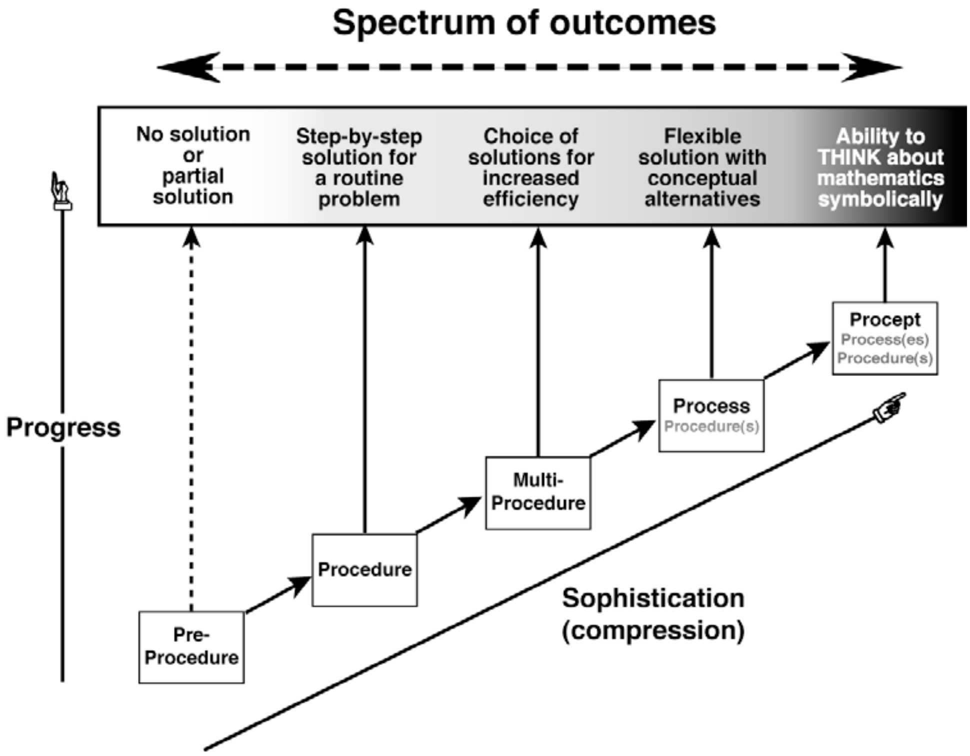

Figure 2: Spectrum of outcomes from increasing compression of symbolism (expanded from Gray, Pitta, Pinto & Tall, 1999, p.121).

The earlier work of Dubinsky and his colleagues (e.g. Cottrill et al., 1996) focused mainly on a symbolic approach by programming a procedure as a function and using the function as the input to another function. The data shows that, while the process level was often attained, encapsulation from process to object was more problematic.

The symbolic compression from procedure to process to object has an embodied counterpart. The move from procedure to process simply involves shifting the focus of attention from the *steps* of a procedure to the *effect* of the procedure. For example, a translation of an object on a plane is an action in which each point of the object is moved in the same direction by the same magnitude. Any arrow of that given magnitude and direction represents the effect of the action and can be imagined as a single free vector that may be moved to any point to show how that point moves. The free vector is a conceptual embodiment of the vector translation as an object. Adding free vectors as objects by placing them nose to tail gives the unique free vector that has the same effect as the two following one after the other.

In this way, we see a parallel between symbolic compression in APOS theory and embodied compression through shifting attention from the steps of an action to the effect and imagining the effect as an embodied object (figure 3). This link between symbolism and embodiment can play its part in the compression of process into object, enabling the individual to refer mentally to the encapsulated process as a conceptual embodiment. From this viewpoint, conceptual knowledge makes links between thinkable concepts, not only with ‘real world’ applications, but also within and between proceptual symbolism and conceptual embodiment (figure 4).

