The transition from embodied thought experiment and symbolic manipulation to formal proof

David Tall

Mathematics Education Research Centre
University of Warwick, UK
e-mail: David.Tall@warwick.ac.uk

Formal mathematical proof, which students meet at university when they are introduced to the culture of pure mathematics, is built on earlier experiences that the learners have met before. These are the more fundamental conceptual embodiments that occur in thought experiments imagining mental situations and in the experience of manipulating symbols in arithmetic, algebra and symbolic calculus. In this paper we consider the ways in which embodiment and manipulation of symbols underpins formal proof and the elements that may support or act as obstacles to formal thinking.

Introduction

When students begin to study formal proof at university, they already have a wealth of preceding experience on which to build. In mathematics there is the use of visual diagrams, dynamic images and thought experiment on the one hand and the use of symbols of arithmetic, algebra and the calculus on the other. Formal mathematics builds on a combination of embodied and symbolic thought (figure 1).

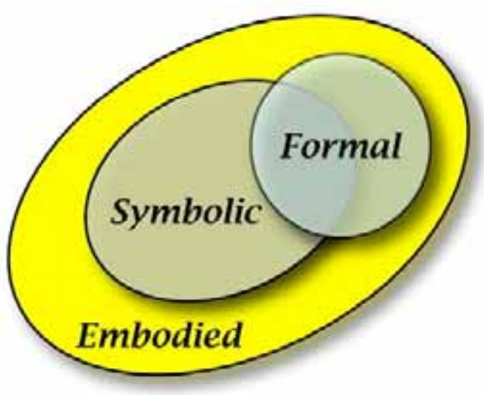


Figure 1: Formal mathematics building on embodied and symbolic thought

The terms ‘embodied’. ‘symbolic’, ‘formal’ here have specific meanings described briefly as follows:

- Embodiment refers to *conceptual embodiment* in which we reflect on our sensory perceptions and imagine relationships through thought experiment.

- Symbolism here refers not to symbols in general, but to those symbols used in mathematical calculation and manipulation in arithmetic, algebra and subsequent developments. These arise through *actions* on objects (such as counting) symbolised and manipulated as concepts (such as number). A symbol used dually to represent process (such as addition) and concept (such as sum) is called a *procept* (Gray & Tall, 1994).
- Formalism refers to the formal theory defining mathematical concepts as axiomatic structures whose properties are deduced by formal proof.

Tall (2004) theorizes that this categorization of mathematical thought involves three substantially different worlds of mathematics:

- An object-based *conceptual-embodied world* reflecting on the senses to observe, describe, define and deduce properties developing from thought experiment to Euclidean proof.
- An action-based *proceptual-symbolic world* that compresses action-schemas into thinkable concepts operating dually as process and concept (procept).
- A property-based *formal-axiomatic world* of concept definitions and set-theoretic proof.

These will be referred to as *embodied, symbolic* and *formal* in the remainder of this paper.

It transpires that each world has its own development of deductive argument. For instance, in the embodied world, we can *see* that addition is commutative by re-arranging 5 objects as 3+2 or 2+3. In the symbolic world, a child can *calculate* that the two sums give the same answer. In the formal world, $x + y = y + x$ is true *because it is an axiom*.

Furthermore, each world has its own manner of development in sophistication. The embodied world is based on sensory perception but its percepts are then analyzed, described, defined and verbal arguments adeveloped to formulate inferences typified by Euclidean geometry. The symbolic world shifts from a focus on action to increasingly sophisticated procedures and on to the conceptual structure of arithmetic and the generalized symbolism of algebra. Formal thinking reverses experience. Instead of analyzing existing concepts to determine their properties, it *begins* with selected properties as axioms and constructs other properties of the structure through formal proof.

Each world also has its own special use of language. In the embodied world, language is used in increasing sophistication to describe properties of objects, then to categorise and define. In the symbolic world, the symbolism has a new part of speech, the procept, acting dually as process and concept in a special way that extends everyday language use of verb participles such as ‘writing’ in ‘I am writing’ to gerunds acting as nouns, as in ‘writing is a mode of communication’. In the formal world, language modifies its role once again, using set-theoretic technical terms to define concepts in a special tone of meaning that Alcock & Simpson (1999) call ‘the rigour prefix’.

