Developing a theory of mathematical growth

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Abstract*: In this paper I formulate a basic theoretical framework for the ways in which mathematical thinking grows as the child develops and matures into an adult. There is an essential need to focus on important phenomena, to name them and reflect on them to build rich concepts that are both powerful in use and yet simple to connect to other concepts. The child begins with human perception and action, linking them together in a coherent way. Symbols are introduced to denote mathematical processes (such as addition) that can be compressed as mathematical concepts such as sum to give symbols that operate flexibly as process and concept (procept). Knowledge becomes more sophisticated through building on experiences met before, focussing on relationships between properties, leading eventually to the advanced mathematics of concept definition and deduction. This gives a theoretical framework in which three modes of operation develop in succession and each grows in sophistication from* ***conceptual-embodiment*** *using thought experiments, to* ***proceptual-symbolism*** *using computation and symbol manipulation, then on to* ***axiomatic-formalism*** *based on concept definitions and formal proof.*

ZDM-Classification: C30

Introduction

It is an honour to pay tribute to Hans-Georg Steiner. He was a man who built links between countries, organised and contributed to conferences throughout the world, from the education focus in the International Congress of Mathematicians (ICM), through the International Congress of Mathematics Education (ICME), the International Group for the Psychology of mathematics education (PME), and his own Theory of Mathematics Education (TME). Yet he also had time for individuals, amongst whom I am privileged to count myself.

At a crucial time in my career over twenty years ago, my mind was alive with ideas, running in several directions, and he took me under his wing and invited me to Bielefeld where I had the opportunity to talk with him to share ideas with his group and with other members of the IDM.

In those days I was a mathematician interested in developing ways to help students conceptualise mathematical ideas and I had begun by programming software to give students new ways to visualize ideas in calculus. At that time I was rethinking mathematical concepts to take account of the new technologies and had practical ideas for calculus, proof and wider aspects of advanced mathematical thinking, but had not yet attempted to develop a coherent overall theory.

Working at Bielefeld gave me the impetus to seek a larger theoretical framework as I worked successively on visualisation in calculus, and later on symbolisation in arithmetic and algebra, and advanced mathematical thinking through concept definition and proof.

In the last five years all these ideas have come together to give a theoretical framework encompassing three distinct modes of mental operation which give rise to three interlinked worlds of mathematics: the *conceptual-embodied* (based on perception of and reflection on objects), the *proceptual-symbolic* (based on actions such as counting that are symbolised as concepts to give flexible 'procepts') and the *axiomatic-formal* (based on formal definitions and proof). This framework, in turn, builds from an even simpler basis relating to the way human beings perceive things, act on things and reflect on their perceptions and actions. This shifts the focus of my thinking to the way mathematical thinking grows in the individual, building on previous experience, to use perception, action and reflection to construct mathematical ides in a way that gives rise naturally to a theoretical framework of cognitive growth through three worlds of mathematics. I dedicate this theory to the memory of Hans-Georg Steiner in conjunction with all those iconic figures in mathematics education at whose feet I reflected on the ideas presented here.

The journey begins

Forty years ago, when mathematics education was in its infancy, the choice of a framework for long-term cognitive development was between behaviourism of Skinner or the genetic epistemology of Piaget, accompanied by other specific theories related to mathematics, such as those of Bruner, Skemp, Dienes, Freudenthal, and Fischbein.

Freudenthal and Steiner combined to participate in the early meetings of the International Congress of Mathematics Education and invited Fischbein to lead the meetings. Thirty years ago at the meeting of ICME in Karlsruhe 1976, PME was born, and among its founding fathers were Fischbein, Freudenthal, Steiner, Skemp and others. Skemp was my PhD supervisor at the time and together we attended the first meeting of PME in 1977.

It was from Skemp that I had my first direct experience of any theory of mathematics education. I was a mathematician with a PhD supervised by Fields Medallist Michael Atiyah and, on transferring my interests to mathematics education, began a second PhD with Richard Skemp.

