What does equation mean? A brainstorm of the concept

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Abstract In this paper we analyse data from conceptual maps designed by 14-16 year-old Brazilian students in order to understand their conceptions of equation and its solving. We claim that it is necessary that students give meaning to equations and algebraic symbols in three different ways: embodied, symbolical and formal. Data show that the absence of meaning lead students to further difficulties on solving equations.

1. Introduction

According to Bazzini and Tsamir [1] official documents that guide teaching and curriculum design suggest that students should understand the meaning of algebraic symbolism, expressions, equations, and also how to represent real world situations using those symbols. Brazilian documents [2] also emphasise this need.

However, it is not clear what kind of meaning students should give to symbols. Tall [6] suggests that there are different kinds of cognitive developments for different kinds of mathematical concepts. This distinction leads us to hypothesise that there are different meanings attached to mathematical symbolism and that they would affect the way students understand algebra.

In this paper, we present a study with 14-16 year-old Brazilian students in which we are looking for what kind of meaning they give to equations, their symbolism and the rules used to solve them. We hypothesise that their understanding of the concept of equation is related to the actions of solving it, rather than to its formal characteristics of equality and equivalence.

2. Theoretical Framework

Tall [6] proposed a categorisation of mathematical cognitive development into three different but interacting ways of thinking, which lead to three different worlds of Mathematics.

A *conceptual-embodied world* of perceptions, in which individuals make sense of properties of objects by observing and verbalizing them through descriptions.

A *proceptual-symbolic world* in which mathematical entities are symbolized and the actions performed as procedures can be seen in a more flexible way by some students who have the flexibility to see the symbols either as process or concept (procept).

A *formal-axiomatic world* based on axioms, definitions and theorems that deduces mathematical structure from the axioms and definitions by formal proof.

In school mathematics, the notion of equation may be presented initially in a *conceptual embodied* form as a balance. This has the advantage of giving embodied meaning of adding and taking equal quantities from both sides to maintain balance, but the disadvantage of not transferring easily when the equations involve negative quantities (see [8]). An alternative *functional embodiment* occurs in the way that quantities may be imagined as being 'picked up' and transferred to the other side, accompanied by a change in sign (see [7]). This can

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often bring about an operation which is conceptualised as an embodied shifting of symbols which is often performed in a rote-learned manner without any conceptual meaning. Formal meaning is rarely appropriate at this level, being more central to formal axiomatic thinking in pure mathematics at university, however, some aspects of formality are often introduced to the students by the teachers, in terms, say, of the notion of 'equivalence' of the two sides of the equation, a concept which again may lack meaning for many school students.

3. Methodology

As part of a broader research project investigating how algebra is taught and learnt, the first author worked in cooperation with six mathematics teachers from São Paulo meeting on a weekly basis, discussing both mathematical and pedagogical aspects of equations. In an initial study, four classes of students were asked to build a conceptual map from words of their own choice that come to their mind related to the word EQUATION.

The four classes consist of 39 Public School 8th graders of approximately 14 years-old (denoted as C14), two classes from another public school, one of 32 fifteen-year old first graders (C15), the other, 28 sixteen-year old second graders (C16), and a class of 18 sixteen-year old second graders from a private school (P16).

3.1 Conceptual Map

The session begins with the teacher putting the word EQUATION on the blackboard and initiates a brain-storming session asking students to give at least one word that comes to mind when seeing the word. All words are written on the board, randomly placed and the session continues until each student has given at least one word. The students are then subdivided into groups of four or five and are requested to carry out the following activities:

- Separate words from the brainstorm in at least 3 categories. All words are supposed to be used and each word should be in only one category.
- Name the categories.
- Design a scheme or diagram relating the names for the categories to the word EQUATION.
- Write a sentence or two to explain the designed scheme.

4. Results

It is important to analyse both the words that have risen from the brainstorm and the map and text. The first is supposed to present the ideas the students as a whole have of equation, the last, how small groups of students interpret those words.

4.1 Brainstorm words

The four classes are from three different schools and different ages. However, many words appeared in all brainstorms. Figure 1 shows an example of how the blackboard ended up afterwards.

