TWO STUDENTS: WHY DOES ONE SUCCEED AND THE OTHER FAIL?

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This article considers two students who both pass a preliminary course in college algebra at the same level; one proceeds satisfactorily in the next course whilst the other finds it impossibly difficult. We analyse the responses of the students in semi-structured interviews on topics from the first course to seek reasons for this difference in later performance. We propose a theory that explains the difference in terms of the nature of the students' mental connections and the richness of their mental concepts as revealed in their interviews. We suggest that this theory can be used to analyse students' thinking in terms of the cognitive units they have to use and reflect upon in their mental structures which prove to be rich and well-connected in those who succeed, but limited and poorly connected in those who eventually fail.

INTRODUCTION

Attainment is increasingly being measured in terms of criterion based tests, to decide whether students have the required pre-requisites to succeed in a given course. Invariably end-of-course tests focus on what students have learned in order to pass an exam, rather than whether they are likely to succeed in a subsequent course. In this article we consider two students, both with a grade "B" in their intermediate algebra course, yet who fare very differently in the following college algebra (pre-calculus) course. We analyse interviews on material from their preliminary course to see if there is any evidence of higher level thinking that will explain reasons for their different performances on the subsequent course. We weave the empirical evidence with a theory of cognitive development relating to existing theories that give insight into the different performance of individuals in a variety of different contexts. This theory is founded on Barnard and Tall's (1997) idea of 'cognitive units' (the elements of thought that an individual manipulates in his or her conscious focus of attention), the richness of interior structure of these units (in the sense of Skemp, 1969) and a range of other theories from the analysis of Krutetskii (1976) through the notions of procedural and conceptual knowledge (Hiebert and Lefevre, 1986, Hiebert and Carpenter, 1992), and the nature of cognitive structure and conceptual linkages.

We hypothesize that the student who is ready to proceed to the next course has fundamental differences in his or her knowledge structure from the student who is not ready, and that these fundamental differences are not necessarily evident from performance on exams in the prerequisite course.

The college intermediate algebra course in this study is for all the students enrolled in it a preliminary course for entry to a college algebra course (pre-calculus). These community college students are adults who may have failed at an earlier stage, have often been traumatised by their previous experiences, and are committed to attempt to overcome their previous difficulties. Nevertheless, we hypothesise that our general theory of cognitive structure is of value in a far wider domain of mathematical and more general cognitive development. It concerns the difference between those individuals who reflect on their knowledge in ways that create rich, mentally manipulable concepts and productive linkages that are powerful, and those who merely acquire knowledge in bits and pieces without organizing them into a connected structure. In this sense, we claim that the theory has a wider general applicability than simply the performance of students in remedial algebra courses.

BACKGROUND

> "I'm telling you, I'm not good at math…I don't really know. I hate math. I have a bad attitude with math."
>
> "Math is my hardest subject, all through high school and everything…"
>
> "I'm not too good in math, I'll tell you that…I've just never been real good. I've never gotten hold of basics in high school too good"

All of these comments were made by community college students during interviews covering intermediate algebra topics. Each of the students struggled in intermediate algebra, was ultimately successful (some with grades as high as "B"), but then had serious difficulty in the next course, the one required for university general studies. Colleagues of the author frequently describe the students as "stupid", "lazy" and "unmotivated", with "no carry-over from one course to the next".

Leaving aside the pejorative nature of these comments, we should ask a deeper question: why it is that some students who have done "well" in intermediate algebra have extreme difficulty in pre-calculus, while others—who have done equally "well"—find it routine?

Our search for an explanation has drawn us to an investigation of student cognitive structures—what do students actually *do* when they are working problems, and *how* do they do it? What is it about their cognitive structure, as evidenced by their problem-solving processes, that allows one to succeed while the other does not? Does the successful student merely have more available procedures, or is there a fundamental difference in his or her cognitive activities?

We use semi-structured interviews to gain some insight into why some are ready to progress, while others are not. A major focus in such an interview would be to look at not only *what* the student does, but *how* and *why*. However, before we address the specifics of such a quest, it is necessary to review a number of major ideas already well-established in the literature.