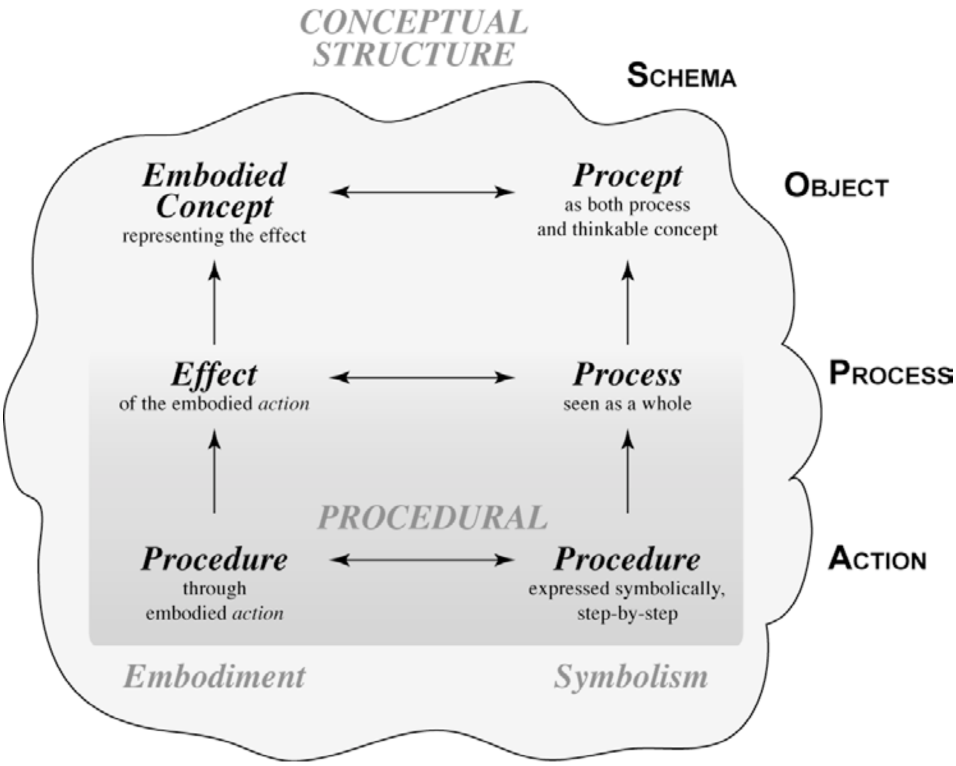

Figure 3: Procedural knowledge as part of conceptual knowledge (from Tall, 2006)

As different individuals follow through a mathematics curriculum that introduces ideas in increasing levels of sophistication, they cope with it in different ways. Piaget hypothesised that all individuals pass through the same sequence of stages at different rates but Gray and Tall (1994) observed the *proceptual divide* in which children develop in different ways, some clinging to the security of known step-by-step procedures, while others compress their knowledge into the flexible use of symbols as process and concept (procepts). Procedures occur in time and work in limited cases but are not sufficiently compressed into thinkable concepts to be used flexibly for more sophisticated thinking.

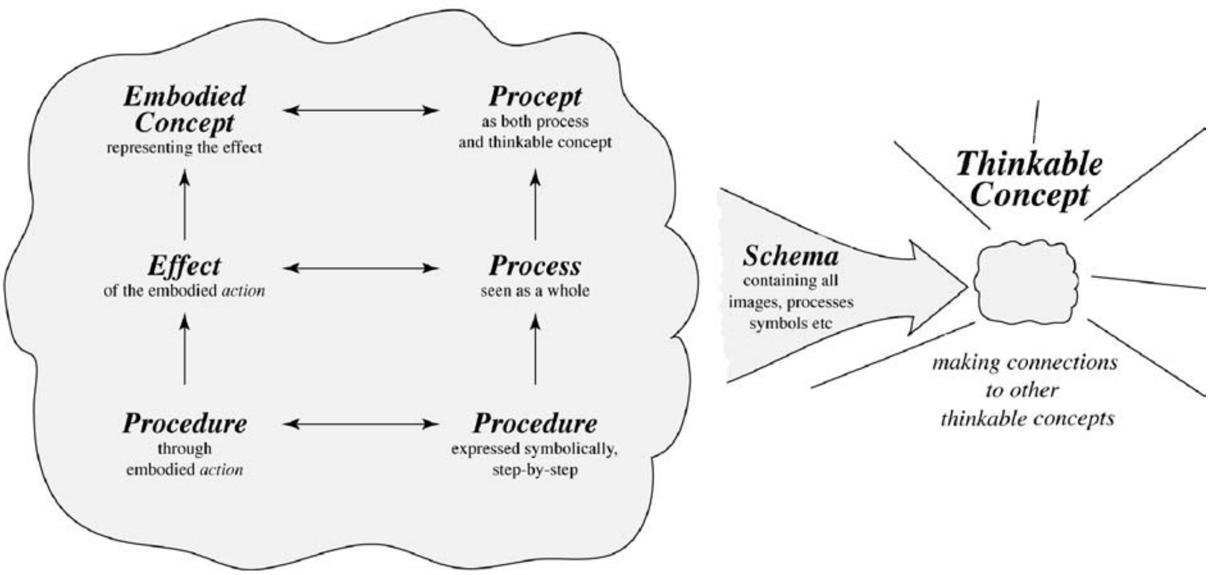

Figure 4: compressing a schema into a thinkable concept (from Tall, 2006)

More generally, a schema may be compressed into a thinkable concept by naming it. For example, 'whole number arithmetic' names the full range of operations and concepts a person builds from the arithmetic of whole numbers. However, such schemas need not be thinkable mental objects that can themselves be operated upon.

In the formal world, a further compression is possible, in which a schema such as whole number arithmetic may be described as a list of axioms, such as the Peano axioms, giving rise to the thinkable concept $\mathbb{N}$ which can be an object in a higher theoretical framework, such as category theory. Likewise, other thinkable concepts can have representations in different worlds, such as the real numbers as an embodied number line, infinite decimals as a proceptual symbolic system and the axiomatic system $\mathbb{R}$. I note with interest that Lakoff and Nunez come to the same three aspects by ideas analysis from intellectual reflection, though I see these building upwards from an interplay between embodiment and symbolism followed by formalism.

Set-befores and met-befores

Long-term human learning is based on a combination of facilities **set-before** birth in the genes and builds on successive constructions based on conceptions **met-before** in development. For instance, the visual structure of the brain has built-in systems to identify colours and shades, with structures to see changes in shade, identifying edges, coordinating the edges to see objects stand out from the visual background. Thus the child is born with a generic system to recognise small numbers of objects (one, two, or perhaps three) which gives a set-before for the concept of 'twoness' before building the counting schema that is compressed into the number concept.