Formal thinking necessarily builds on the students’ prior experience of embodiment and symbolism. For instance the notion of vector space defined as an axiomatic system is embodied geometrically in two and three dimensions and symbolised as *n*-tuples in $\mathbb{R}^n$. This is typical of cognitive development in which embodiment offers an insightful meaning into a mathematical concept in 2 and 3 dimensions, but requires the symbolism of $\mathbb{R}^2$ and $\mathbb{R}^3$ to imagine generalizations to $\mathbb{R}^n$ and subsequently to the formal definition of a vector space.

The same occurs in many areas, so that identities such as $x(y+z) = xy + xz$ can have simple embodiments for positive values of x, y, z but soon become more complicated if these variables take on positive or negative values of different sizes.

In general, therefore, the trend in elementary mathematics is for successful students to move from embodiment, which is meaningful in simple cases, to symbolism that is powerful in more sophisticated contexts (Krutetskii, 1976; Presmeg, 1986; Gray, Pitta, Pinto, Tall, 1999). However, embodiments continue to have powerful effects on meaning in formal mathematics that can be beneficial in some instances and deceptive in others.

An example

We begin with an example of a mathematics student attacking the proof of a theorem that can be approached by a variety of different methods (Mejia & Tall, 2005). Grad is a competent student at a high-ranked university who had completed three years of study and found mathematics difficult. He used embodiment to gain insight into the theorem, making links that were plausible but not formally sound, used symbolism in a way that lacked confidence yet had been sufficient for him to pass his degree, and acknowledged the need to give fully formal proofs.

Grad was given the following task (based on a problem from Raman (2002)).

Task:

> Determine whether the statement below is true or false.
> Explain your answer by proving or disproving the statement.
>
> *The derivative of a differentiable even function is odd.*

As he read the statement out loud, Grad drew a parabolic shape in the air with his finger (figure 2) and thought for a few seconds. He continued as follows:

First the even function. … I don't think the derivative can be even […]. It's symmetric to the *y*-axis (gesturing with his hand to show a vertical axis) … effectively, I'm talking about two dimensional … so it's (err) … quadratic function (draws a parabola in a form similar to $y = x_2$ on the desk with his finger) … so the derivative is decreasing all the way (traces the parabola again) … so it can't get the same value twice, so it must be odd, so from that it's definitely not even.

Figure 2: Grad imagines a parabola, drawing it in the air

In this excerpt he appears to shift from the general notion of even function ("symmetric to the *y*-axis") to a more specific even *power*, (perhaps $y = x^2$) and deduces that its derivative can't be even so it must be odd. This deduction is false for a general even function, but is true for even and odd powers that seem to be his current focus of attention. Without further discussion, he concludes, saying, "generally I think it's true, but [*laughs*] I'm not so sure".

Analysing Grad's use of embodiment (through enactive drawing) and symbolism (through focusing on even and odd powers), we see that both lead to him making deductions that are true in specific cases but which do not hold in a general formal proof.

When asked for a proof of the statement, he used the two-sided definition of derivative, and manipulated it to give his version of the proof (figure 3). The use of the two-sided derivative would be justified because the statement claimed that the function is differentiable, but Grad did not seem to be aware of this. His manipulation lacked fluency but he was able to give a proof that had the fundamental essentials.

$$f(x) = f(-x)$$

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} = 0$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h}$$

$$=$$

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x-h)}{2h} = \lim_{h\to 0} \frac{f(x-h) - f(x+h)}{2h}$$

$$f'(x) + f'(-x) = \lim_{h\to 0} \frac{0}{2h}$$

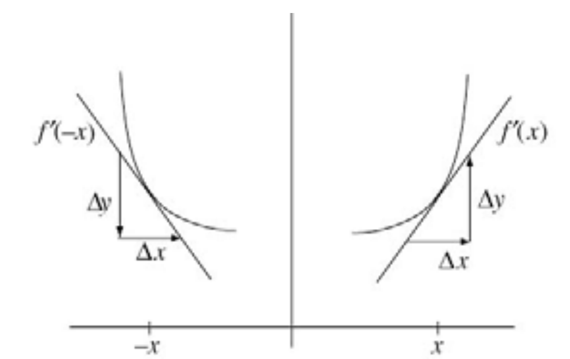
$$= 0$$

Figure 3: Grad's symbolic solution

Grad was then shown a number of pre-prepared responses to the problem, from which we focus on three:

Response A:

If $f(x)$ is an even function it is symmetric over the y-axis. So the slope at any point x is the opposite of the slope at $-x$. In other words $f'(-x) = -f'(x)$, which means the derivative of the function is odd.



Response B:

Want to show if $f(x) = f(-x)$ then $f'(-x) = -f'(x)$.

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h}$$ by the definition of the derivative.

$$f'(-x) = \lim_{h\to 0} \frac{f(x-h) - f(x)}{h}$$ since f is even.

Let $t = -h$

$$f'(-x) = \lim_{t\to 0} \frac{f(x+t) - f(x)}{-t}$$

$$= -\lim_{t\to 0} \frac{f(x+t) - f(x)}{t}$$

$f'(-x) = -f'(x)$, as required.

Response C:

Given $f(x)$ is even, so $f(x) = f(-x)$. Take the derivative of both sides. $f(x) = f(-x)$ by the chain rule. So $f'(x)$ is odd.

Grad considered the embodied visual proof (A) to show the best understanding and considered that a school pupil giving this response should be given a mark of 100% while a university student would probably get less than 50% because it needed further explanation. He thought response (B) was more formal and very convincing, but still in need of more explanation, so the mark given would "depend on the marker". He felt Response (A) and (C) were the best arguments, suggesting that Response (C) would get full marks, because it is more convincing and has "… less steps, the less mistakes you can make … less assumptions made … yeah, straightforward."

Grad is an example of a student who works to come to terms with formal ideas using embodied thought experiments yet is aware of the public aspect of formal proof and the need to build a proof in a way that will be approved by the mathematical community.

The role of prior experience and the notion of 'met-before'

Grad is just one individual responding to the notion of proof in his own way. However, *all* of us respond to new ideas based on our prior experiences. In Tall (2004). I introduced the notion of 'met-before' as a current cognitive structure that arises from previous experience and which is evoked to make sense of a current situation.

Met-befores are essential in curriculum building and curriculum builders necessarily sequence topics so that later topics build on earlier ones. However, such sequences are often conceived in a logical way, building new mathematical structures on previously introduced mathematical ideas. In practice, the student builds on his or her own personal met-befores in ways that may not fit with the intended new developments.

Different approaches to proof based on embodiment and symbolism

The shift to the formal world initiates students into the culture of mathematicians who base their public communications in terms of formal definitions and proof. Students trying to make sense of this new culture must build on their experience of embodiment and symbolism.

Pinto (1998) found that, when introduced to the formal limit concept in analysis, some students built explicitly on their previous embodied images in an attempt to *give meaning* to the formal idea from their current knowledge structure. She called this a *natural* approach. Others attempted to gain insight by focusing on the definition and the way it is used in formal proofs to *extract meaning* from the definition to give a *formal* approach.

Chris used a natural approach to build from his embodied imagery. He imagined the definition arising from the picture of a graph of a sequence approaching a limiting value L by noting that given any desired error ε, there is a value of N such that the values of the sequence to the right of N lie in the range $L \pm \varepsilon$. Chris was able to build from his imagery to build relationships about limits, continuity in various contexts that both make sense to him and also satisfy the mathematical community. In some instances, such as the constant sequence 1, 1, 1, …, he could sense that it did not fit in with his met-before that a sequence *approached* a limit, nevertheless, he was perfectly happy that it fitted the definition (Pinto & Tall, 2002).

Ross, on the other hand, approached the task formally by repeating the definition until he could say it in full detail and then carefully reading proofs to see how they were constructed logically. Indeed, he made sense of the notion of limit from his experience of considering the convergence of sequences. In the case of the constant sequence 1, 1, 1, … he thought about the *speed* of convergence and remarked that some sequences converged faster than others, and that the constant sequence converged the fastest of all. While the natural thinker Chris adds the constant sequence to his imagery as a special case, the formal thinker Ross places it centrally in his concept of limit.