At that time, mathematics education began to build into the vast enterprise that it is now. And yet, it seems to have created more problems than it solves. The vastly increasing literature of mathematics education grows like Topsy, almost without limit. In such a complicated world, I began to crave a simple theory of how we think mathematically. I think of this in terms of a quotation from another of my icons, my colleague, and the founding Chairman of the Mathematics Institute at Warwick, Christopher Zeeman, who said:

> "*Technical skill is mastery of complexity, while creativity is mastery of simplicity.*"

At the time I visited Bielefeld I was faced with a huge complexity of ideas as I wrote software to support the learning of calculus. I knew students saw limits as unending processes rather than limit concepts (Tall, 1976) and believed that the standard approaches to teaching calculus violated their intuitive understanding. I knew about non-standard analysis and that—in a precise technical sense—the infinite magnification of a function differentiable at a point had an image that was exactly a real straight line (Tall, 1980). So I developed software to magnify graphs using a practical approach I called 'local straightness'. As I was ignorant of computer software, I did not know that every other program written at the time required the input of functions in computer notation such as x^2*SIN(3*x) instead of $x^2 \sin 3x$. I considered this an unnecessary technicality in a calculus class of the day and wrote a routine to put in powers using the cursor keys to shift up and down to give the formula as

$$x^2 \sin 3x$$

with an internal routine to translate it into x^2*SIN(3*x) invisible to the user.

It was with this *Graphic Calculus* software that I visited Bielefeld to find an opportunity to link the practicality of the computer software to a wider theoretical framework of how humans learn to think mathematically.

At this time Steiner was a great support to me. He expressed incredulity at how I had managed to do so much work alone and compared my output with the million-dollar research of Schoenfeld who had programming support to develop a single program to allow the learner to interplay with visual and symbolic representations of lines and circles while I had single-handedly created the whole of *Graphic Calculus*. My software came with a theoretical perspective and empirical evidence as to

- how the learner can gain insight through looking closely at graphs to see that graphs that are 'locally straight' have a changing slope that gives rise to a new function: the derivative;
- how this relates to first order differential equations where the slope is known and the graph to be found by following the changing direction given by the differential equation;
- how solutions of simultaneous differential equations involving *dx/dt* and *dy/dt* can be constructed visually by following the direction given in (t, x, y)-space and how this is a prototype to visualise systems of higher order differential equations;
- how a continuous function can be 'pulled flat' by taking a thin vertical strip and stretching it horizontally, so that the change in area under the graph to within a practical error equals the height times the strip-width—giving the rate of change of the area equals the original function.

This was a full theory relating differentiation and integration that coped not only with differentiable functions, but with functions that were nowhere differentiable. Fundamentally it begins with human perception of the visual notion of local straightness in a way that leads naturally to the symbolic theory of calculus and the formal ideas of continuity, differentiation and integration.

Steiner remarked that such a development required a team with a mathematics educator, a mathematician, a software designer, a programmer, a schoolteacher, a researcher, an author for the documentation, a secretary, and a team-leader. In my case, he commented that there was such a team and it consisted of a single person.

I remember too how Michael Otte, after several glasses of wine, talked about three deep thinkers in mathematics education: Otte himself in Germany, Brousseau in France and Tall in England. It was all heady stuff and very encouraging for a lone mathematician turned mathematics educator who, for the first time, was exposed to a whole department who cooperated together on a larger vision.

This inspired me to seek a simple theory involving not just the visualisation and symbolisation of the calculus, but a wider theoretical framework building from human perception and action to introduce symbolism to develop mathematical thinking that eventually leads on to formal definitions and proof.

Towards a theoretical framework

In the early nineties, I was privileged to work with Eddie Gray on how young children do arithmetic (Gray & Tall, 1991, 1994). He had the idea of asking young children simple problems such as 6+8, then, if they did not know the answer immediately, he asked them how they worked it out. As we discussed the various thinking processes of the children he had interviewed, we began to realise that we were speaking of the symbol 6+8 in more than one way. While for an adult it was simply the answer, 14, for different children it evoked a range of strategies of calculation: counting out two sets of 6 and 8 and then counting them all, counting on 8 after 6, or switching them round and counting 6 after 8, knowing that 8+2 is 10, and breaking 6 down into 2+4, rearranging the sum to 10 and 4 to get 14. The symbol 6+8 evoked a whole range of meanings from long counting procedures to simple number concepts. At that moment we realised we needed a word to encapsulate this flexible meaning of the symbol that was both process (in possibly many ways) and also concept and the term 'procept' came to mind (Gray & Tall, 1991).