In our analysis we will mainly focus on met-befores where previously constructed cognitive connections are used to interpret new situations. Sometimes a met-before is consistent with the new situation, sometimes it is inconsistent. For instance, the met-before '2+2 makes 4' is experienced in whole number arithmetic and continues to be consistent with the arithmetic of fractions, positive and negative integers, rationals, reals and complex numbers. But the met-before 'taking away gives less' remains consistent with (positive) fractions, but is inconsistent with negatives where taking away –2 gives more. The same met-before works consistently with finite sets, where taking away a subset leaves a smaller number of elements, but is inconsistent in the context of infinite sets, where removing the even numbers from the counting numbers still leaves the odd numbers with the same cardinality. In this way, met-befores can operate covertly affecting the way that individuals interpret new mathematics, causing internal confusion that impedes learning.

As we look at the framework of development through three distinct worlds of mathematics, we need to take into account the actual learning of students at successive stages and the met-befores they have available to make sense of new experiences, particularly those which become inconsistent with a new experience. Written curricula almost always focus on met-befores that remain consistent in the new context; problems occur with subtle met-befores that are inconsistent.

ILLUSTRATIONS OF THE FRAMEWORK IN ACTION

To illustrate the framework of three worlds and the related ideas of cognitive compression and met-before, we consider college algebra, calculus, and proof.

College Algebra

Algebra is a nightmare for many adults:

> For some, audits and root canals hurt less than algebra. Brian White hated it. It made Julie Beall cry. Tim Broneck got an F-minus. Tina Casale failed seven times. And Mollie Burrows just never saw the point. This is not a collection of wayward students, of unproductive losers in life. They are regular people […] with jobs and families, hobbies and homes. And a common nightmare in their past.
>
> (Deb Kollar, *Sacramento Bee* (California), December 11, 2000.)

Why does algebra cause so much anguish? Its predecessor, arithmetic is built on *embodiment*: collecting objects into sets and counting them, putting them together to add, dividing them into equal size subsets to share, putting them in order of size, measuring lengths, adding lengths by putting them one after another. Some aspects of algebra can be embodied, for example, the expression $2a+3b+4a$ can be simplified by 'picking up the $4a$ and moving it next to the $2a$' then grouping them together as $6a$, to give $6a+3b$. This 'fruit salad' version of algebra, treating letters as objects (apples and bananas), works in simple cases but soon fails. If we have 'six apples and three bananas' then we have 'nine apples and bananas', but do we write it as '9 *a b*'. What does $6a-3b$ mean? How can we take 3 bananas from 6 apples? Expressions like $3+2x$ may not be understood and the student may do what s/he knows (adding 3 and 2), leaving the x that makes no sense to write $5x$. For many struggling to find meaning, algebra is a minefield of dysfunctional met-befores.

Equations bring new problems. There is the long-standing observation (christened the 'didactic cut' by Filloy and Rojano, 1989) that an equation such as $5x+3=13$ with an expression equal to a number is easier to solve than an equation with the unknown on both sides such $5x+3=9x-5$. The former may be seen as an operation which can be 'undone' by taking off 3 from the 13, and then dividing by 5 to get x is 2. According to APOS theory, the latter would be more sophisticated because it requires the two sides to be seen as equal expressions that need to be manipulated as objects.

Instead of a process-object interpretation, equations can be seen as a 'balance', with the operations on both sides embodied as a strategy to maintain the balance. This makes sense to a wide range of the population when algebraic equations are first introduced (Vlassis, 2002). However, it soon fails with equations with negative terms or negative solutions. Introducing this embodiment can act as a met-before that *enhances* meaning for those that focus on the principle 'do the same thing to both sides' but acts *as an impediment* for cannot imagine it working with negative terms.

Lima & Tall (2006) reveals data that suggests that neither process-object compression nor the embodied balance approach covers the full range of cases. The students were taught to maintain the balance by 'doing the same thing to both sides'. In interview, it transpired that many students focused not on the general principle, but on two specific principles: shifting 3 in the equation $2x+3=9$ to the other side by 'change sides, change sign' and shifting 2 to the other side in $2x=6$ by 'shift it over and put it underneath'. Instead of the balance embodiment, many students combined an embodied shifting of terms with added 'magic' of rules that made no sense to them. The 'didactic cut' (and the related APOS interpretation) was not applicable because the students had similar proportion of success and failure solving the two equations $5t-3=8$ and $3x-1=3+x$.

While the students did not appear to be using a *conceptual embodiment* such as a balance, they were performing a mental action corresponding to shifting the symbols around from one place in the equation to another, with added rules. Lakoff (1987, p.12,13) makes a distinction between *conceptual embodiment* and *functional embodiment*. He does not expand on this distinction later in the book, nor in his other books (Lakoff & Johnson, 1999; Lakoff & Nunez, 2000). However, if 'conceptual embodiment' is interpreted in terms of thought experiments and 'functional embodiment' in terms of functioning as a human being, then the mental shifting of terms may be a functional embodiment. In this way there may be a broader link between the three worlds of mathematics and Lakoff's theory. However, Lakoff makes no explicit mention of compression of knowledge and APOS theory focuses more on compression of symbolic knowledge rather than embodiment.

After thinking about the teaching of algebra for many years, I have a sense that both embodiment and symbolism play essential roles. The met-befores from arithmetic often have embodied underpinnings while the embodiments applied to algebra—such as the balance model for equations, or a pictorial representation of a^2-b^2 as the difference of two squares —only copes with positive values. Students with a proceptual sense of arithmetic are very likely to find algebra a natural generalisation of their arithmetic knowledge, but those already limited to procedural operations and hampered by a lack of embodied meaning are likely to be limited to the fragility of learned procedures supported by meaningless rules such as 'change sides, change signs' or 'move it over and put it underneath.'

