A LIFE-TIME'S JOURNEY FROM DEFINITION AND DEDUCTION TO AMBIGUITY AND INSIGHT

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In this paper I speak of a personal journey in mathematics and mathematical thinking that began in a mathematical world of precision and certainty and found a world of mathematical thinking full of ambiguity and insight. It is a journey on which I have had many fellow travellers from whom I have learnt most of what I know, particularly my colleague Eddie Gray and our research students, and other very special co-authors. They have accompanied me on various quests in a life-time's journey by way of 'concept image', 'measuring number', 'local straightness', 'generic organiser', 'cognitive root', 'procept', 'cognitive unit', 'advanced mathematical thinking', to 'three worlds of mathematics' where the route to the world of formal mathematics is by way of two other mental worlds— conceptual embodiment (thought experiments based on perceptions) and proceptual symbolism (actions, such as counting, symbolised as concepts, such as number). On the journey I also discovered new insights, at least for myself: that the proliferation of mathematics education research is constructing a growing environment for individuals to build theories and careers, leading to a welcome level of productivity, but that this productivity has led to a wealth of complication that needs compression into an insightful simplicity. We know more than we knew thirty years ago, but our knowledge has not produced universal success in teaching mathematics. Looking back, I see my journey as a quest to seek the underlying simplicity that enables us to think in a powerful mathematical way in our increasingly complicated lives.

STARTING FROM WHERE WE ARE

We all begin our journey from where we are at the start. My own journey in mathematics education began from my position as a mathematics lecturer in a university mathematics department which coloured the ways in which I viewed mathematics and mathematical thinking. I sought clear definitions, clear deductions and the construction of a coherent theory of mathematics education.

In the early 1970s, the main cognitive frameworks available were stimulus-response behaviourism (hardly appropriate for the subtleties of mathematical thinking) and Piaget's epistemological approach to child development. The simple but profound book on *Psychology of Learning Mathematics* by Richard Skemp (1971) came as a breath of fresh air. I was so enamoured of his work that I invited him to speak to the Mathematics Department at Warwick and the mathematicians were so impressed that, when the Education Professor resigned that year, Richard was invited to apply for the post and became Professor of Education.

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When Richard was asked to review Freudenthal's book *Mathematics as an Educational Task*, having already bought his own copy and not wanting another, he passed the invitation to me. To review a work of the great Freudenthal was a huge task for a young mathematics lecturer and I sought advice from a senior colleague, James Eels, who knew him well. He confirmed that I should say *exactly* what I felt and, emboldened by his advice, I wrote a welcoming but critical essay. I received a post-card from Freudenthal after the review appeared: "thank you for the review which I enjoyed, *especially* the critical parts."

My position in the Mathematics Department was as 'a lecturer in mathematics with special interests in education'. Rolph Schwarzenberger, a professor and a friend who gave me guidance, encouraged me to begin research into the learning of undergraduates and I began a study on students' understanding of limit processes (Tall, 1976) where I found that most students believed that '$0.\dot{9}$ is just less than one'. I had data, but where was the theory?

I still saw mathematical thinking from the viewpoint of a mathematician, becoming enamoured of catastrophe theory through the research of my departmental chair Christopher Zeeman (1977). The paper I presented at the very first meeting of PME (Tall, 1977) used catastrophe theory to describe how the brain could leap suddenly from one viewpoint that becomes untenable to another that is more stable.

This paper interested Shlomo Vinner and I travelled to Israel in 1979 to work with him. There I met Efraim Fischbein, the first President of PME, who was at that time working on the concept of infinity. As a mathematician, I suggested to him that the conflicts in the data he had collected related in part to two different ways of looking at infinity: cardinal infinity arising as an infinite extension of counting and the infinities and infinitesimals of the calculus of Leibniz that I suggested were infinite extensions of 'measuring' (Tall, 1980a). He told me he wanted to 'see' an infinitesimal and I suggested that rational functions could be ordered by looking at their graphs and defining $f(x) > g(x)$ if the graph of f is above the graph of g for sufficiently large values of x. Functions such as $y = x$ or $y = x^2$ are, in this sense, greater than zero but ultimately below any positive constant graph so they are 'infinitesimals'. He protested saying that he wanted to 'see' an infinitesimal as a tiny value and these rational functions did not look 'small' to him. He remained unmoved when I suggested we shift our attention to a vertical line far off to the right to see that constant functions $y = c$ met the line in fixed points, but variable functions met the line in variable points where those that were positive but ultimately lower than any given positive constant were infinitesimal. I wrote about this in a paper (Tall 1980b) and was met with the wrath of Israel when Tommy Dreyfus alerted me that Fischbein was concerned that he had more strength in psychology than in mathematics and that, in referring to his rejection of mathematical constructions, I had touched a sensitive spot. I wrote an immediate letter of apology to Efraim and he and his student Dina Tirosh became long-term friends and continued to keep in contact over the years. When he passed away, I was invited to give his eulogy at PME.