Many students made only partial attempts at natural or formal approaches. Cliff tried to make sense of the ideas based on his embodied imagery. For instance, his image of continuity was based on the met-before that a continuous graph is drawn without taking the pencil off the paper. When the lecturer proved that an arbitrary function defined on the integers is continuous, Cliff did not believe it because the graph of the function consisted of disconnected dots over integer values. He failed to overcome the conflict between his embodied imagery and the formal theory.

Another student, Rolf, built upon his experience working with symbolic calculations and saw his initial task to gain fluency at using the definition of the limit of a sequence. This entailed working with specific sequence, for example, $a_n = 1/n^2$, and for a specific numerical value of ε, say, 10^{-6}, then he could calculate the value of $N = 10^3$ for which $a_n < \varepsilon$ when $n > N$. However, this strategy is not sufficient to give the limit concept its full logical meaning A problem which proves difficult for such procedural students is to show that if $a_n \to 1$, then for n beyond some value N, the terms a_n must be bigger than $\frac{3}{4}$. Not knowing the formula for a_n in this problem means that it is not possible to carry out a *numerical* calculation to find N (Pinto 1998).

Weber (2004) refined this analysis by a qualitative case study on a particular analysis lecturer and his students. He found that the lecturer began with an initial *logico-structural* teaching style in which he guided the students into constructing a sequence of deductions to prove a theorem. He divided his working space on the board into two columns, with the left column to be filled in with the text of the proof and the right column as ‘scratch work’. He wrote the definitions at the top of the left column and the final statement at the bottom, then used the scratch work area to translate information across and to think about the possible deductions that would lead from the assumptions to the final result. Later in the course, he became more streamlined in his proofs, working in a more sequential *procedural* style, writing the proof down in the left column and using the right column to work out detail such as routine manipulation of symbols. At a later stage, he used what Weber termed a *semantic* style, teaching topological ideas building on visual diagrams to give meaning, then translating this embodiment into formal proof.

His students learning approaches were analysed into three types, building on the theory of Pinto:

- a *natural* approach involved giving an intuitive description and using it to lead to formal proof,
- a *formal* approach where students had little initial intuition but could logically justify their proofs,
- a *procedural* approach where students learnt the proofs given them by the professor by rote without being able to given any formal justification.

The term ‘natural’ corresponds to that of Pinto in terms of giving meaning from intuitive (embodied) knowledge, ‘formal’ now refers to those who are *successful* in following a formal approach and ‘procedural’ refers to those who attempt to learn the formal proofs by rote without either embodied or logico-structural meaning. Thus the students mentioned in Pinto’s research, Chris is successful in giving embodied meaning to formal theory via a ‘natural’ route. Ross is successful in a ‘formal’ approach in extracting meaning from the definitions and the logical structure of theorems. Cliff is prevented from making sense of the formal procedures because they conflict with his embodied imagery. Rolf attempts to extract meaning from the definitions based on his symbolic experience and remains ‘procedural’ in the sense of Weber. In essence Weber’s ‘procedural’ route is taken by both Cliff and Rolf, but there is a difference: Cliff experienced a sense of conflict because it contrasted with his embodied ideas, but Rolf was happy to learn procedures by rote.

Weber’s data shows that students can vary in approach dependent on the context in which they work. Six students interviewed after the course all responded in a natural manner to a topological question (where topology had been taught in a semantic manner building from visual imagery). However, in two other questions about functions and limits, only *one* student responded naturally. The other responses to a question on functions were 4 formal and 1 procedural, and to a question on limits were 2 formal and 3 procedural.

From formal definitions back to embodiment and symbolism

We now have evidence that students moving towards formal proof have a variety of ways of building on their previous experience of embodiment and symbolism as they attempt to make sense of formal proof as shared by the community of mathematicians. Tall (2002) analyses in general how previous experiences provide concept images for thought experiments that may support formal proof. In parallel, concept definitions that arise are used as a basis for formal proof of a succession of theorems. A natural approach builds formalism on intuitive embodied imagery, supported by experiences in calculation and computation. A formal approach focuses less on embodiment and more on the logical structure, (figure 4).