RELEVANT RESEARCH LITERATURE

Hiebert and Lefevre (1986) distinguish between procedural knowledge following step-by-step instructions and conceptual knowledge that is "a connected web of knowledge, or network". Hiebert and Carpenter (1992) further discuss these ideas in the following terms:

> We believe it is useful to think about the networks in terms of two metaphors … structured like vertical hierarchies or … like webs. When networks are structured like hierarchies, some representations subsume other representations, representations fit as details underneath or within more general representations. Generalisations are examples of overarching or umbrella representations, whereas special cases are examples of details. In the second metaphor a network may be structured like a spider's web. The junctures, or nodes, can be thought of as the pieces or represented information, and the threads between them as the connections or relationships. Hiebert & Carpenter (1992, p. 67)

While these ideas have been part of mathematics education theory for many years, they have been used primarily as general philosophy rather than specifics about cognitive structure. We plan to use the distinction between procedural and conceptual knowledge and extend the ideas to an analysis of student cognitive structure in college algebra courses.

In our analysis a particular idea proves to be helpful. Barnard and Tall (1997) introduced the idea of a "cognitive unit" as a "piece of cognitive structure that can be held in the focus of attention all at one time", which may be considered as nodes in the web metaphor of Hiebert and Carpenter. However, additional insight is possible. While these nodes may be viewed individually as cognitive units, a single node may be unpacked to reveal an internal structure which is again a web of connected cognitive units. This shift from an individual node to a web of nodes, and back again, was described by Skemp (1979) as a "varifocal learning theory", in which the nodes of webs are subtly connected conceptual schemas. It is important to be able to "zoom in" and "zoom out", to compress a collection of related ideas, each of which is a cognitive unit, into a single cognitive unit. According to Barnard (1999), the entire entity can—if necessary—be conceived of as a unit "small enough to be held in the focus of attention all at one time". This describes a form of mental compression, in which you "can file…away, recall…quickly and completely…, and use…as one step in some other mental process" a concept which can be utilised as a single entity or unpacked as a whole schema of ideas (Thurston, 1990). This is discussed further in educational terms by Gray & Tall, (1994) as what enables the student to use a whole complex of ideas from one context as a foundational unit in a subsequent context.

Problem 1

For example, consider the equation $y = mx + b$. One can consider it a concept as a network of internal ideas: m is the slope, b is the y-intercept; any linear equation can be represented by substituting appropriate numbers in for m and b; one can draw the graph if there are two points available, or one point and the slope, etc. A student may thus see $y = mx + b$ as a *single entity* rich with properties and links easily brought into the focus of attention.

However, consider the following:

The equation $y = 2x + 3$,
The equation $y - 5 = 2(x - 1)$,
The graph of $y = 2x + 3$,
The line through $(0,3)$ with slope 2,
The line through the points $(1,5)$ and $(3,9)$,

each of which may be considered individually as a cognitive unit which can be linked with any or all of the rest as representing the same underlying concept—the same straight line or equivalent linear relationship between x and y. We could express this diagrammatically as five separate nodes with appropriate connections as in Figure 1.

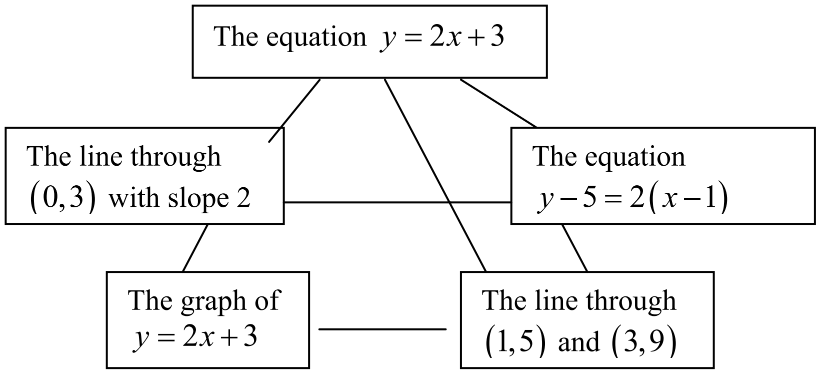


Figure 1. Relationships

To seek insight we will explore the cognitive structure demonstrated in interviews with two students working problems involving graphs of linear functions from the Intermediate Algebra course. They took place when the students were enrolled in the pre-calculus course, so would already show longer-term understanding rather than the knowledge the students may have learned at the time of an exam.