Calculus

The categorisation of mathematical thinking into embodied, symbolic and formal is particularly appropriate in the calculus. Reform calculus in the USA builds on combining graphic, symbolic and analytic representations of functions using computer software and graphical calculators. However, those of us occupied in research in undergraduate mathematics need to look a little deeper into how the concepts of calculus are constructed. Mathematicians, who live in a world built on the met-before of the limit concept have a view of calculus that sees the need to introduce the limit concept explicitly at the beginning of the calculus sequence. My own view is different. For students building on the embodiment and symbolism of school mathematics, I see a more natural route into the calculus that has the full potential to lead either to standard mathematical analysis, non-standard infinitesimal analysis, or practical calculus in applications.

There is an essential difference between the *embodied* notion of local straightness and the *symbolic* notion of local linearity. Local straightness involves an embodied thought experiment looking closely at graphs to see that, as small portions of certain graphs are highly magnified, they look straight. Of course, this is difficult to formalise at first encounter. But it makes sense to students as they look at a computer screen successively magnifying a graph of a familiar function composed of polynomials, trigonometric functions, exponentials or logarithms. For instance it is enactively and visually evident that the slope function of the cosine is *minus* sine because its graph is the graph of $\sin x$ upside down (figure 5).

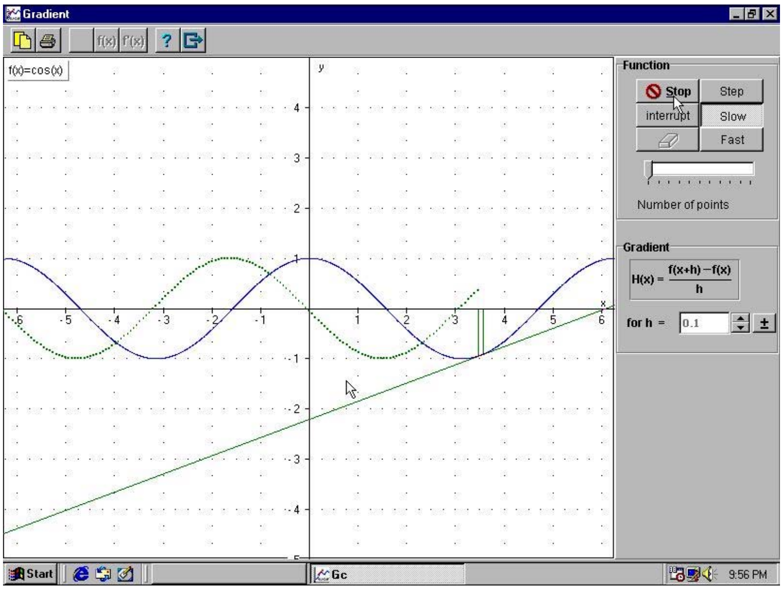

Figure 5: The gradient of $\cos x$ is *minus* $\sin x$ ($\sin x$ upside down), drawn using Blokland & Giessen, 2000

It also makes sense that a function like $|\sin x|$ has a corner at every multiple of π so that on can begin to imagine not only local straightness, but also situations that are *not* locally straight. It is also relatively simple to give an embodied proof with hand gestures, that the recursive blancmange function is everywhere continuous, but *nowhere* differentiable. Here magnification of the graph shows tiny blancmanges growing everywhere, so the magnification *never* looks straight (figure 6).

The arguments and pictures are found in several of my papers (see for example, Tall 1982, 2003). Defining the 'nasty function' $n(x) = bl(1000x)/1000$ then $\sin x$, and $\sin x + n(x)$ *look the same* when drawn on a computer over a range say –5 to 5, but one is differentiable everywhere and the other is differentiable nowhere! This gives an embodied insight into the concept of differentiability as a global phenomenon: it is the slope *of the graph* and you can *see* the changing slope as the eye follows the curve looking at its changing slope as a function of the position of the graph.

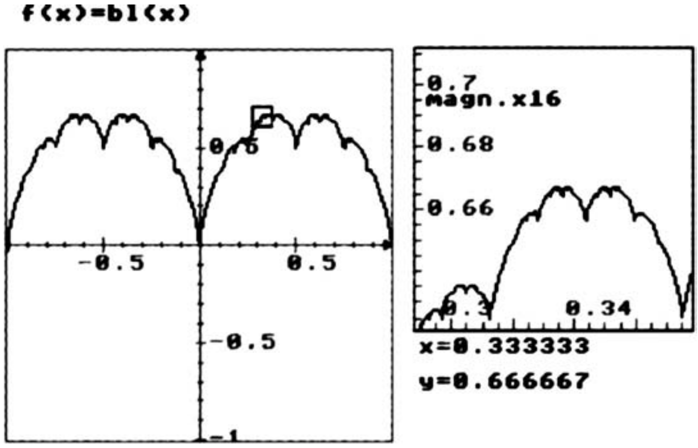

Figure 6: A graph that nowhere looks straight under magnification

Local linearity, on the other hand, is a *symbolic* concept, seeking the best linear approximation to the curve at a single point. It involves an explicit limiting concept from the start instead of an implicit limiting concept that occurs when zooming in to see how steep the curve is over a short interval. Non-differentiability is the non-existence of a limit, which lacks the immediacy of the embodied idea of not being locally straight, which applies just as easily at a point as it does over an interval.