This shift in focus from a process (counting) to a concept (number) was a special case of the notion of compression of knowledge formulated by Thurston (1990) in which experience with a complicated phenomenon led to familiarity and the opportunity to conceive it in total as a simple idea that could operate as part of a new system of ideas.

We began to consider other ways in which the complicated details of a particular situation could be compressed to give a rich simplicity in mathematics. We had identified how complicated counting procedures in arithmetic are compressed into simple number concepts, and the symbol can act dually as process or concept, but we realised that not all mathematical concepts are procepts.

A triangle is not a procept, it is arises from our perceptions and actions on physical objects we can see, touch and talk about. Compression of knowledge in geometry acts in a different way. Certain phenomena are observed that are named and talked about. A 'triangle' is a figure with three straight sides and three corners or vertices. Triangles come in different shapes and sizes and can be classified in various ways as scalene, isosceles, equilateral, acute-angled, right-angled, obtuse-angled. It is the coherence of perceptions of—and actions on and with— triangles that gives human meaning. It is by giving the *name* 'triangle' that compression begins to take place, allowing the single word to apply to all of these examples and to distinguish different kinds of triangle and properties of triangles, for instance, that the angles of a triangle always add up to 180°.

In a totally different context, as I shifted from early arithmetic and geometry to write the final chapter of *Advanced Mathematical Thinking* (Tall, 1991); I saw my colleagues writing in two quite different ways: some, such as Dubinsky (1991), talked about process-object encapsulation, others, such as Vinner (1991), talked about concept images and mental pictures and how they related to logical forms of thinking based on definitions and deductions.

This revealed two complementary modes of operation: one typical of geometry focusing on thought experiments on physical and mental objects, the other focuses on actions that are symbolised and the symbols manipulated as if they are mental objects. In each case, reflections on the properties concerned enables connections to be made between concepts and, eventually, the properties themselves can become the basis of a third, formal-axiomatic, approach.

As I reflected on these ideas over time, I began to relate these observations to other existing theories, I began to seek broad similarities that were likely to provide a simple structure common to all rather than the complicated details which were more likely to set them apart.

Relationships between theories

Piaget formulated long-term development in his stage theory through sensori-motor/concrete-operational/formal-operational, Bruner (1966) in his enactive-iconic-symbolic modes, Fischbein (1987) in his intuitive / algorithmic / formal approaches, and Skemp (1979) through his theory of two levels, delta-one and delta-two, where each involves *perception* (input) and *action* (output) and the shift from delta-one to delta-two occurs through *reflection* on perception and action in the actual world to produce conceptions at the higher inner delta-two level. Just has Skemp distinguished perception and action, Piaget (1985) distinguished two different forms of abstraction: empirical abstraction from properties of objects themselves and pseudo-empirical abstraction by focusing on the properties arising from actions on objects, each leading to a third form of abstraction based on the mental actions performed which he termed reflective abstraction.

I thought large and long about perception, action and reflection in a Skempian way that related to the theories of others. This was not, however, in the form of some great literature review, for that would involve the technical mastery of complexity, not the creative insight of simplicity. On many occasions as I encountered new data from working with others (particularly my doctoral students), I reflected on the new evidence to see how it related to underlying ways that we think mathematically.

From a biological viewpoint, we develop by making connections between neuronal groups in our brains. However, we can only manage the complexity by filtering out most of the detail and focusing on the salient features that need to be considered, which is greatly enhanced by the use of language to name those features. As we develop in new areas, useful connections are strengthened and older ones that are unused may fade. In this way we grow from new-born children reliant on our mother's milk, through our childhood, making new links, more subtle concepts, towards the mature thinking of adults.