INTERVIEWS

We asked the students to answer the question, "What is the slope of the line on the graph here?" (shown in figure 2)

Natasha had a flexible solution.

Natasha: What is the slope on the graph here? It goes down…do you want me to solve it using the points?

Researcher: I don't care how you solve it.

Natasha: Because you can just really look at it and tell…it's a negative 2 slope.

Researcher: So you did it by counting squares.

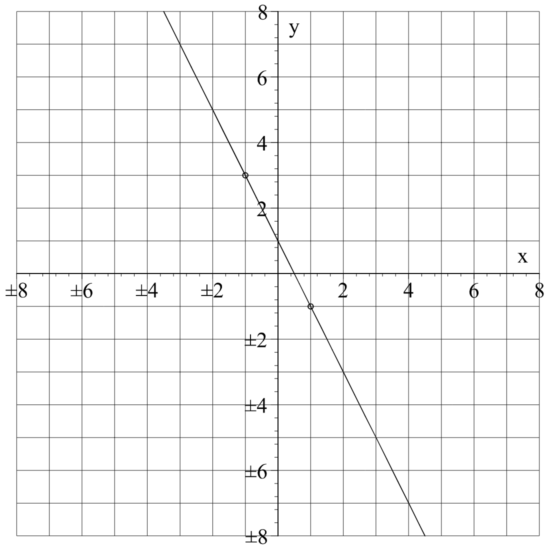


Figure 2. The Graph

Natasha: Yeah, just by looking at it. I mean, if the squares weren't there, I could do it by taking the two points and finding the slope like I've done…

Kathy's solution process (in figure 3) was less straightforward:

Kathy: One, negative one, and negative one, three. …shoot.

Researcher: *(pointing at the left and right sides of the equation.)* So how did you get from here to here?

Kathy: By subtracting…you've got me all bamboozled. My handy-dandy calculator, I rely on that. That would be four.

Researcher: Four over two? Does that look right?

Kathy: I think it does, but I could be wrong.

Researcher: Well, you made a mistake, not a very big one…

Kathy: One I can't catch, though. It'll make me feel really stupid later.

Researcher: *(Encouraging.)* I'm just trying to protect you from making a dumb mistake on your exam! … The denominator, what's minus one minus one?

Kathy: Zero.

Researcher: No, it's not.

Kathy: Two, is that it?

Researcher: It's negative two, isn't it?

Kathy: Yeah.

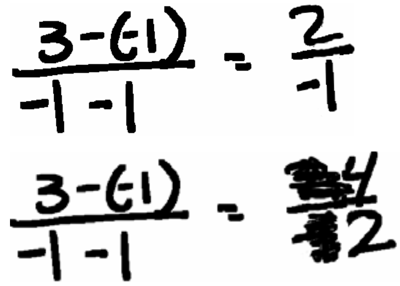


Figure 3. Kathy's solution

Kathy had great difficulty with negatives and fractions; she froze when asked to manipulate them without a calculator. When she had a calculator, she used it to do her arithmetic and only then was able to cope with simple calculations. Her solution process is illustrated in the figure 4.

Analysis

Natasha had flexibility; she could either read the slope off the graph, if that was easy, or she could find the slope by identifying two points and using the slope formula. This flexibility saved her a lot of effort on this particular problem. Kathy only evoked a formula for the slope, which required two points. She was unable to link the slope to the change in y over the change in x from the graph. This problem was, for her, consequently much more complicated, ultimately involving computations with negatives, which caused her much difficulty. She was also insecure when she finished the problem; she always checked with an authority figure—the instructor, the

