THE CONCEPT OF EQUATIONS: WHAT HAVE STUDENTS MET BEFORE?

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In this paper, we present results from a research study designed in collaboration with teachers to investigate how Brazilian 15-16-year old students interpret the concept of equation and its solution. Data from a questionnaire, an equation-solving exercise and interviews with selected students are reported and analysed in terms of how the students are affected by their earlier experiences in arithmetic and algebra.

Introduction

Teaching and learning algebra has long been seen as a source of difficulty. The situation in Brazil reveals problems similar to those in the literature. Freitas (2002) categorised student errors solving linear equations in terms of misunderstanding algebraic rules. Our purpose here is to use a theory of long-term mathematical growth involving embodiment, symbolism and proof (Tall, 2004) to seek deeper reasons for these phenomena.

Literature review

Kieran (1981) gave evidence that the equals symbol is often seen as a "do something symbol" rather than a sign to represent equivalence between the two sides of an equation: '2+3=5' means 'add 2 and 3 to get 5' and an equation such as $4x-1=7$ is seen as an operation to find a number which when multiplied by 4 and 1 is subtracted, gives 7. Filloy & Rojano (1989) emphasised the difficulty when the unknown appears on both sides of the equation, by naming it 'the didactic cut' between arithmetic and algebra.

As process-object encapsulation theories appeared, Linchevski and Sfard (1991) suggested that a major problem is that students view algebraic expressions as procedures of evaluation rather than as mental entities that can be manipulated.

Tall & Thomas (2001) distinguished three levels of algebra: *evaluation algebra* (the evaluation of algebraic expressions such as 4*A1+3 as in spreadsheets or in the initial stages of learning algebra), *manipulation algebra* (where algebraic expressions are manipulated to solve equations), and *axiomatic algebra* (where algebraic systems such as vector spaces or systems of linear equations are handled by definition and formal proof).

[1]The first author was supported by the CAPES Foundation, Ministry of Education of Brazil.

The story emerging from these theories tells how operations in arithmetic are expressed as generalized expressions of evaluation, which in turn become mental entities for manipulation, later to be translated into formal terms.

Reflecting on this development, Tall (2004a) theorized that this development is a life-long journey through three distinct worlds of mathematics:

> A *conceptual-embodied world* of perception in which sense making becomes increasingly sophisticated by verbalizing properties of objects through description, definition, thought experiment and (Euclidean) proof.
>
> A *proceptual-symbolic world* of action-schemas, such as counting, that are symbolized and routinized as procedures, where they may remain to give procedural thinking or be seen as an overall process symbolized as an entity, such as number, whose symbol is used flexibly as process or concept (procept).
>
> A *formal-axiomatic world* of formal definition and mathematical proof.

These three worlds will be named 'embodied', 'symbolic' and 'formal' in the remainder of this paper. It is theorized that the embodied and symbolic worlds develop in parallel, but operate in different ways. Human meaning begins from coherent embodiment of connections between concepts. Action-schemas, however, can be routinized and learnt by rote. We hypothesise that symbolic meaning comes from two distinct sources: from relationships with meaningful embodiment and from the internal coherence of the symbolism.

Watson (2002) revealed a parallel between compression of knowledge in the embodied and symbolic worlds, which arises through a shift of attention from the steps of an action-schema to its overall *effect*. For instance, $2x+4$ is a different sequence of actions (double the value and add 4) from $2(x+2)$ (double the result of adding two to the value), but has the same underlying effect. Such a viewpoint gives a practical way of conceptualizing the shift from procedure of evaluation to flexible algebraic manipulation.

One further element in long-term learning is the effect of prior knowledge, based on structures 'set-before' in our genes or 'met-before' in our experience. Tall (2004a) termed a current structure resulting from earlier experiences a *met-before*. Some met-befores—such as those in a well-designed curriculum—can be a positive foundation for successful development, others, such as epistemological obstacles studied by the French School (Brousseau, 1997), can cause conflict in a new context and have a negative effect on learning. The theory of met-befores therefore represents these positive and negative aspects in a single theory. It is our purpose in this paper to use this framework to analyse the conceptions developed by students studying algebra.