A significant breakthrough came when I was working with Anna Poynter on her research into students thinking about vectors (Poynter, 2004). She was considering the links between the geometrical representations of vectors (where vectors have magnitude and direction and are added by the triangle law), the symbolic representation of vectors, (represented by coordinates in column vectors, and matrix operations), and we discussed these together with the later set-theoretic definitions of vector spaces from which deductions were made by formal proof.

In a moment, I saw three quite different ways of conceptualizing mathematical concepts: the *embodiment* that comes from mentally representing physical objects and actions and reflecting on the mental concepts, the *symbolism* that came from representing actions as symbols and manipulating them as mental objects, and the *formalism* that came from verbalizing properties and making them the basis of a coherent system of definition and deduction.

Anna also gave me the insight into parallel acts of compression of process into object in the conceptual and symbolic worlds. She noted that translations of vectors could be performed by different actions (eg shift from A to B, then B to C, or shift straight from A to C). What mattered was not the action but the *effect* of the action. This shift of attention from action to effect proved to be an embodied form of compression that corresponded naturally to symbolic compression from procedure to process and on to concept.

For instance, the different actions of dividing something into four equal pieces and selecting two, or dividing it into six equal pieces and selecting three, produces a different number of pieces, but the quantity produced is the same. Focusing on the quantity, the effect of the two actions is the same and this effect gives rise to the notion of a fraction. This is done cognitively, not through the mathematical formulation of equivalence of two fractions, but through the mental process of changing focus from the action to its effect, to embody the fraction as the quantity produced in the process of sharing.

Communicating ideas to others

As I discussed these ideas with others, I found that terms like embodiment, symbolism and formalism had a range of different meanings. I used the terms with one meaning while others often had a quite different interpretation. Did 'embodiment' refer to the everyday use relating a physical interpretation to an abstract concept like 'Mother Theresa is the embodiment of Christian Charity' or did it refer to the a more technical sense used by Lakoff (1987) in which *all* human thought is embodied? Did 'symbolism' refer to the mathematical use of symbols in arithmetic, algebra, and so on, or to the wider sense of human symbols as in the theories of Peirce and Saussure? Was 'formal' used in a cognitive Piagetian sense or a mathematical sense as in the formal approach to mathematics advocated by Hilbert? My search for simplicity was suddenly becoming very complicated!

However, I had the insight of my friend Richard Skemp to guide me. He believed that words should be used in ways that evoke natural meaning, yet put together in ways that provoke new ways of thinking. Thus 'understanding' by itself has a natural meaning that has proved hard for educators to categorise, but 'instrumental understanding' and 'relational understanding' have new connotations that enable the originator of the concepts to draw attention to different types of understanding (Skemp, 1976). Likewise the terms 'concept' and 'image' have general connotations but afford the opportunity of formulating a new notion by putting them together as 'concept image' in contrast to 'concept definition'.

So here I gave the three aspects of mathematical thinking double-barrelled names: 'conceptual-embodied', 'proceptual-symbolic' and 'axiomatic-formal' to refine the meanings and, once the particular meaning I had in mind was expressible to others, the terms could be shortened to 'embodied', 'symbolic' and 'formal', now carrying more specific connotations.

The three different meanings relate to the emphases placed on the three aspects of perception, action and reflection. The conceptual-embodied world focuses on objects including the figures in geometry, collections that can be counted, lines that can be measured, graphs that can be investigated to derterme their properties; they have actions performed on them by making constructions to tease out the properties of the figures, to verbalise their properties and to reflect on the relationships that arise. For instance, an isosceles triangle is initially a specific kind of figure as a whole gestalt, with two equal sides, two equal angles, and symmetry when cut out of paper and folded down the middle. At first all these properties are seen to occur together, not as a consequence one of the other. Later, when a triangle is only given as having two equal sides, it may be split into two congruent triangles by dropping a line from the apex to the midpoint of the base as part of a procedure to establish congruent triangles to deduce that the base angles are equal.

In summary, the conceptual-embodied world begins by interacting with *objects*, teasing out their properties, to begin to describe them and to formulate definitions for mental concepts that may be used in the deduction of relationships and continues into a mental platonic world of Euclidean proof.