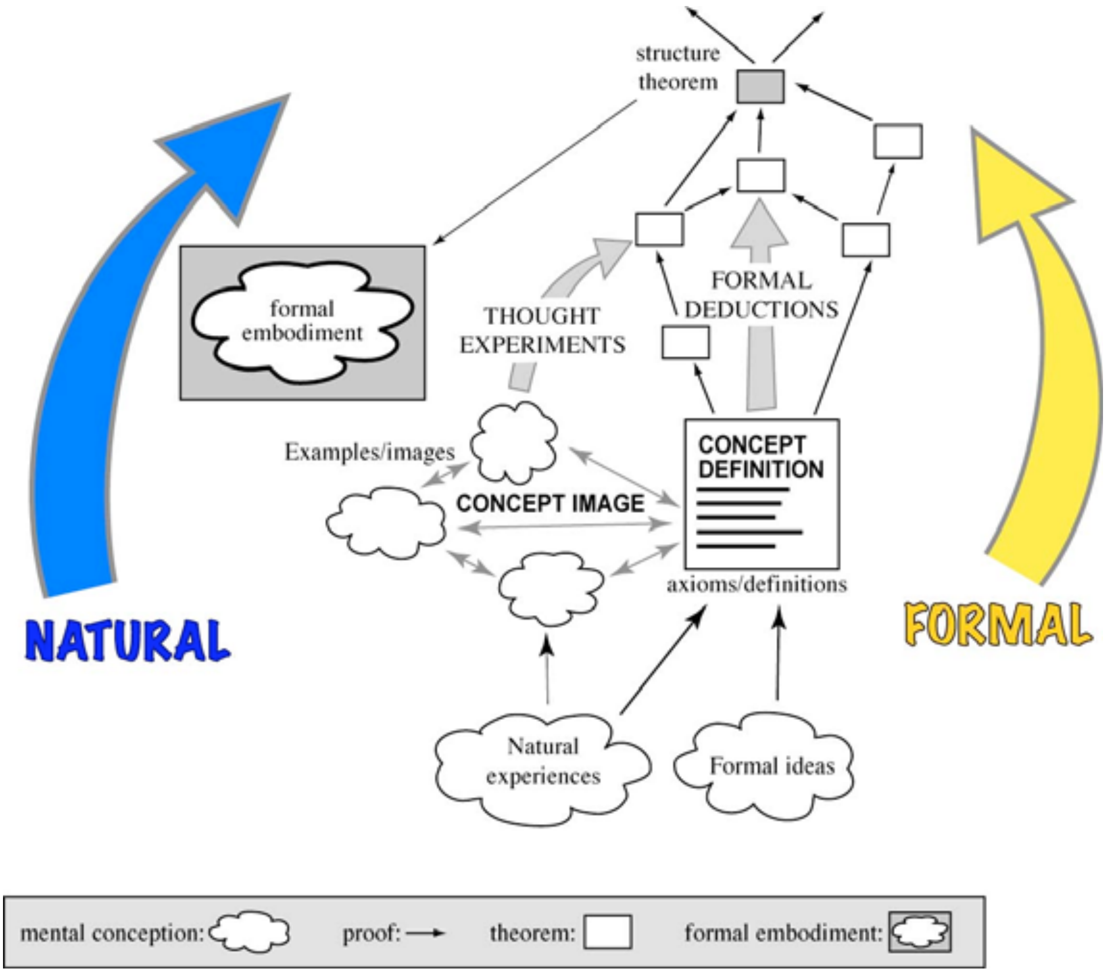


Figure 4: From embodiment to formalism and back again (Tall, 2002)

However, as successive theorems are proved by formal deduction, there may come a special type of theorem called a *structure theorem* that gives an insight to the structure of the axiomatic system itself. Typical examples of such structure theorems are as follows:

- An equivalence relation on a set A corresponds to a partition of A;
- A finite dimensional vector space over a field F is isomorphic to F^n;
- Every finite group is isomorphic to a group of permutations;
- Any complete ordered field is isomorphic to the real numbers.