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Back in Warwick, at the age of around forty, I started a second PhD in Mathematics Education with Richard Skemp who was about to become the second President of PME. He was such an inspiration with his clear and simple approach to theory, and gave me great insight into the psychological side of mathematical thinking. I was invited to present the eulogy in his memory at PME too.

When Shlomo Vinner visited me at Warwick at Easter in 1980, I had a great deal of data available and no way of making sense of it because the students' answers weren't mathematically coherent. He showed me his paper with Rina Hershkowitz (1980) prepared for PME later that year in which students' interpretations of geometric concepts were analysed in terms of the new notion of 'concept image' and 'concept definition'. It was exactly what I needed to make sense of my data. I began writing with great speed and energy and wrote the first draft of a paper in a day, finishing it off in subsequent discussion with Shlomo (Tall & Vinner, 1981).

This became my second source of embarrassment with my Israeli friends. Following the custom in my experience as a mathematician, authors' names were written in alphabetical order and so it happened that 'Tall and Vinner' became quoted as the origin of 'concept image' and 'concept definition', when it really was the invention of Vinner, shared earlier with Rina Hershkowitz. In my defence, I recalled Richard Skemp's immortalization of 'instrumental and relational understanding' using terms formulated by his friend Stieg Mellin-Olsen in a social setting, which Richard converted to a psychological setting. His paper became a classic. In the case of 'concept image' and 'concept definition', I took Shlomo's definition of the terms in two separate compartments in the philosophical mind and turned them into mental conceptions in the biological brain. This paper became a classic too. Publication is all. The credit goes to those who publish first, even if they acknowledge earlier sources.

As I worked on my doctoral thesis in the early eighties, computers arrived and I began to bring together my mathematical knowledge with my personal version of cognitive psychology. I knew that students had a serious difficulty with limits, which they saw as potential processes that were never finished rather than fixed concepts. I had used infinitesimal concepts in teaching a course on 'Development of Mathematical Concepts' long ago in the early seventies and a young David Pimm—then a mathematics undergraduate—had persuaded me to teach a mathematics course on non-standard analysis. I therefore knew the mathematical theorem that when the graph of a differentiable function is magnified by an infinite quantity, with infinitesimals too small and infinite elements too far away to see, the resulting image is precisely an infinite straight real line (The technical details are in Tall, 1980c). So I was on mathematically solid ground when I introduced students to 'locally straight' functions that looked straight under appropriately high magnification. This led to my programming the software for *Graphic Calculus*, which was the first approach to the calculus using a balanced combination of mathematical correctness and visual conceptual meaning.

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As I studied for a PhD in education under Richard Skemp, my earlier PhD in mathematics entitled me to have my own PhD student, John Monaghan. He radically changed my earlier analysis of infinite concepts by showing me that his students thought that √2 was an infinite number too. It is infinite *in extent* because its decimal goes on and on forever, 1.414… . He found that students conceived of 'proper' numbers that could be calculated precisely, such as integers, fractions and finite decimals, and 'improper' numbers that could not. (Monaghan, PhD[1]).

The eighties became a period of great excitement as I travelled to show *Graphic Calculus*, and developed more theory relating to the visualisation of concepts. I theorised that a picture can be *specific*, such as the graph of $\sin x$ and its slope function, looking like $\cos x$. It can also be imagined as being typical of more general locally straight functions whose slope function could be seen by looking along the changing slope of the graph. So I formulated the notion of 'generic organiser' as an environment that enables the learner to manipulate examples and (if possible) non-examples of a specific mathematical concept or a related system of concepts (Tall 1989). This was considered as a complementary construction to the notion of advance organizer (Ausubel *et al,* 1978), a higher-level structure used to organize future learning from above, whereas a generic organiser builds up generalisations from below. I also proposed the notion of 'cognitive root', which is 'an anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built' (Tall, 1989). A good example of a cognitive root is the concept of 'local straightness'.