Likewise, the proceptual-symbolic world is initially an *action*-based world, starting with actions on physical objects. By symbolising the actions and using the symbols as manipulable objects the system changes focus to the symbols themselves as procepts. Reflection on relationships then leads to building connections between the properties of these procepts. Process-object theories such as those of Dubinsky and his colleagues (e.g. Asiala et al. 1996l) originally referred mainly to process-object encapsulation without relating it to embodiment. But now, by switching the focus of attention from the steps of an embodied action to the effect of the action, we can see that symbolic process-object compression has a corresponding form of conceptually embodied compression.

As I thought through these ideas, I realised that the three modes of operation had different characteristics that made them distinct. Reflection in the embodied world of perception and action involves performing practical experiments or imagining thought experiments to see if one aspect of a situation necessarily leads to another. Working in the symbolic world of procepts entails deriving results from calculating with numbers in arithmetic or manipulating formulae in algebra. The formal-axiomatic world develops theorems from definitions and formal mathematical proof.

Language is used in distinct ways in each world. In the embodied world it is used to name concepts and formulate their properties, to organise them into named categories that themselves become thinkable concepts with properties that can subsequently be built into rich deductive systems.

In the symbolic world of procepts, ordinary language is used to formulate problems and to talk about what to do, and also to describe new concepts such as 'product', 'prime number', or 'solution of an equation'. But the major feature of this world is a new part of speech—the procept—that operates dually as process or concept in a way that is not described at all in classical grammar.

As I was a young mathematician, I remember Christopher Zeeman giving a lecture describing how a participle, such as 'calculating' in 'I am calculating' can operate as a noun (technically a gerund) as in 'calculating is an important skill'. Languages such as English allow words to function flexibly as adjectives, verbs, nouns. However, the proceptual properties of symbolism in arithmetic, algebra, symbolic calculus, and so on are far more subtle, enabling us to compress subtle operations such as working out the area under a curve $y = f(x)$ from a to b as a symbol $\int_a^b f(x)\,dx$ that represents both the process of integration and the concept of integral.

The language in the formal world changes yet again, to a new precise formulation where the terms are defined to have specific set-theoretic properties in such a way that they can be used for precise logical deduction in proving theorems.

I confess that, at the time, I was both amazed and fearful of my insight. Was it an illusion, or was mathematical thinking genuinely based on perception, action and reflection, developing through (at least) three distinct ways of thinking.

I thought of the 'three worlds' of Karl Popper, and the rise and fall in interest in his theoretical framework. I therefore did not initially publish anything on these ideas in formal journals. Instead, I took the opportunity of talking about them in seminars with students and at conferences where I could speculate freely and note the reactions of those present. By interacting with others, I found in what ways I needed to clarify and modify my ideas to give an increasingly coherent theory.

Further links with other theories

As I reflected on other theories mentioned earlier, I began to see more clearly the deeper links between theories as distinct from superficial differences that set them apart. Embodiment relates to a combination of enactive and iconic in Bruner's theory, while he subdivided his symbolic mode into 'numbers' and 'logic' which relate to the proceptual and formal. The three worlds of mathematics relate also to the sequence of Piaget's sensori-motor, pre-conceptual, concrete operational and formal operational levels, though the notion of 'formal' that I use refers to the more sophisticated set-theoretic idea of axiomatic mathematics rather than Piaget's formal level which is more concerned with performing thought experiments on things that are not currently present.

There are echoes of Fischbein's 'intuitive, algorithmic and formal' too, although I see 'embodied' as more than simple intuition, with the embodied world of thought growing in sophistication as we interpret our perceptions and actions using increasingly sophisticated language. Likewise, his 'formal' is more akin to that of Piaget than that of Hilbert. More significantly, however, Fischbein's theory can be seen as to how one may look at a single piece of mathematics and how it may be approached from three different viewpoints. My insight was not like this at all. I was interested in the long-term growth of knowledge and in elementary mathematics, only the embodied and symbolic may be present.