Similarities between brainstorms suggest us that all four classes are very similar and probably with a broadly similar understanding of equations. Words like "calculation", "number", "addition", "subtraction", "multiplication", "division", "signs", "solution", "answer", "unknown", "rules", "results" tell us that it is likely that those students see an equation as a calculation in which it is necessary to use rules to find the solution, the value for the unknown. It is possible that the learning has been based on rules, although we do not have data to affirm that. On the other hand, there are no words related to real-life problem solving.

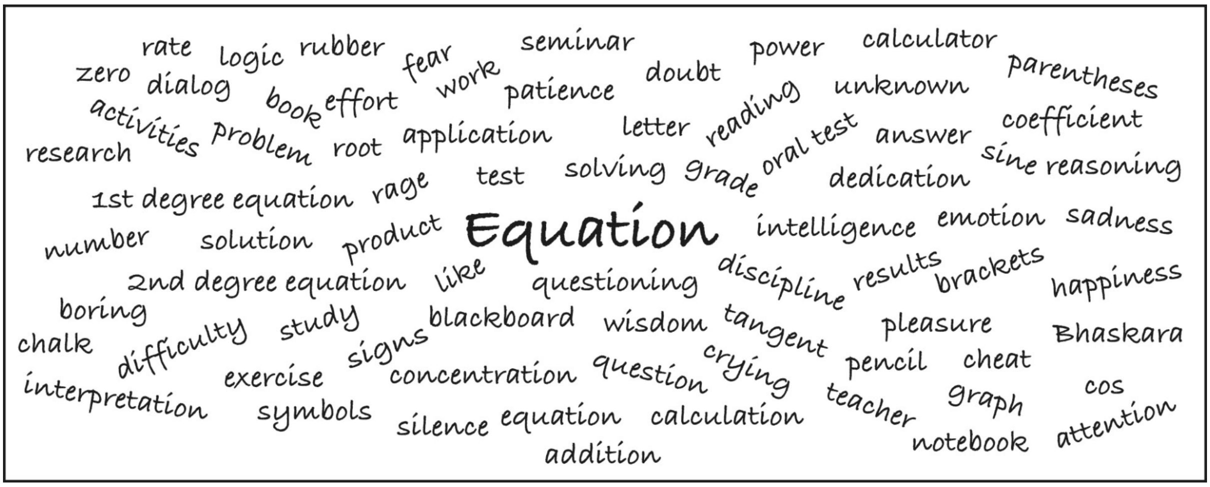

Figure 1: Words from P16 class

Another characteristic common to all brainstorms is that the responses include words like “school”, “teacher”, “students”, “tests”, “assessment”, “exercises” and “examples”. This is consistent with a view that sees equations as a topic relating to activities in school. There is no evidence that students give conceptual embodied meaning to the equations in terms of ‘balancing’ two sides.

Words like “number”, “operations”, “calculation” and those related to solving methods, like “rules” or “formula”, are mention in all brainstorms. This shows us that those students are likely to relate equations to the act of doing something in order to get the “solution” (a word that also appears in all brainstorms). Other evidence of the actual solutions given by the same students in [5], suggests, on the contrary, that these students attempt to solve equations by the functional embodiment of picking up elements to ‘change sides, change signs’ in a way that has little meaning for them.

Dreyfus and Hoch [3] reported that most of Israeli students participating in their research mention the unknown as an important characteristic of equations. In the case of our research, the word “unknown” occurs only in two brainstorms (C14 and P16). However, the other two (C15 and C16) mention the word “letter”, C15 say “x and y” and C16 say “variable”. This suggests that such students may be aware of the need of a letter that stands for a number.

Only in the brainstorm of the class C15 does a student say the word “equals”. No other words related to equals sign, equality, equivalence, or balance have been found in any of the four brainstorms. We believe that it may be related to the status of equals sign for these students. Kieran [4] reported that the equals sign is seen as a “do something” signal by the subjects in her research. We believe that this gives supporting evidence that these students do not give any formal meaning to the equations.

Brainstorms from both C14 and P16 present words that represent emotions like “panic”, “fear”, “happiness” and “wish”, and also words that show the need of abilities, such as “reasoning”, “concentration”, “patience” and “dedication”. We believe that these classes relate to equations more than just mathematical components, but also their feelings and worries which may affect their work in Mathematics.

4.2 Maps and Texts

Looking at the maps and explanations designed by the small groups may enable us to look more closely at how students relate their words and give meaning to equations.