In every case, the structure theorem tells us that the formally defined axiomatic structure can be conceived an embodied way and in many cases there is a corresponding manipulable symbolism. For instance, an equivalence relation on a set A—axiomatized as reflexive, symmetric and transitive—corresponds to an embodiment that partitions the set. Any (finite dimensional) vector space is essentially a space of n-tuples that can (in dimensions 2 and 3) be given an embodiment and (in all dimensions) can be handled using manipulable symbolism. Any group can be manipulated symbolically as permutations and embodied as a group of permutations on a set. A complete ordered field specified as a formal axiomatic system corresponds precisely to the symbolic system of infinite decimals and to the embodied visualisation of the number line. Thus, not only do embodiment and symbolism act as a foundation for ideas that are formalized in the formal-axiomatic world, structure theorems can also lead back from the formal world to the worlds of embodiment and symbolism. These new embodiments are now fundamentally different, for their structure is built using concept definitions and formal deduction.

New embodiment and symbolism may be a springboard for imagining new developments and new theorems; it may not. For instance, the embodied interpretation that a complete ordered field is the real line gave generations of mathematicians the belief that including the irrationals completed the real line geometrically by 'filling in the gaps between rationals'. This is not true, for it is possible to imagine (as did earlier generations) that the embodied number line has yet more elements that are infinitesimally close, but not equal, to real numbers. Using a symbolic representation of an ordered field containing the real numbers (such as the field of rational functions with an appropriate order) it is a simple matter to construct an ordered field containing infinitesimals. The structure theorem tells us that such a field not technically complete.

The structure theorem that every group is isomorphic to a group of permutations does not really help us solve problems symbolically for the use of permutations in large finite groups becomes so unwieldy that other techniques need to be developed to prove more sophisticated theorems. However, the structure theorem does provide an insight that brings some sense of unification between the theory of three worlds of mathematics.

Subtle embodiments of formal definitions

Embodiment is a natural mode of operation for human beings. Axiomatic structures arise from everyday concepts of counting, measuring, sorting, ordering, sharing, moving, categorising, thinking. In his famous lecture given at the turn of the twentieth century, Hilbert (1900) asserted:

To new concepts correspond, necessarily, new signs. These we choose in such a way that they remind us of the phenomena which were the occasion for the formation of the new concepts. So the geometrical figures are signs or mnemonic symbols of space intuition and are used as such by all mathematicians. Who does not always use along with the double inequality $a > b > c$ the picture of three points following one another on a straight line as the geometrical picture of the idea "between"?

However, not only are embodied ideas an inspiration for formal theories, we know that they can also act as met-befores causing obstacles in understanding.

Definitions are built up from individual parts that I call 'definitional elements'. For instance, an equivalence relation has three definitional elements:

R: $a{\sim}a$ for all a;

S: $a{\sim}b$ implies $b{\sim}c$;

T: $a{\sim}b$ and $b{\sim}c$ implies $a{\sim}c$.

Another possible definitional element might be:

T*: $a{\sim}c$ and $b{\sim}c$ implies $a{\sim}b$.

It is easy to prove that the definition **RST** gives the same structure as **RST***. However, the definitional elements **T** and **T*** are *not* the same; they have different meanings in themselves and in other systems. For instance, **T** is a definitional element for an order relation ($a<b$ and $b<c$ implies $a<c$) but **T*** is not (since $a<c$ and $b<c$ does *not* imply $a<c$).

Asghari (2004) investigated how individuals who had not met the concept of equivalence might write down their own rules to formulate the structure and found that an entirely unexpected law arose. His problem concerned a mad dictator who restricted travel between the ten cities in his country, so that a 'visiting-city' that one is allowed to visit must obey two conditions:

1. When you are in a particular city, you are allowed to visit other people in that city.

2. For each pair of cities, either their visiting-cities are identical or they mustn't have any visiting-cities in common.