Each world I described becomes increasingly sophisticated as the individual makes more subtle connections and compresses complicated detail into simpler concepts that can be mentally manipulated in the mind. For instance, the embodied world begins with perceptions of objects as gestalts, but then, as significant features are noticed and described, our perceptions of the objects become more sophisticated. A triangle is no longer a perceived shape, it is a three-sided figure whose internal angles add up to 180°. A theoretical framework is built using the notion of congruent triangles to provide a euclidean proof of properties, such as the properties that the angle bisectors of a triangle meet in a single point. Our visual imagination was enhanced by our connected knowledge of the properties of objects that build up into a coherent global theory of Euclidean geometry.

I confess that I was elated and disappointed with Lakoff's notion of 'embodiment' as if he hijacked the term and made it so general to promote an overall description of all human thinking, thus limiting analysis of mathematical thinking to a single category of embodiment.

On one occasion, Lakoff (1987, pp. 12–13.) made a distinction between conceptual and functional embodiment that could prove helpful, though the complex linguistic descriptions given by him are descriptive rather than definitional, so I cannot be sure that how I read his definitions is what he intended. If 'conceptual embodiment' is used to refer to embodied conceptions in terms of mental objects and 'functional embodiment' as (possibly unconscious) ways of functioning as human beings then we can see 'conceptual embodiment' in terms of the first world of mathematics, and 'functional embodiment' occurring in all worlds in general and the proceptual world in particular. Furthermore, the growth of this conceptual-embodied world resonates with the geometric development of van Hiele (1986), where objects are first perceived as whole gestalts, then broadly described, with language growing more sophisticated so that descriptions became definitions suitable for deduction and proof.

Thinkable concepts

In essence the simple idea essential to the development of powerful thinking in mathematics is the idea of compression of knowledge into thinkable concepts. Eddie Gray and I began to use the term 'thinkable concept' (Tall & Gray, 2006) to refer to some phenomenon that has been named so that we can talk and think about it, such as number, food, warmth, rain, mountain, triangle, brother, fear, black, love, mathematics, category theory. We soon realised that the phrase 'thinkable concept' is a tautology, for a named phenomenon is a concept and is therefore thinkable. Nevertheless, we found it useful to use this double-barrelled description because the word 'concept' is open to many different meanings. Concepts begin to take form before they are named. When they are in the process of being constructed, properties and connections are perceived before the name is given, but it is only when a phenomenon is verbalised with a name that we begin to acquire power over it to enable us to think about it in a serious analytic way.

As an example we considered the concept of 'procept' itself. We had ideas about the flexibility of relationships between process and concept long before we formulated the term 'procept', but having given it a name, we were able to look at different kinds of procept that occur in mathematics (Tall, Gray, et al, 2001).

The search for a simple theory of mathematical development began to become clearer: what is necessary is to identify important phenomena, name them and focus on their properties to enrich them as thinkable concepts that can be related together in increasingly sophisticated theories.

Set-befores and met-befores

Thinking about the notion of metaphor, so essential in the theories of Lakoff (1987) and Sfard (1991), I felt in my bones that this term was so aesthetic as to not truly represent the everyday thinking of children. As a play-on-words, I mused on the possible spelling 'met-afore' meaning thinking about something that one had met before, then this transmuted into 'met-before' and a verbal distinction between met**A**phor and met**B**efore. I shared my joke with others and found them smiling and using the term 'met-before' in their own discussion of the ideas. In particular, when I discussed the ideas informally with teachers I found they took to them with enthusiasm and some began to use the language of met-before to talk to the children they taught about the differences between their earlier experiences and the new ideas they were meeting at the time. On one occasion I had a short talk with a young teacher about how the children's 'met-befores' affect their current learning, looking at examples from arithmetic. A year later I met him again and he told me excitedly that he had used the idea every day since, talking to young children about their 'met-befores' and sharing these insights with his colleagues. Here was an idea that could be transferred effectively from educational theory to classroom practice.

I therefore defined the term *met-before* to refer to some aspect of thought that had been met on an earlier occasion, and is now part of the individual's current way of thinking, affecting the way in which they view new ideas.