RESEARCH METHODS AND DATA COLLECTION

The first author worked in collaboration with five secondary Brazilian teachers to discuss issues concerning equations and to design instruments for collecting data. The research involved 77 students in three groups of 15-16 year-old high school students: 26 first year and 32 second year from a public school in Guarulhos/SP, 19 second year students from a private school in São Paulo/SP. Three instruments were designed by teachers and researchers in collaboration and a further test was inserted by the researcher to clarify issues arising during analysis of data, as follows:

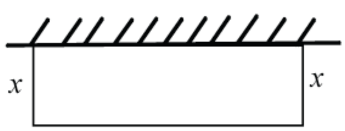
A brainstorming session to categorize words used in algebra starting with EQUATION, conducted by the teacher in class, observed by the researchers.

A written questionnaire concerning the notion of equation, its solution and its use in solving problems, administered in class by the teachers (table 1).

A written equation-solving task, added by the researcher after reviewing the data from written questionnaire, administered by the teachers (table 2).

Interviews with selected students conducted by the researcher in the presence of an observer, based on aspects arising from the earlier data.

1. What is an equation?
2. What is an equation for?
3. Give an example of an equation.
4. What does the solution of an equation mean?
5. Solve the equation $t^2 - 2t = 0$ for real numbers, and explain the steps of your solution.
6. Solve the equation $(y-3)\cdot(y-2)=0$ for real numbers, and explain the steps of your solution.
7. Ulisses likes to grow flowers. In his backyard there is an available area, near the wall, so he wishes to build a rectangular flower bed and, to fence it, he intends to use 40 m of fence he has got. He has not decided yet the size of it, so he made the following drawing:



What are the flower-bed side measures to get an area of 200 m^2?

8. To solve the equation $(x-3)\cdot(x-2)=0$ for real numbers, John answered in one line: "$x = 3$ or $x = 2$". Is the answer correct? Analyse and comment on it.

Table 1: Questionnaire

Solve:	$3x - 1 = 3 + x$	$5t - 3 = 8$	$2m = 4m$	
	$m^2 = 9$	$3l^2 - l = 0$	$a^2 - 2a - 3 = 0$	$r^2 - r = 2$

Table 2: Equation-solving task

ANALYSIS OF DATA

Here we analyse the data collected in the Questionnaire, Equation Solving Task and the Interviews, supplemented by the initial brainstorming task.

Questionnaire

Students' most frequent responses to question 1 (What is an equation?) were of the form: *"It is a mathematical calculation"* or *"It is a calculation you do to find the solution, to find x"*. These suggest that most students seem to see an equation either as an arithmetic calculation, or as a calculation in which it is necessary to find the value of *x*, meaning the unknown. (Similar results were found by Dreyfus and Hoch, 2004.) In addition, while some students (21 out of 77) referred to the unknown as an important feature of equations, *no student explicitly mentioned the equals sign*. This is consistent with the likelihood that students do not see the equals sign as an integral part of an equation, but as "sign to do something" in the sense of Kieran (1981). This "action to be performed" is almost certainly a met-before from the students' previous experience of arithmetic, where the equals sign indicates that an arithmetic calculation is to be performed.

The responses to question 2 (What is an equation for?) relate mainly to mathematical contexts such as *"to find the unknown value"*, rather than real-life problem solving (*"Not much in daily life, but may be useful to people who like maths"*).

Question 3 (Give an example of an equation) had 47 valid responses such as '$3x+x+9=21$', or '$x^2-2x+3=0$' and 23 other responses that included '$30-20+15\cdot 5-(-5+1)=$' or '$15x+\{5+(+3x+7x)+5\}+(+3)=$'. The latter reveal an equation as a numeric process to be calculated or an algebraic expression to be evaluated to find a value to give the right-hand side. Forty-six students actually *solve* the equations or evaluate the expressions given.

In question 4, (What does the solution of an equation mean?), students responded in terms of "*The solution to a mathematics problem*", or "*The unknown value"*. Some responses involved *"The calculation of angles and measures*", where equations expressed known facts such as two angles adding to a right angle. Every case involved an expression to be evaluated. No student related equations to a real-world problems.

Question 5, 6 required the solutions of $t^2-2t=0$ and $(y-3)\cdot(y-2)=0$. Analysis of the solutions suggested the need to study a wider range of problems, which are analysed in the equation-solving task, discussed below.