At this time in the mid-eighties I was becoming considered as an 'expert' in visualization, though I knew little about the wider cognitive aspects of the topic. I worked with the idea of linking visualization and symbolism with my second PhD student Michael Thomas who has continued to work with me ever since. He built up the idea of 'versatile thinking' in which visualization and symbolism were used in complementary ways (Thomas PhD, Tall & Thomas, 1991) and together we began to build a 'principle of selective construction' where the computer could be used to carry part of the burden of internal computation while the learner could focus on the higher level relationships. In doing so, we began to be interested in the relationship between process and concept.

At this point there was a veritable explosion of related ideas. Ed Dubinsky (1991) was developing his APOS theory whereby ACTIONS were routinized into PROCESSES, encapsulated into OBJECTS to become part of a larger SCHEMA of thought, and Anna Sfard was developing a theory with complementary aspects of STRUCTURAL and OPERATIONAL mathematics (Sfard 1991). I spent time with both, but we never came close enough to write joint papers in process-object theory, although my thinking was greatly influenced by both of them. I misunderstood Anna's notion of 'structural'

[1] In this paper, all PhD theses are given in a supplementary list at the end in chronological order.

because she related it to the way Ian Stewart and I had written about complex numbers as ordered pairs of real numbers and I naturally related the term to the structural approach of Bourbaki with axiomatic definitions and formal deduction.

It was some time before the light dawned. Eddie Gray and I had been colleagues for many years at Warwick when he began his own research in the late eighties. His first major work (Gray, 1991) studied the responses of young children to simple arithmetic problems and he agreed to submit his current research for a PhD. One late Thursday afternoon in December 1990 as we looked at his data on the relationship between counting processes and number concepts, it became clear that we did not have a word to cover *both* possibilities. Dubinsky (1991) had his notion of process that could be encapsulated into object and de-encapsulated into process, Sfard had her notion of process that could be reified into object, but she referred to process and object as 'two sides of the same coin' (Sfard 1991), and asked, "How can anything be a process and an object at the same time?" Suddenly I suggested we call this 'thing' that can be both process *and* concept, a *procept*. It was a moment of supreme revelation, comparable with the time that Shlomo showed me his paper on concept image. All the disparate pieces fell into place. Some children were thinking flexibly, shifting seamlessly from process to concept, others were fixed in the procedures of counting. Immediately we saw applications in algebra (expressions as process or concept) and analysis (limits as process or concept). This was huge!

At that very moment, Rolph Schwarzenberger, currently the head of our department, walked into my office. Before he could speak, I said, "On your knees, Schwarzenberger! You were not there when Leibniz first said 'functio', or when Cantor first said, 'set'. But you *were* there when Tall said 'procept'!" He smiled indulgently and said, "I am not sure of the etymology of the word," meaning that the prefix 'pro' was a Greek term with a different usage. I looked back and said, "If you have a delicate new plant, the last thing you do is to prune it, instead you nurture it and pour manure all over it." (Except that I did not say the word 'manure'.)

That evening we were delighted with our work and considered all the possibilities that lay before us. But then, as I sat in my car in the car park in the gathering gloom, I suddenly lost heart and said to Eddie: "I'm not so sure; all we have is a word." He leaned in through the car window and said, "We have more than that, we have *duality*, *ambiguity* and *flexibility*!" This most profound insight became the title of our first paper together: 'Duality, Ambiguity and Flexibility in Thinking' (Gray & Tall, 1991). It was sometime later that Eddie's wife Mareea christened it the DAFT paper.

Eddie used the theory to suggest the idea of the 'proceptual divide' in the spectrum of performance between those children who cling to remembered procedures and those who become increasingly flexible through the use of proceptual thinking. The notion of procept was the product of an equal partnership in which we shared different experiences to produce something new that was genuinely greater than the sum of its parts. It began a productive relationship over a decade and a half, in which we worked with our doctoral students to develop and enrich the theoretical framework.