The form of our cognitive structure is first laid down in our genes, but develops throughout our life by the connections we make as met-befores. To allow for the two aspects of genetic nature and social nurture, I use the term *set-before* for a cognitive structure that is 'set before' our birth. Some mathematical conceptions are probably set-before, such as the notion of 'numerosity' to distinguish between the size of small sets containing 1, 2 or possibly 3 objects. However, much of our mathematical knowledge is built on 'met-befores'.

Met-befores can have both helpful and confusing effects in new learning. Knowing that 2 and 2 makes 4 in whole number arithmetic remains valuable in dealing with fractions, negatives and real numbers. But sensing that 'adding two numbers always gives a bigger result' or 'taking something away always leaves less' can cause great confusion when negative numbers are encountered.

Thus a study of students' met-befores can help the teacher gain insight into the sources of some of their students' difficulties in coping with new ideas.

Complication, Complexity and Simplicity

Working with doctoral student Hatice Akkoç, (Akkoç & Tall, 2002) we were able to take a fundamental step forward in drawing together these ideas. The main idea is that simplicity eventually arises from having rich thinkable concepts to operate with. Akkoç distinguished clearly between the meaning of the terms complication and complexity. When a new situation is encountered, it may very well have many aspects that make it appear extremely complicated. What matters, with appropriate focus of attention, is to organise and compress these ideas into rich thinkable concepts to shift from the original complication to simple complexity.

This distinction featured in the doctoral thesis of Bayazit (2006) who reported two very different approaches teaching the concept of function by two teachers.

Teacher Ahmet focused on the *simplicity* of the function concept: it is a relationship between two sets A and B that satisfies one basic property: it relates each element in A to one element in B, *and that's all!* Everything he did related to this idea, including a discussion that a function need not be one-one or onto, or introducing the vertical line test as a pictorial translation of the basic property, or the possibility that the function had an inverse, but only when it was both one-one and onto.

Teacher Burak was also aware of the difficulties of the concept, but he responded by teaching the students what he considered that they must do to answer the questions on the examination.

For example, in considering the notion of inverse function, Ahmet looked at examples where the basic idea could be satisfied in the reverse direction between the related pairs, bringing out the need for a function to be one-one and onto for it to have an inverse. Burak approached the inverse function by working an example, looking for the inverse a linear transformation such as $y-3=2x$, explaining that to express x in terms of y requires subtracting 3 from both sides to get $y-3=2x$, dividing both sides by 2 and swapping sides to get $x=(y-3)/2$, then exchanging x and y to get the inverse as $y=(x-3)/2$.

Ahmet focuses first on the underlying simple meaning then develops the techniques to cope, Burak goes straight to focusing on the procedural actions required to pass the exam. Ahmet's students proved to be significantly more successful than those of Burak.

Procedural and conceptual learning

The distinction between procedural and conceptual learning has long been a topic of discussion in mathematics education. The theoretical framework given here suggests a key distinction between procedural and conceptual thinking is the idea of compressing phenomena into thinkable concepts to enable the individual to make links between them. Students who learn to routinize procedures as actions in time may learn to perform those procedures in routine situations but may fail when they are given a question in an unfamiliar context.

An example of this occurred in the research of Rosana Nogueira (Noguiera & Tall, 2006) in classes where the teachers felt that their students were struggling in linear equations and addressed the more sophisticated concept of quadratic equations by teaching them how to use the quadratic formula. The students had difficulty using the formula, but the difficulties were compounded when an equation was not given in the form $ax^2+bx+c=0$ for numerical values of *a*, *b*, *c*. For example, 77 students were given the equation $(x-3)(x-2)=0$ and told "John says the solutions are 2, 3; is he correct, and why?" Only three substituted the values and correctly checked the arithmetic, sixteen others attempted to solve the equation, of whom only six correctly multiplied out the brackets and only five of them attempted to use the quadratic formula. *None* of them said that if one bracket was zero then the product was zero. They had been taught a specific technique and had no flexibility to begin to know what to do when a problem did not fit the pattern.

International repercussions

The short-term success and long-term failure of procedural teaching occurs widely around the world (Gray & Tall, 2006).