The problem was for his officials to formulate valid visiting laws, which they demonstrate on a 10×10 grid (figure 6). While the diagonal and reflection in the diagonal occurred often, the transitive law was more opaque and several respondents suggested an alternative that might be called 'the box law' (figure 7):

If three corners of a box (with horizontal and vertical sides) are in the relation, then so is the fourth corner.

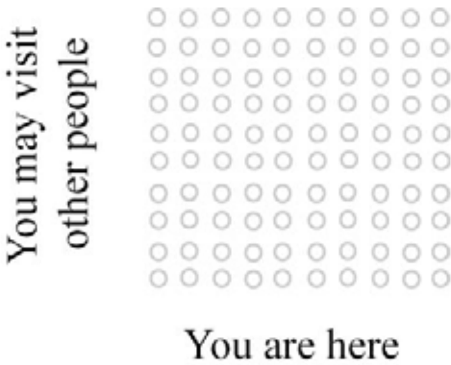


Figure 6: The problem grid

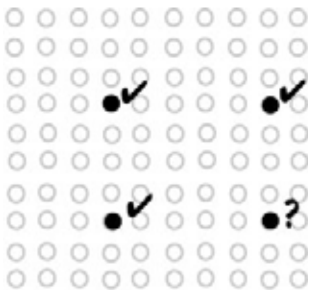


Figure 7: The 'box law'

It is left to the reader to find the relationship between this box law and the formulations **RST** and **RST*** for an equivalence relation mentioned earlier. A fundamental idea taken from this research is that the formal definitions given in mathematics are not necessarily evident for the learner and may not be 'natural' ideas that arise from the students' prior knowledge.

For many years, at the University of Warwick, students were first introduced to a 'foundations' course focusing on the development of the formal elements of mathematics. Over all this time, the topic that was consistently considered the most difficult was the introduction of relations and specific types of relations such as functions, order relations and equivalence relations. We were mystified why a simple idea like an equivalence relation should provoke such a reaction when it involved only the universal quantifier "for all" with none of the apparent difficulties coordinating multiple quantifiers in analysis.

Chin (2002) investigated the situation and it became apparent that the notion of relation was embodied very differently from the notion of equivalence relation. Whereas a relation from A to B has a natural representation as a subset of $A \times B$ and this is inherited by the concept of function from A to B, apparently students do not see an equivalence relation on A being naturally represented as a subset of $A \times A$. While the reflexive law **R** is easily embodied as the diagonal of $A \times A$ and the symmetric law **S** as reflection in the diagonal, the transitive law **T** is altogether more subtle.