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Meanwhile, a development that had begun back at the beginning of the 1980s and had run in parallel for years began to bear fruit. The papers presented in the first years of PME were mainly about mathematics education at elementary level. I presented a paper in 1981 which sounded the clarion call to build a theory of mathematics education from early beginnings right through to university undergraduate and research level (Tall, 1981). Gontran Ervynck was also working on this idea and coined the phrase 'advanced mathematical thinking', proposing a working group of that title and inviting me to share in its organisation. I had no talent for administration and he did it all.

By 1979 we had agreed to write a book that I edited and it appeared as *Advanced Mathematical Thinking* in 1991. As I wrote the epilogue to that book, I saw the contrast between two distinct strands of thought leading to formal mathematics.

> There are therefore (at least) two different kinds of mathematics. One builds from gestalts, through identification of properties and their coherence, on to definition and deduction at advanced levels of mathematical thinking. The other continually encapsulates processes as concepts, to build up arithmetic, then generalizes these ideas in algebra before formalizing them as definitions and deductive theorems in the advanced mathematics of abstract algebra. David Tall, 1991, p.254

At the same time, PhD student Md.Nor Bakar had introduced me to the theory of prototypes (Smith, 1988), which resonated with my thoughts on concept images and generic examples and led to our realisation that students' conceptions of functions were very much influenced—often unconsciously—by their previous experiences (Bakar, PhD; Bakar & Tall, 1991).

Meanwhile, Norman Blackett studied trigonometric ratios as process and concept and researched how students might gain more conceptual insight using interactive visual software to explore the properties of trigonometric ratios in right-angled triangles. As a by-product it gave a significant gender difference, with the girls cooperating while the boys took the activity less seriously so that a previous advantage enjoyed by the boys was turned around (Blackett, PhD; Blackett & Tall, 1991).

In 1993 I was struck down by an illness (sarcoidosis) which left me exhausted and I had fourteen months off work. I continued to see my research students during this time, but everything else stopped. On attempting to return to work, it was clear that I could cope for a day, perhaps two, but then the duties began to fall behind schedule and within a week it had all ground to a halt. So I took early retirement in 1994, while continuing to work at the university on a one-third timetable. The remainder of my working life was the best time of all. I had a stream of willing PhD students, supported by Eddie as second supervisor and we developed a range of new ideas as I shifted from writing a majority of solo papers to joint papers with others.

My three American PhD Students each gave me insight into the nature of American College Mathematics. Phil Demarois studied the relationship between different representations (facets) and different levels (layers) of encapsulation in the function concept (Demarois PhD, Demarois & Tall, 1996). Mercedes McGowen studied the cognitive collages of knowledge construction produced by college students to show how the students concept maps changed with time, as the more successful built new knowledge on old while the less successful had more fragile structure which failed to build on previous knowledge (McGowen PhD, McGowen & Tall 1999). Lillie Crowley revealed how students could be awarded the same grade at one stage, yet diverge in the next stage with the more successful producing coherent knowledge structures while the less successful remained with limited procedures that trapped them in less appropriate ways of working (Crowley PhD, Crowley and Tall, 1999).

Malaysian PhD student Yudariah Yusof took me along a different track by investigating student attitudes to problem-solving (Yusof, PhD; Yusof & Tall, 1995). Amazingly, she found that the qualities that mathematicians desired of their students were encouraged in problem-solving, but suppressed in regular mathematics lectures.

Maselan bin Ali studied how Malaysian students coped with the symbolic routines of differentiation, showing that the more successful tended to have two or more procedures available for a given problem (classified as multi-procedure or process) while the less successful were more likely to have at most one (classified as procedural) (Ali, PhD; Ali & Tall, 1996).

Robin Foster studied young children's solutions of equations of the form $3+4=\square$, $3+\square=7$ and $\square+4=7$. We hypothesised together that the first could be solved by any method of addition, the second solved by 'counting-on' from 3, but the third would be more difficult for children at the 'count-all' stage because it would require them to guess a starting point from which to count-on, to see if the desired objective was reached (Foster, 1994). He also considered the way in which children used Dienes' Multibase Blocks to solve subtraction problems and found that while the most successful would use the materials to illustrate what they knew, a second group might use the materials to see the underlying mathematics, a third group might function satisfactorily with the materials, but not without, and a fourth group might fail entirely. (Foster, PhD). This has widespread importance in using statistics in any controlled experiment where one group are given a special treatment: the statistics only measure the change in *one* sub-group—the second sub-group who have a genuine improvement in long-term learning—while the first, third and fourth have no permanent long-term effect at all.