As governments press for higher attainment in mathematics, teachers are being set 'wish lists' of desirable levels of performance in examinations. In England, despite the force of law being used to impress the National Curriculum in classrooms, pupils are not achieving the levels desired. The theoretical framework suggested in this paper explains why the failure continues.

Growing sophistication requires the compression of phenomena into thinkable concepts. Procedural learning enables some students to rote-learn a particular algorithm where success means being able to carry out the algorithm routinely without error. Such procedures occur in time and the mental processes become subconscious to help make them work. As such they enable students to *do* routines but not necessarily to be able to *think* about them. Without thinkable concepts, there can be no relationships between them and no conceptual understanding.

In arithmetic, the difference between those who flexibly compress arithmetic procedures into number concepts and those who remain with increasingly complicated step-by-step procedures was called *the proceptual divide* (Gray & Tall, 1994). One may suspect that what is happening more broadly is that students who fail to turn complicated situations into more simple structures by compressing knowledge into thinkable concepts will find it extremely difficult to build a coherent connected mental structure to put significant ideas together. Meanwhile those who successfully compress knowledge into thinkable concepts will often have a generative structure in mind that will construct new relationships: I know 3+5 is 8, so I also know 23+5 is 28 and 300+500 is 800. The student with thinkable concepts can use them to build new knowledge.

Long-term learning

A major problem in the teaching of mathematics is the success of long-term learning. Conceptions at one stage can sow met-befores that cause serious difficulties later on. Noting that multiplying numbers always gives a bigger result is fine for counting numbers but causes confusion with fractions. Treating algebraic symbols as objects in 'fruit salad algebra' where 2*a*+3*b*+4*a* is 2 apples, 3 bananas and 4 apples can easily lead to putting all the apples together to get

$6a+3b$. But this interpretation fails to give a meaning to $6a - 3b$. (How can you have 6 apples minus 3 bananas?) Likewise, treating an equation as a physical balance is fine for simple equations but can cause huge conceptual difficulties when negative terms are involved. Thus using physical ideas to embody mathematical concepts is fine at one level but can produce serious difficulties later.

We therefore need to have greater understanding of the developing relationship between embodiment and symbolism and at a later stage, how these relate to subsequent formalism.

In the calculus, I have shown that embodiment can provide a powerful foundation for later symbolic and formal development. But in linear algebra, conceptual embodiment works in well two and three dimensions, but it does not extend naturally to higher dimensions. On the other hand, the symbolism of linear equations in two and three unknowns extends easily to n unknowns.

Embodiment provides fundamental human support for many elementary mathematical concepts but some embodied met-befores may often have aspects that do not hold in more sophisticated systems.

We therefore cannot solve the difficulties of long-term mathematical learning by always building from embodiment to symbolism. Those who succeed in mathematics long-term move over and use symbolism itself in more sophisticated ways (Gray, Pitta, Pinto & Tall, 1999).

This increased sophistication invariably comes from compressing subtle phenomena into thinkable concepts represented symbolically.

To make increasingly sophisticated mathematics simple, this viewpoint makes it evident that we need teachers to act as mentors so that when they have appropriate learning experiences, the focus is turned to essential ideas that lead to simple but powerful concepts. This will not be achieved simply by teaching students to practice specific algorithms to pass specific tests. This may work for those who have a mental structure to set these procedures in context and have the fortune to be able to focus on appropriate thinkable concepts. It will limit the possible growth of those who practice procedures for specific purposes without compressing them into thinkable concepts.

In order to develop mathematical thinking that is both powerful and simple, we need to know much more about the met-befores that students bring to new tasks and how we may mentor them to use their knowledge to produce thinkable concepts that may be connected together in powerful ways.

This needs to be set in the broader context of the natural growth of mathematics in the individual which begins with human perception and action in a conceptually-embodied world of operation, shifts attention to a world of proceptual symbolism by compressing processes occurring in time into thinkable concepts, and, for those who go on to study mathematics at an advanced level, to build mathematical theories in an axiomatic-formal world based on set-theoretic concept definitions and formal deduction.

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