More generally, the function $a(x)=\int_0^x bl(t)\,dt$ is differentiable once everywhere and twice nowhere. When I showed a class of students the graph of $a(x)$ calculated numerically by a computer program, one of the students (not a mathematics major) said, 'you mean that function is differentiable once but not twice.' If you know of any other mathematics professor who has had a student imagine a function that is differentiable once and not twice, tell him or her to e-mail me.

Local straightness is particularly apt when dealing with differential equations. A differential equation $dy/dx=F(x,y)$ tells us the slope of a locally straight curve at a point (x,y) is $F(x,y)$, so it is easy to program software to draw a small segment of the appropriate slope when the mouse points to (x,y) and by depositing such a solution end to end, this constructs an approximate solution. This was done in the *Solution Sketcher* (Tall, 1990) and has been implemented in the currently available *Graphic Calculus* software (Blokland & Giessen, 2000, figure 7).

The Reform Calculus Movement in the USA focuses on the notion of *local linearity*, where the derivative is introduced as the best linear approximation to the curve at a single point. It seeks a *symbolic* representation at a point, using a limiting procedure to calculate the best linear fit perhaps even with a *formal* epsilon-delta construction. Then the fixed point is varied to give the global derivative function. I cannot imagine a worse approach to the concept to present to beginning calculus students.

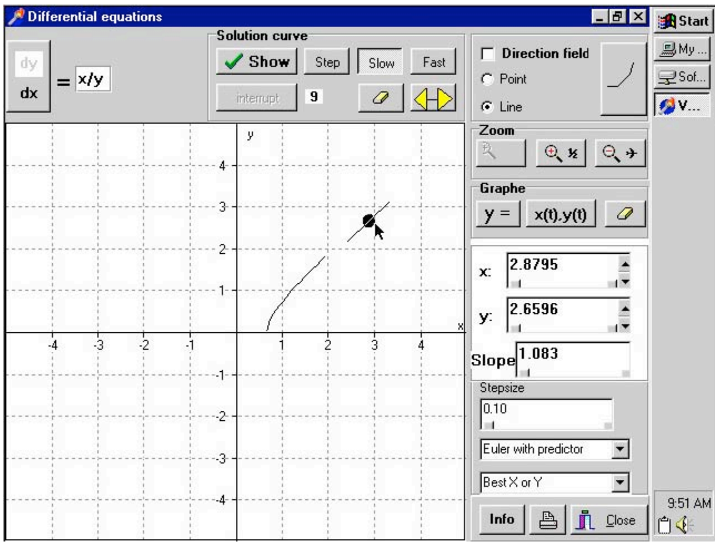

Figure 7: building the solution of a differential equation
by following its given slope (Blokland & Giessen, 2000).

Thurston (1994) imaginatively suggests seven different ways of thinking about the derivative, as distinct from different logical definitions:

(1) **Infinitesimal**: the ratio of the infinitesimal change in the value of a function to the infinitesimal change in a function.

(2) **Symbolic**: the derivative of x^n is nx^{n-1}, the derivative of sin(x) is cos(x), the derivative of $f \circ g$ is $f' \circ g * g'$, etc.

(3) **Logical**: $f'(x) = d$ if and only if for every ε there is a δ such that when

$0 < |\Delta x| < \delta$, then $\left| \dfrac{f(x+\Delta x) - f(x)}{\Delta x} - d \right| < \delta$.

(4) **Geometric**: the derivative is the slope of a line tangent to the graph of the function, if the graph has a tangent.

(5) **Rate**: the instantaneous speed of f(t), when t is time.

(6) **Approximation**: The derivative of a function is the best linear approximation to the function near a point.

(7) **Microscopic**: The derivative of a function is the limit of what you get by looking at it under a microscope of higher and higher power. (from Thurston, 1994)

Such a list is built by a great mathematician looking down from the formal world at a range of possible meaning which include local straightness (item 7). However, I suggested long ago (Tall, 1982) that the conception of derivative of a real function can be built from an even more primitive notion, from which all others grow:

(0) **Embodied**: the (changing) slope of *the function* itself, seen by magnifying the graph.

Mathematicians, with their met-befores based on the limit concept have long passed beyond this missing level 0. Learners without experience of the limit concept may benefit from such an embodied introduction. In a range of papers, I have shown how such an embodied beginning can lead either to a standard analysis approach, a non-standard infinitesimal approach or a more practical combination of embodiment and symbolism taken in applications by engineers, biologists, economists and so on. Those applying the calculus are more likely to use a combination of embodiment to imagine a situation and symbolism to model it to seek a solution while rarely using the formalism of mathematical analysis.

It is my contention (Mejia and Tall, 2004) that the calculus belongs not to the formal world of analysis, 'looking down' on it from above; it belongs in the vision of Newton and Leibniz, *looking up* from met-befores in embodiment and symbolism used appropriately. The framework of embodiment, symbolism and formalism suggests how learners may be mentored to comprehend the calculus, building *up* to the limit concept from experience rather than *down* from the formal definition.

Using a framework of embodiment and symbolism, Hahkiöniëmi (2006) studied his own calculus teaching to find students following different developments, including an embodied route, a symbolic route and various combinations of the two. He found that 'the embodied world offers powerful thinking tools for students' who 'consider the derivative as an object at an early stage.'

This simple observation is at variance with APOS theory suggesting the building up of the limit concept from (symbolic) Action to Process and then to Object. It questions Sfard's (1991) theory of structural and operational thinking that suggests that operational thinking invariably must precede structural. Using a computer to zoom in to magnify a graph, students *do* perform actions and *do* operate and then begin to conceptualise the graph of the changing slope as an *object* in itself. But it is still an *embodied* object in a thought experiment imagining the relationship between the graph and its slope. If one can *see* it, then one can attempt to calculate it, numerically or symbolically. Here embodiment gives meaning and symbolism gives precision of numeric computation and symbolic representation.