The practical "fence" problem in question 7 produced a few responses using the numbers given in an equation such as $40+40+x+x=200$, but *no one* gave an equation in the form $x\cdot(40-x)=200$ that would lead to the required solution. One student only gave a correct numerical solution with sides 10m and 20m, writing the answer straight down, probably because he noticed that these numerical values satisfied the condition. Thus *none* of the students symbolised the physical problem as an algebraic equation.

In question 8, where students were asked to analyse and comment on John's one-line solution of the equation $(x-3)\cdot(x-2)=0$, three students checked that the solution was correct by explicitly substituting the values for x and checking the arithmetic. Apart from this, the most common response from sixteen students was to try to solve the equation and compare results. Only 3 of these were correct. Solution methods varied, with 14 students beginning by multiplying out the brackets, performed correctly by only 6. Five students attempted to use the quadratic formula. No student mentioned the fact that 'if the product of two numbers is zero, then one of them must be zero.

The two methods illustrated were either the remembered quadratic formula, or the use of arithmetic method of checking an equation by carrying out a calculation. The former is procedural; the latter need only treat the equation as a calculation process giving a numerical result. Both are clearly met-befores: the use of the formula to solve the equation, and the experience of checking a calculation to verify that it is correct. Neither goes beyond the procedural calculation of evaluation algebra to move to the flexible use of algebraic expressions as process or concept characteristic of manipulation algebra (in the sense of Tall & Thomas, 2001).

Equation-solving task

This task, added by the researcher, to supplement questions 5 and 6 above began with three linear equations:

$$3x-1=x+3,\ 5t-3=8 \text{ and } 2m=4m.$$

The most used and successful met-befores to solve them were the rules of "change side change sign", transforming $3x-1=x+3$ into $3x-x=3+1$ and, on reaching an equation of the form $2x=4$, to "move the coefficient of x to the other side of the equals sign and divide by it", in this case giving $x=4/2$. Such solutions involve a movement of the symbols, together with an extra technical element (such as changing the sign) to give the correct result. As such they could easily be rote-learnt as meaningless embodied actions, shifting symbols and doing something else at the same time. Such operations may be fragile and applied inappropriately, for instance, students may change sides without changing signs, or change the sign of the coefficient of x as they shift it to the other side, or change $ax=b$ erroneously to $x=a/b$. These errors were also noted by Freitas (2002) and theorised by Linchevski and Sfard (1991) as 'pseudo-conceptual entities'.

Other errors in interpreting linear equations related to the equals sign. Several students interpreted $2m=4m$ as a sum, giving $6m$. Perhaps students needed to "do something", so they perform an operation. Some students also need to find the value of m and $2m=4m$ was turned into $6m$, then $m=-6$.

In the case of quadratic equations, four new equations were given:

$$m^2=9,\ 3l^2-l=0,\ a^2-2a-3=0,\ r^2-r=2$$

and analysed together with

$$t^2-2t=0,\ (y-3)\cdot(y-2)=0$$

from the original questionnaire.

The first equation $m^2=9$ was often seen as a problem to find the square root knowing the square is 9, so the solution is $m=\sqrt{9}$ and so $m=3$. The other equations were approached either by testing numeric values to see if they were solutions or by using the quadratic formula. None of the students used the property that if the product of two factors is zero, then one of the must be zero, even in the case of $(y-3)\cdot(y-2)=0$. In interview, students did not seem to believe it. The only met-befores seem to be numeric 'guess and test' to seek solutions, or an attempt to use the quadratic formula. The students are therefore at a procedural level relying on a single procedure, without the appreciation of several procedures to give alternative approaches and certainly not approaching a flexible level of moving between expressions as processes to evaluate and concepts to be manipulated. They respond at a fragile procedural level rather than proceptual.

Interviews

Fifteen students were selected for interview to give a spectrum of levels of response, including mainly those who used non-standard algebraic manipulations. They were asked to talk about their responses. The equals sign (which was not mentioned in the written responses) arose in two responses, however, it was still regarded as a sign to give a result. Calculations were often made in a fragile way that led to error, for instance, some students said that t^2 is equal $2t$ because it is $t\cdot t$, which is the same as $2t$, so $t^2-2t=0$. When a fuller explanation of their understanding of equation was requested, the responses again indicate mainly a focus on the calculation involved and on the need to find the value of the unknown.