As the word "equals" only appeared in the class C15 brainstorm, we looked more closely at how it was categorized. All small groups from C15 that used the word, placed it in categories named "Symbols", "Signs" or "Operations". It is not mentioned in any accompanying explanation. This is consistent with it being conceived as part of a functional action of assignment rather than as an equivalence between two expressions.

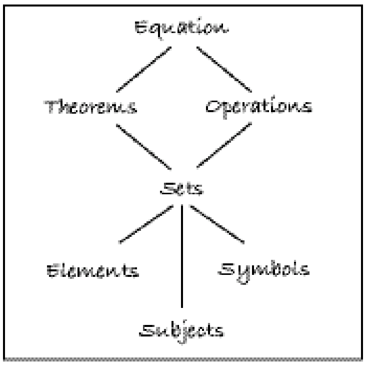

Figure 2 – Conceptual map of a small group in C15.

Looking at the categories constructed by small groups in class C15, some similarities are evident in all of them. In general, categories are "symbols", "formulas", "subjects", "school" and "material" (or words related to them). In a "symbols" category are words like "brackets", "plus", "unknown", "numbers". For "formulas", the words include "rules", "quadratic formula". As in the brainstorm of this class, many other subjects appeared, such as "Physics" and "Biology", placed in a "subjects" category. "School" included words related to the classroom and assessments; "material" included objects like "pencil" and "rubber".

The conceptual map designed by one group in C15 shows that, although they have mentioned words for other subjects, these are not related closely to equations, so we conjecture that they are unlikely to see equations as a way of modelling problems. Words that are closely related to equations are operations and theorems (in the form of rules or formulas) that relate to the action of solving an equation.

Some groups in class C16 classify words as "closely related", "related" and "not related" to equations. This kind of categorisation has centred all words related to Mathematics in only one category. It is possible that these students relate equations very much to procedures, as the text of one group says, *"The equation is basic to everything in mathematics. It involves: sum, division, subtraction, multiplication, number, solving, and above all, reasoning."*

The most important characteristic presented in conceptual maps in class C14 is the category in which the word "unknown" is placed. Four of eight groups put this word in categories that do not relate to equations at all. One put the word in a category called "foolishness". This may be evidence that, even if students use words such as "unknown", "letter" or "variable", this might not mean that they understand their role in equations and their solution.

The groups in class P16 seem to be more concerned about the learning of equations than with the concept itself in the design of their maps. Their words and explanations reveal their feelings, problems with previous teacher and efforts to learn. This can be seen in the text *"Equation is part of Mathematics that we learn in school. The school has a methodology with which one obtains the learning which involve feelings. With dedication and enthusiasm."*

5. Discussion

Vlassis [8] reports the importance of the balance model approach in the teaching of equations in order to give the meaning of equivalence to the equals sign. We hypothesise that this approach gives conceptual embodied meaning to equations. Although the teachers with whom we have worked with declare they discuss such approach, there is no evidence in their students' work that they give this embodied meaning to equations. In further data collected with the same students (see [5]), it is possible to hypothesise that they give functional embodied meaning to equations, as they speak of picking a number from one side and putting it in the other, adding a change in sign. The absence of meaning to equations presented in this paper ended up leading students to a lack of flexibility to use procedures in solving equations.

If we look closely on what those students do with words like "unknown", it is possible to claim that they probably do not give symbolical meaning to symbols involved in equations as well, as they seem not to understand its role in the concept of equations and its solving. Relations between equation and symbols or signs are made, but the categories in which those words are in also have words related to rules, operations and action, which guide us to claim that those students perform actions on symbols, not necessarily understanding their meaning.

Words related to equivalence or equality do not appear in many brainstorms. This is further evidence that the formal meaning of equivalence is not familiar to many of these students, neither in an embodied form as a balance, nor in a formal sense as an equivalence of two expressions when an appropriate value is substituted for the variable.

We sense that what is happening is a cumulative consequence of failure to develop flexible meaning in using symbolism. Students who do not have a flexible meaning for arithmetic as process or concept are coerced into the rote-learning of procedures which become inflexible and fragile and their mathematics gets increasingly complicated. If the symbols for whole numbers do not have flexible meaning as process and concept, there will be increasing difficulties with fractions and negatives and serious conceptual problems with algebra. They may make some progress using the embodied notion of balance, but this may be a short-term policy which fails to solve problems for students who have difficulty with negative quantities (see [8]).

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