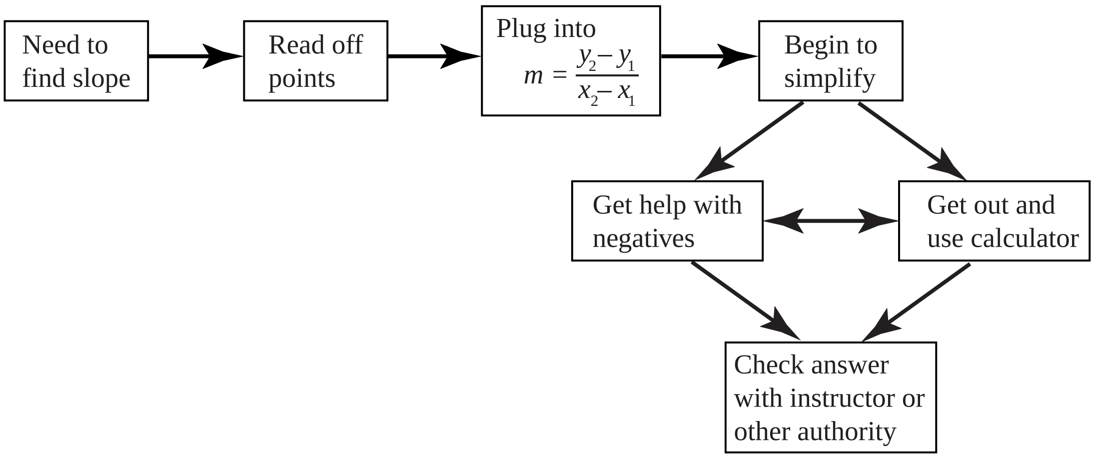
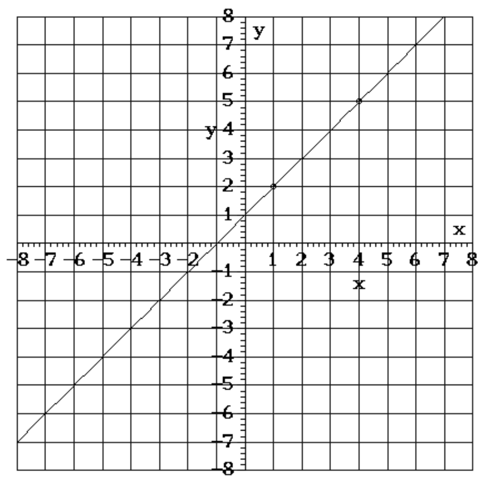


Figure 4. Kathy's Solution Process

interviewer, or the answers in the back of the book—for assurance that she had found the correct answer.

Problem 2

In a second problem to write the equation of a straight-line graph (figure 5), Natasha's solution was not the most efficient. She was able to read off the slope, but then used the point-slope formula to write the equation, whereas it would have been more efficient to simply use the *y*-intercept from the graph. Nevertheless, she demonstrated flexibility in finding the slope, and again in checking to ensure that the point she was using in the formula was correct. An outline of her strategy is given in figure 6.



Write the equation of the line.

Figure 5. Problem 2

Meanwhile, Kathy approached the problem by reading off two points (1,2), (4,5), using the formula for the slope to find it is 1, and then the formula for a line through (1,2) with slope 1. She too made an error, but made no effort to correct it until she was prompted by the researcher.

Problem 3

A third problem to find the equation of a straight-line graph had the same format of problem 2, but with the negative slope from problem 1. Natasha again showed flexibility; she read the slope m=–1 by inspection and then read off the *y*-intercept b numerically to give the equation in the form $y = -1x + 1$.