At University Level, Marcia Pinto from Brazil considered how students at Warwick coped with a first course of university analysis, distinguishing between those who rely on their concept imagery to give meaning to the definition (natural learners) and those who give meaning to the definition through studying the formal definitions and related proofs to derive properties (formal learners) (Pinto, PhD; Pinto & Tall, 1999).

Erh-Tsung Chin (Abe) from Taiwan studied how students made sense of equivalence relations, revealing that underlying embodiments interfered with their understanding of the three properties of an equivalence relation (Chin, PhD; Chin & Tall, 2002).

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Soo Duck Chae from Korea studied the use of computer software in investigating the meaning of the bifurcation of solutions of iteration of the equation $f(x) = \lambda x(1-x)$ as λ increases. Initially the iteration successively replacing x by $f(x)$ tends to a limit for $0 < \lambda \leq 3$ then bifurcates to an orbit of period 2 at a value of $\lambda = \lambda_1$, and to orbits of period 4, 8, 16, … at λ_2, λ_3, λ_4, … to give a sequence that converges to the Feigenbaum constant λ_∞. According to Dubinsky's original APOS theory, it is possible to distinguish three stages of encapsulation: from the process of performing the iteration encapsulated as a final orbit, from the varying orbit to the sequence of bifurcation points, and from the sequence of bifurcation points to the Feigenbaum limit. APOS theory as originally formulated implies that encapsulation need be performed at each stage so that the objects formed at that stage could be used at the next. Yet, Soo Duck found that for $\lambda \leq 3$ many students were still at the process stage getting closer and closer to the limit, while for $\lambda > 3$, they switched focus to the visual picture of the orbit and used this as an object which bifurcated to reveal the sequence (λ_n) whose approximate numerical values looked as if they would converge geometrically. This confirmed for me that a symbolic APOS theory required complementing with embodied visualisation and human action to explain how mathematical learning occurred. On the other hand, the embodied theory of Lakoff (Lakoff & Nunez, 2000) did not focus on process-object encapsulation at all. In 2001, Eddie and I discussed the relationship between embodied objects and symbolic procepts (Gray and Tall, 2001).

The major step in my journey to link embodiment and the symbolic compression from process to concept occurred when Anna Poynter (previously Anna Watson) revealed the insight of a student Joshua. In talking about the sum of two vectors, he explained that the sum was a single vector 'had the same effect' as the combination of two individual vectors. This key unlocked the door of the relationship between embodiment and process-object encapsulation. Embodied encapsulation involves a 'delicate shift of attention' (in the sense of Mason, 1980) from the *action* being performed to the *effect* of that action (Poynter, PhD; Watson, 2004).

This parallel between embodiment and symbolism led me immediately to the idea of *three distinct worlds of mathematics* (Tall 2004). We begin in *a world of conceptual embodiment* focusing on (real-world) objects and actions and by thought experiments focusing on generic properties, we construct hierarchies of mental objects through to the platonic world of Euclidean geometry and visual representations of algebra and the calculus. *The world of proceptual symbolism* compresses actions from processes into thinkable concepts (procepts) to lead to a hierarchical development in arithmetic, algebra and symbolic calculus. The two combined lead by natural or formal thinking to *the formal axiomatic world* of concept definition and formal proof.

I had begun thirty years before in the formal world of mathematics and I had backtracked and found a route from the perceptions of a child to the conceptions of mathematicians. And virtually every insight came from someone else! As Richard Skemp taught me, 'pleasure is a signpost, not a destination'. 'The journey is the reward'. The journey still continues, clarifying issues in the theoretical framework, for instance, in the way in which learners build on previous knowledge to produce what I call 'met-befores' in the concept image that can be helpful in some contexts, but act as cognitive obstacles in others. We already have a map of the cognitive growth of procepts through arithmetic, algebra, calculus and on to the formal definitions of analysis and other formal mathematics (Tall, Gray, et al 2000).

The journey uses the natural strengths and limitations of the biological brain to compress complicated detail into the simplicity of thinkable concepts that can be handled by the limited focus of attention (Akkoc & Tall 2004).