Given the complexity of the concept of the derivative, human meaning needs to be created in a way that *makes sense*. Hahkiöniëmi proposes a learning framework in which the teacher is responsible for guiding the student through the ideas, taking account of different possible conceptual routes rather than seeking a single genetic decomposition characteristic of APOS theory (figure 8).

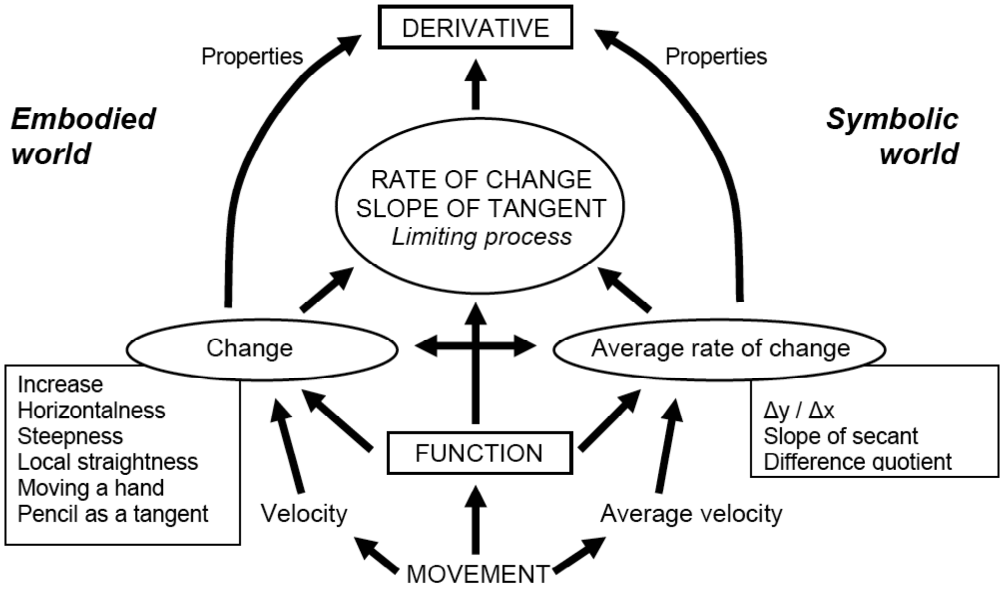

Figure 8: Hypothesised learning framework (Hahkiöniëmi, 2006).

Proof

Proof is handled differently in each of the three worlds. In the embodied world it is handled initially in terms of *thought experiment* using specific, then generic pictures, and later, as language takes over from description to definition, the properties of figures and their relationships are verbalised in *Euclidean proof*. In the symbolic world, proof is first by arithmetic calculation, (first specific then generic), then by general algebraic manipulation. There are connections between embodiment and symbolism in the embodied symbolic world. Finally, in the formal world, *formal proof* is based on concept definition and formal deduction (figure 9).

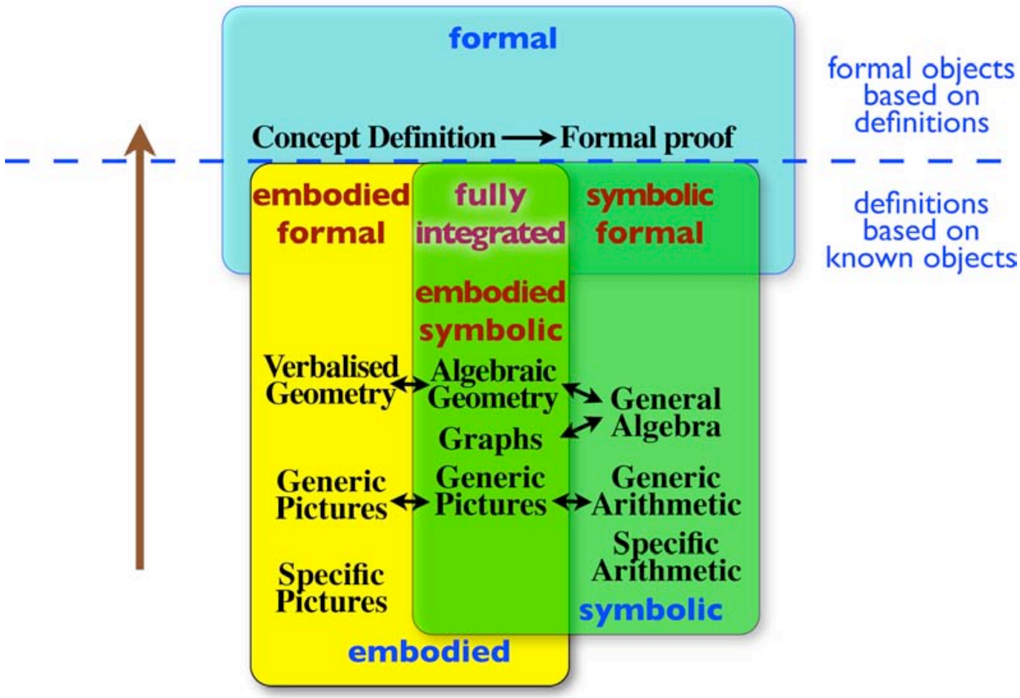

Figure 9: cognitive development of proof

I leave it to others to compare this map of cognitive development with the proof schemes of Harel & Sowder (1998) and subsequent developments. For instance, the framework as presented does not include authoritative proof. However, it should be remembered that the whole cognitive development is recursive and students are trying to comprehend what mathematicians *who have already passed through such a development* are conveying to them. Procedural thinking fits into the framework as a primitive form of rote-learning by repetition to reproduce written proofs from memory.