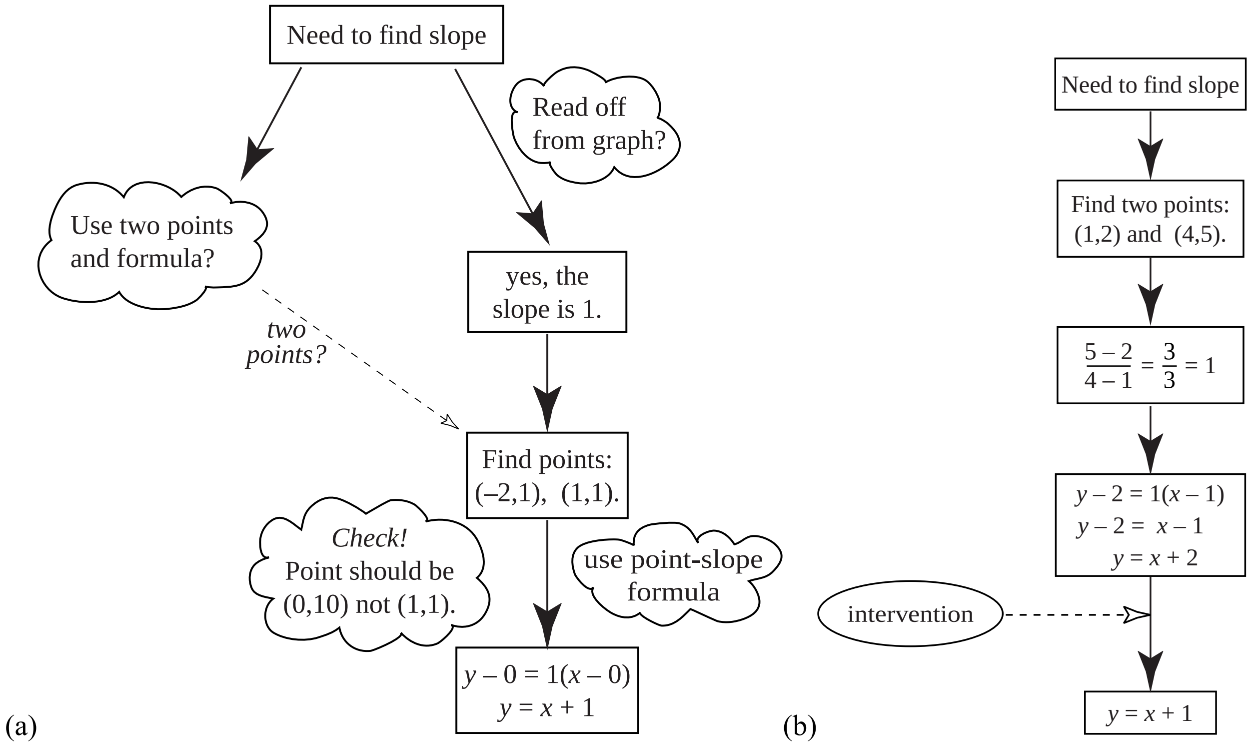


Figure 6. (a) Natasha's and (b) Kathy's Solution Processes

Kathy did not exhibit the same flexibility or checking mechanisms:

Kathy: Three, negative two…I need to find another point…

Researcher: That's a good plan.

Kathy: Four, negative three…

Researcher: What are you going to do now?

Kathy: Find the slope first. Negative three…

She then uses the formula incorrectly (figure 7) …

Researcher: Now wait a minute. Why did you…this is right. Why did you tell me that $-3-2$ is –1?

Kathy: Wait, is it negative 5?

Researcher: Yes, however, you made a mistake earlier. Where you had $-3-2$, it's $-3-(-2)$, isn't it?

Kathy: Oh, yes.

Researcher: So then what do you want up here?

Kathy: That would be positive 5?

Researcher: Isn't minus a minus a plus?

Kathy: Yes.

Researcher: Minus three plus two is minus one.

Kathy: Which is what I got, I just didn't have the negative sign.

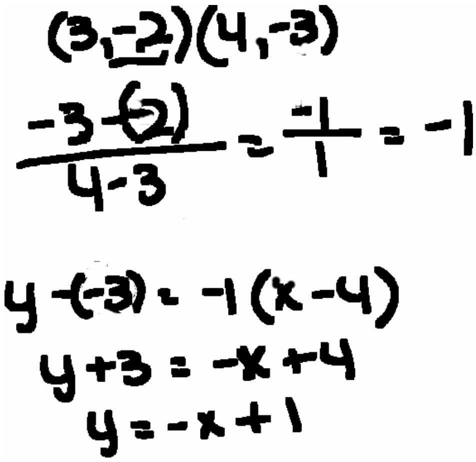


Figure 7. Kathy's writing

(note: she had made canceling errors)

Researcher: So you've got the slope, now what are we going to?

Kathy: Use the…

Researcher: What's the *y*-intercept on that line?

Kathy: The *y*-intercept?