The research continued as Amir Asghari produced a highly personal Lakotos-style thesis in which he challenged not only my work in textbooks with Ian Stewart and research with Abe (Erh-Tsung) Chin, but also analysed that of the Greeks, Gauss, Russell to point out subtle flaws in the theories of Dienes and Skemp. Once again, challenge has led to personal enrichment. (Asghari, PhD; Asghari & Tall, 2005).

The journey has been enhanced by the companionship and insight of many other companions who have taught me so much. Other than my full PhD students, Tony Barnard shared with me his concept of 'cognitive unit', Dina Tirosh her research on infinity, John Pegg his research in SOLO and Van Hiele Theory; Ian Stewart collaborated in the writing of three mathematics texts, and a host of others shared ideas including Gary Davis, Demetra Pitta, Shakar Rasslan, Juan Pablo Meija-Ramos, my two Brazilian visiting students, Victor Giraldo and Rosana Nogueira de Lima, and Adrian Simpson, the organiser of this volume and the related celebrations. But it is to my research students I give the greatest thanks, particularly Eddie Gray, John Monaghan and Michael Thomas, all of whom have distinguished careers supervising doctoral students in a direct line from Richard Skemp through myself and them to succeeding generations.

Christopher Zeeman once said to me that the main test for PhD students to be awarded the degree is that they have taught their supervisor something important. It is an interesting idea.

An individual cannot be the source of limitless power of thought. In the real world there is no such thing as perpetual motion. To develop one needs new sources of energy and that energy comes from other people freely giving of their ideas. In the recent review of the first thirty years of PME (Guttiérrez & Boero, 2006), the ideas discussed in this paper have the largest number of references for a single person in the whole volume. I take quiet pride in this statistic for someone who retired over a decade ago on ill-health. Of course, it is not the work of a single person, but the contributions of a whole family working for a common purpose. As can be seen from this review, almost everything I have done has been gifted to me willingly by others. My work has only been made possible by the support of all my collaborators, especially the PhD students at Warwick University who all earned the award of a doctorate for teaching their supervisor something important.

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Norman Blackett (1985–90). Developing Understanding of Trigonometry in Boys and Girls using a Computer to Link Numerical and Visual Representations.

Md.Nor Bakar (January 1991–December 1993). What Do Students Learn About Function? A Cross-cultural Study in England and Malaysia.

Eddie M. Gray (1990–1993). Qualitatively Different Approaches To Simple Arithmetic.

Yudariah binte Mohammad Yusof* (April 1992–April 1995). Thinking Mathematically: A Framework for Developing Positive Attitudes Amongst Undergraduates, (PhD, 1995).

Maselan bin Ali* (January 1993–June 1996). Symbolic Manipulations Related to Certain Aspects Such as Interpretations of Graphs.

Philip DeMarois* (1994–1998). Facets and Layers of the Function Concept: The Case of College Algebra.

Mercedes McGowen* (1994–1998). Cognitive Units, Concept Images and Cognitive Collages: An Examination of the Process of Knowledge Construction.

Marcia Pinto* (1993–1998). Students' Understanding of Real Analysis.

Richard Beare (1998–1999). Researching, developing and applying the potential of spreadsheets for mathematical modelling in educational contexts).

Robin Foster* (1991–2001). Children's use of Apparatus in the Development of the Concept of Number.

Lillie Crowley* (1995–2000). Cognitive Structures in College Algebra.

Soo Duck Chae** (1997–2002). Imagery and construction of conceptual knowledge in computer experiments with period doubling.

Ehr-Tsung Chin (1998–2002). Building and using concepts of equivalence class and partition.

Anna Poynter* (January, 2000–November, 2004). Effect as a pivot between actions and symbols: the case of vector.

Hatice Akkoç (October 1999 –2003). The Function Concept (EdD).

Nora Zakaria†† (October 1999–2004). A study of the nature of knowledge transfer across subject boundaries comparing procedural students and conceptual students at university level.

Amir Asghari† (October 2001–April 2004). Equivalence.

Victor Giraldo (2002–2004). PhD at Federal University, Rio de Janeiro, with one year at Warwick. Descrições e Conflitos Computacionais o Casa da Derivada.

Juan Pablo Mejia Ramos (in progress, 200 4–). University students' conceptions of proof.

Rosana Nogueira de Lima (in progress, 2005–). Students' conceptions of equation. PhD at Pontifícia Universidade Católica de São Paulo, with one year at Warwick.

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