In considering the development of proof at the undergraduate level, account should be taken of earlier forms of argument, such as embodied arguments using prototypical generic examples on the one hand and symbolic developments starting from specific arithmetic calculations seen as generic arguments and then moving to symbolic arguments using algebraic manipulation.

The major shift in proof occurs from the embodiment and symbolism of school mathematics to the formalism of advanced mathematical thinking (Tall, 1991). Proof in the embodied and symbolic worlds is based on concepts that are given definitions, so the concepts underpin any sense of proof. Proof in the formal world is ostensibly based only on set-theoretic definitions and mathematical deduction. However, as students come to appreciate formal proof, they build on their previous experience.

My colleague and PhD student, Marcia Pinto (1998) followed through students learning concepts in formal mathematical analysis and found there were two distinct routes, one a 'natural' route *giving meaning* to definitions from the met-befores of the individual's concept image (including both embodiment and symbolism), the other a 'formal' route *extracting meaning* from the concept definition (figure 10).

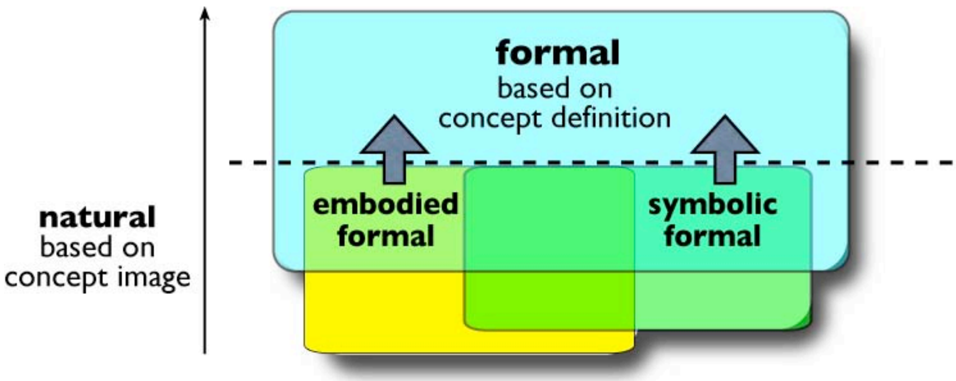

Figure 10: natural thinking building on embodiment and symbolism,
formal thinking building on concept definition

Weber (2004) added to this framework a *procedural* approach that involves learning the proof by rote as was mentioned above. While an attempted formal approach can fall back to procedural learning of formal proofs by rote, a natural approach can lead to conflict between concept image and formal theory, from which rote-learning is again a strategy used in an attempt to cope.

FROM FORMAL PROOF BACK TO EMBODIMENT AND SYMBOLISM

A major goal in building axiomatic theories is to construct a *structure theorem*, which essentially reveals aspects of the mathematical structure in embodied and symbolic ways. Typical examples of such structure theorems are:

- An equivalence relation on a set A corresponds to a partition of A;
- A finite dimensional vector space over a field F is isomorphic to F^n;
- Every finite group is isomorphic to a group of permutations;
- Any complete ordered field is isomorphic to the real numbers.

In every case, the structure theorem tells us that the formally defined axiomatic structure can be conceived an embodied way and in the last three cases there is a corresponding manipulable symbolism.

Thus, not only do embodiment and symbolism act as a foundation for ideas that are formalized in the formal-axiomatic world, structure theorems can also lead back from the formal world to the worlds of embodiment and symbolism. These new embodiments are fundamentally different with their structure built using concept definitions and formal deduction. They lead to greater sophistication and future development leavened with the insights and flaws of human thinking.

REFLECTIONS

The final return of formalism to a more sophisticated form of embodiment and symbolism through structure theorems leads me to see the three worlds of mathematics as a natural structure through which the biological brain builds a mathematical mind.

At the point were undergraduates study mathematics there is a range of questions to address. In college algebra we need to have a far better insight into the underlying problems that cause students anxiety. I suggest that this is a problem in the transition between embodiment and symbolism. The embodiments of arithmetic work well with whole numbers and fractions but need modification for negative numbers and have limited application in algebra. My own view is that the major shift from arithmetic to algebra is far easier when the student has a flexible proceptual view of arithmetic and can easily shift to algebra as generalised arithmetic.

An embodied approach has a so-far-untapped potential to give meaning in college calculus. The met-befores of mathematicians give a view of the subject based at the very start on the limit concept computed at a fixed point that is then allowed to vary. An embodied locally linear approach gives the student the vision to *see* the whole derivative function as the graph of the changing slope.

The development of proof is seen as generic proof in embodiment and manipulative proof in symbolism, first through specific calculations, then generic arithmetic, then general algebra, and, as the framework of relationships between properties grows, it becomes possible to base proofs on set-theoretic definitions of axiomatic systems.

Proof as conceived by university mathematicians grows from embodiment and symbolism and has structure theorems that can take us on to more sophisticated embodiment and symbolism. A theoretical framework of conceptual embodiment, proceptual symbolism and axiomatic formalism provides a rich structure in which to interpret mathematical learning and thinking at all levels, and in particular in undergraduate mathematics.