Researcher: Where does it cross the *y*-axis?

Kathy: One, er, zero, one.

Researcher: So you should get something *x* plus one, shouldn't you?

Kathy: Yeah.

In each problem, Natasha demonstrated flexibility in choosing a route to a solution, thus showing evidence of links between graphs, formulas, and other aspects. She also routinely checked her work using alternative methods, another indication of useful links. She found her own error in the second problem. Kathy, on the other hand, had at most a single procedure in each case and was prone to make mistakes.

SUMMARY

In all the questions considered there is a broad common thread. Natasha demonstrated links between graphical and symbolic representations, as well as links to and between procedures. Although she made mistakes, she had methods of checking and self-correcting. She did not always make the necessary connections and had some fears about negative numbers, but was broadly successful. Kathy obtained the same grade on her examination but merely learned a set of procedures and had

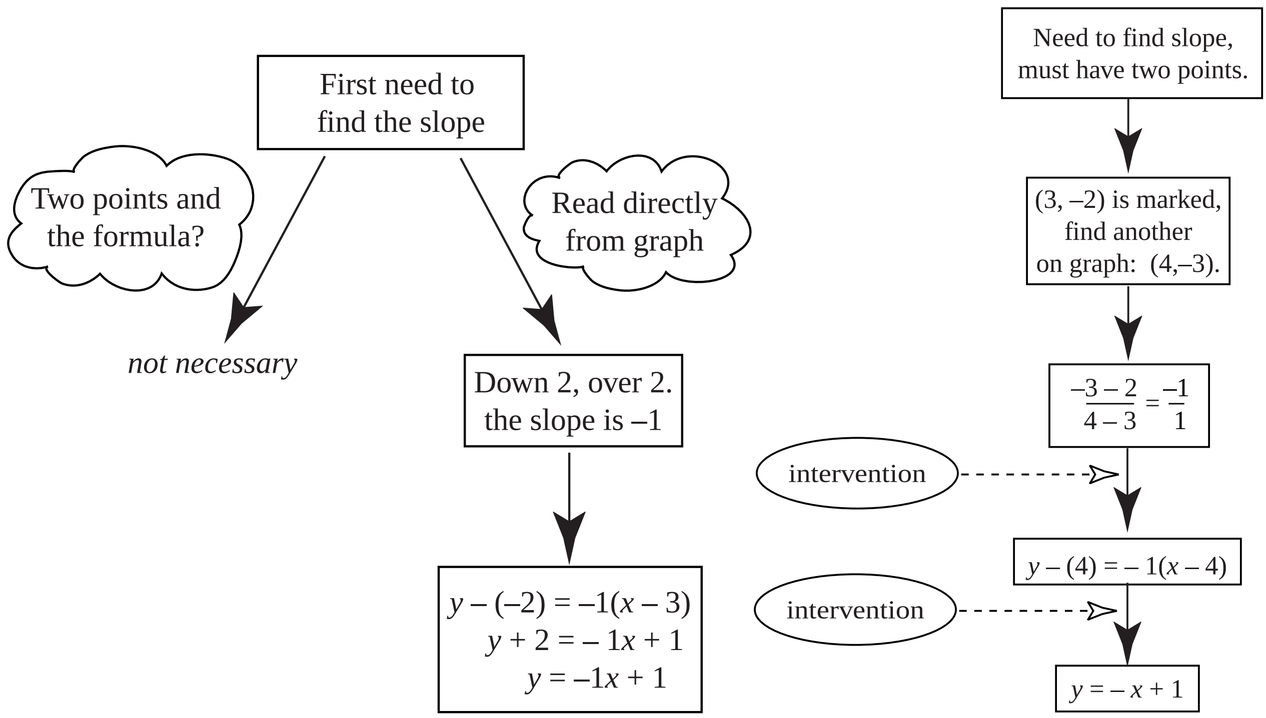


Figure 8. Natasha and Kathy's Solution Processes

difficulties with negative numbers and fractions that she coped with in routine questions by using her calculator. The procedures she has learned have allowed some success, but she must work very hard, and the procedures are not organized in a useful way that would allow her to build on them in the subsequent pre-calculus course.

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