Students often referred to the use of rules to solve equations. None mentioned the idea of performing the same operation in both sides (just as none of them used this technique in the equation-solving exercise). The rules given involved operating on the symbols as "*a rule that must be used to solve an equation, otherwise the right solution will not be found*". The language used often seemed to have an embodied meaning relating to actions performed on the symbols in the equation such as *"pick this number and put it at the other side of the equals sign"*, *"I take off the brackets"*, or *"the power two passes to the other side as a square root"*. These actions have underlying embodied foundations that relate not to real-world activities, but to moving symbols around, with a mysterious twist to make things right. It seems as if students are more comfortable trying to shift symbols rather than to perform the same operation to both sides.

Rules that they have met before in arithmetic were sometimes misapplied. For instance, when solving $t^2 - 2t = 0$, a student wrote $1t^2 - 2t^1 = 0$ and performed the subtraction as $-1t = 0$, because *"you have to subtract powers as well"* (subtracting the constants 2 from 1 and the exponent 1 from 2).

Another student solving the equation $3l^2 - l = 0$ explained, *"I leave 3 aside, pick up 2 (the exponent), then make 2l and put 3 and l together"*, reaching $2l - 3l = 0$. To subtract these terms, she said, *"plus with minus is minus; different signs, add numbers"* and wrote down $-5t = 0$.

Discussion

The data collected shows that these students' conceptions of equations and ways of solution are fundamentally based on arithmetic met-befores, where the equals sign is conceived as "something to do" to "get the solution" and on what they recall from previous experience in algebra. Their main solution method is the quadratic formula, which could give a correct solution whether or not it is fully understood, but was often fragile and applied incorrectly.

There was no aspect of embodiment of real-world contexts in their conceptions of equations. There was no mention of equivalence between the two sides of the equation, nor of applying the same operation to both sides to simplify the equation in the process of moving towards a solution.

Discussion with the teachers revealed that there was a widespread belief that algebra was difficult and so there was a strong focus on the quadratic formula because it was seen as the most efficient way of getting a solution with less possibility of students making mistakes. This focus on a single procedure seems to have the effect of impeding the development of any flexibility to give meaning to equations and their solution. There is no possibility of a shift from procedural methods of evaluation to more flexible operations of manipulation algebra.

We share the widespread belief that the teaching of algebra in general and equations in particular should be based on experiences that give meaning. Embodiment gives human meaning, but does not feature in the experiences of these students. Symbolic meaning arising from the coherent relationships between different methods of solution is also unlikely. Instead the students have limited procedural knowledge that is fragile and prone to error.

References

Brousseau G. (1997). *Theory of Didactical Situations in Mathematic*s, Kluwer, Netherlands.

Dreyfus, T. & Hoch, M. (2004). Equations – a structural approach, *Proceedings of 28th International Conference for the Psychology of Mathematics Education*, Bergen, Norway, vol. 1, p. 152-155.

Filloy, E. and Rojano, T. (1989), Solving equations, the transition from arithmetic to algebra, *For the Learning of Mathematics*, 9(2)**,** 19-25.

Freitas, M. A. de (2002). *Equação do primeiro grau: métodos de resolução e análise de erros no ensino médio*. Master's Dissertation. São Paulo: PUC-SP.

Kieran, C. (1981). *Concepts associated with the equality symbol*. Educational Studies in Mathematics, 12, 317-326.

Linchevski, L. & Sfard, A. (1991). Rules without reasons as processes without objects – The case of equations and inequalities. *Proceedings of 15th International Conference for the Psychology of Mathematics Education*, Assisi: Italy. 2, 317– 324.

Tall, D. O. (2004). Thinking through three worlds of mathematics, *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Bergen, Norway, 4, 281–288.

Tall, D. O. & Thomas, M. O. J. (2001). The long-term cognitive development of symbolic algebra, *International Congress of Mathematical Instruction (ICMI) Working Group Proceedings - The Future of the Teaching and Learning of Algebra*, Melbourne, 2, 590-597.

Watson, A. (2002). Embodied action, effect, and symbol in mathematical growth. *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education*, Norwich, UK, 4, 369–376.