REFERENCES

Blokland, P. & Giessen, C. (2000). *Graphic Calculus for Windows*. [On line.] Available from http://www.vusoft2.nl.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., Vidakovic, D. (1996). Understanding the Limit Concept: Beginning with a Coordinated Process Scheme. *Journal of Mathematical Behavior*, 15 (2), 167–192.

Davis, Robert B. (1983). Complex Mathematical Cognition. In Herbert P. Ginsburg (Ed.) *The Development of Mathematical Thinking*, (pp. 254–290). New York: Academic Press.

Filloy, E. and Rojano, T. (1989). Solving equations: the transition from arithmetic to algebra, *For the Learning of Mathematics*, 9 (2), 19–25.

Gray, E. M., Pitta, D., Pinto M. M. F. & Tall, D. O., (1999). Knowledge Construction and diverging thinking in elementary and advanced mathematics, *Educational Studies in Mathematics*, 38 (1–3), 111–133.

Gray, E. & Tall, D. O.: 1994, Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetic, *The Journal for Research in Mathematics Education*, 26(2), 115–141.

Hahkiöniëmi, M. (2006). *Tools for studying the derivative*. Unpublished PhD, University of Jyväskylä, Finland.

Harel, G., & Sowder, L. (1998). Student's proof schemes: results from exploratory studies. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in Collegiate Mathematics Education* III (pp. 234–283). Providence, RI: American Mathematical Society.

Hilbert, D. (1900). *Mathematische Probleme*, Göttinger Nachrichten, 253-297, translated into English by Mary Winton Newson, retrieved from the word-wide web at http://aleph0.clarku.edu/~djoyce/hilbert/problems.html.

Kollar, D. (2000). A Common Nightmare in their Past, July 10, California: Sacramento Bee.

Lakoff, G. (1987). *Women, Fire and Dangerous Things*. Chicago: Chicago University Press.

Lakoff, G. & Johnson, M. (1999). *Philosophy in the Flesh*. New York: Basic Books.

Lakoff, G. & Nunez, R.: 2000, *Where Mathematics Comes From*. New York: Basic Books.

Lima, R. N. de & Tall, D. O. (2006). The concept of equation: what have students met before? *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* 4, 233–241, Prague, Czech Republic: PME.

Meija-Ramos, J. P. & Tall, D. O. & (2004). Reflecting on Post-Calculus-Reform: Opening Plenary *Topic Group 12: Calculus, International Congress of Mathematics Education*, Copenhagen, Denmark. http://www.icme-organisers.dk/tsg12/papers/tall-mejia-tsg12.pdf

Mejia-Ramos, J. P. & Tall, D. O. (2006). The Long-Term Cognitive Development of Different Types of Reasoning and Proof, presented at the Conference on *Explanation and Proof in Mathematics: Philosophical and Educational Perspectives*, Essen, Germany.

Pinto, M. M. F. (1998). *Students' Understanding of Real Analysis*. Unpublished PhD, University of Warwick.

Pegg, J., & Tall, D. O. (2005). The fundamental cycle of concept construction underlying various theoretical frameworks. *International Reviews on Mathematical Education (ZDM)*, 37 (6), 468-475.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*. 22, 1–36.

Tall, D. O. (1982). The blancmange function, continuous everywhere but differentiable nowhere, *Mathematical Gazette*, 66, 11–22.

Tall, D. O. (1985). Understanding the calculus, *Mathematics Teaching,* 110 49–53.

Tall D. O. (1991). *Real Functions and Graphs* for the BBC computer, Master, Compact, Nimbus PC & Archimedes computer. Cambridge: Cambridge University Press.

Tall, D. O. (Editor), (1991). *Advanced Mathematical Thinking*. Dordrecht: Kluwer, now Berlin: Springer-Verlag.

Tall, D. O. (2003). Using Technology to Support an Embodied Approach to Learning Concepts in Mathematics. In L.M. Carvalho and L.C. Guimarães, *História e Tecnologia no Ensino da Matemática*, vol. 1, pp. 1-28, Rio de Janeiro, Brasil.

Tall, D. O. (2004), The three worlds of mathematics. *For the Learning of Mathematics*, **23** (3), 29–33.

Tall, D. O. (2006). A Theory of Mathematical Growth through Embodiment, Symbolism and Proof. *Annales de Didactique et de Sciences Cognitives*, 11, 195–215. Strasbourg: Irem de Strasbourg.

Tall D. O., Gray, E., Bin Ali, M., Crowley, L., DeMarois, P., McGowen, M., Pitta, D., Pinto, M., Thomas, M., & Yusof, Y. (2001). Symbols and the Bifurcation between Procedural and Conceptual Thinking, *Canadian Journal of Science, Mathematics and Technology Education 1*, 81–104.

Thurston, W. P. (1990). Mathematical Education*, Notices of the American Mathematical society,* 37 (7), 844–850.

Thurston, W. P. (1994). On Proof and Progress in Mathematics, *Bulletin of the American Mathematical Society*, 30 (3) 161–177.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: a case study of one professor's lectures and proofs in an introductory real analysis course, *Journal of Mathematical Behavior*, 23 115–133.

Vlassis, J. (2002). The balance model: hindrance or support for the solving of linear equations with one unknown, *Educational Studies in Mathematics*, 49, 341–359.

Zeeman, E. C. (1977). *Catastrophe theory*. Addison Wesley, London.