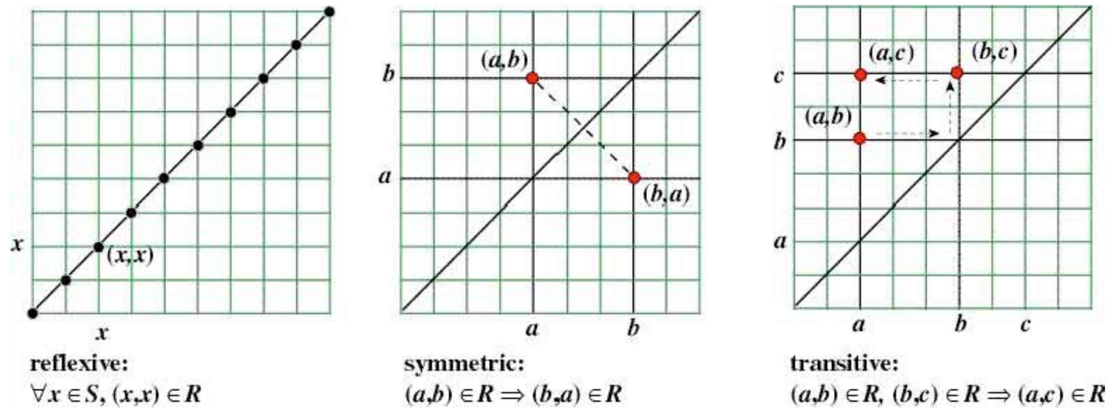


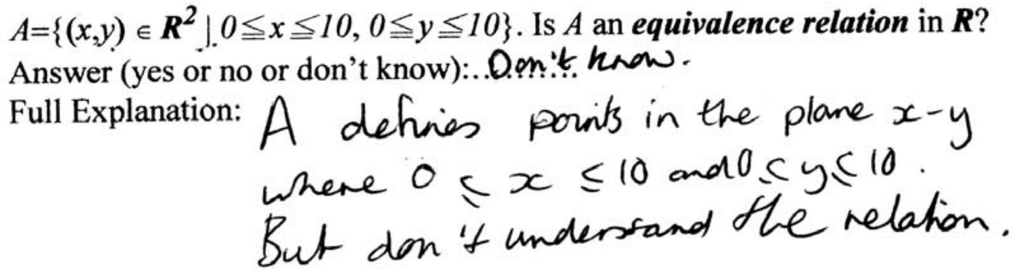
Figure 8: Visual representations of the axioms for an equivalence on a set ***R***.

Chin studied the development of 15 students over the first two years. Their marks for the first year were widely distributed—three over 80, four between 70 to 79, four between 60 to 69, one between 50 to 59, three between 40 to 49 (where an honours degree pass mark is 40 and a first class degree is 70).

He asked the students the following question:

$A = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \le x \le 10,\ 0 \le y < 10\}$. Is A an ***equivalence relation*** on $\mathbb{R}$?

In the first year *no* student responded positively to this question, even though they had been taught an equivalence relation on ***S*** is a subset of $\boldsymbol{S} \times \boldsymbol{S}$. Several wrote explicitly that they did not understand the question:


$A=\{(x,y) \in \boldsymbol{R^2} \,|\, 0 \leqq x \leqq 10,\ 0 \leqq y \leqq 10\}$. Is A an ***equivalence relation*** in ***R***?

Answer (yes or no or don't know): Don't know.

Full Explanation: A defines points in the plane x-y where 0 ≤ x ≤ 10 and 0 ≤ y ≤ 10. But don't understand the relation.

Figure 8: typical response of a first year student (Chin & Tall, 2001)

In the second year, only *one* student, Simon, gave a satisfactory response:

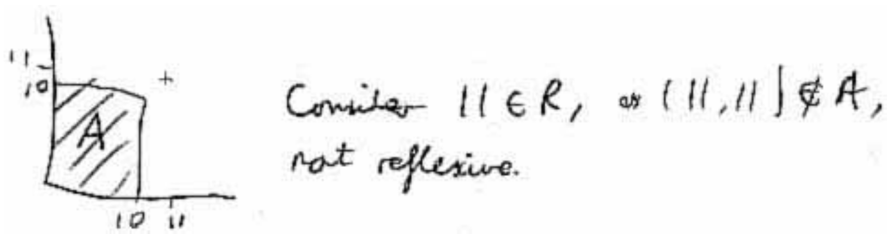


Figure 9: the single correct response in the second year (Chin & Tall, 2001)

Chin's work revealed that while it was usual for a student to have a mental picture for a partition (as a set *A* broken into disjoint subsets) they were far less likely to have a mental picture of an equivalence relation as a subset of $A \times A$.

In this way we find that the different kinds of relations—functions, order relations, equivalence relations—have very different embodiments. A function $f : A \to A$ lives in $A \times A$ while order relations and equivalence relations live in the set *A*.

Even though an order relation and an equivalence relation both live in the underlying set, their embodiments are fundamentally different. The embodiment of an order relation met-before by students is usually in the form $a < b < c$ where *a*, *b*, *c* are three points ordered from left to right. In arithmetic we would rarely write 5 ! 6 because we *know* 5 is *less* than 6. Likewise, we would not write 5 ! 5 because 5 = 5 . Thus in arithmetic we use the *strong* relation < rather than the weak relation ≤. Hence, although the transitive law $a \sim b$, $b \sim c$ implies $a \sim c$ has the same format in the case of an order relation and an equivalence relation, the (strong) order relation < has the property that the three elements *a*, *b*, *c* are *different*.

Chin found that many students found the following question difficult:

> Let $X=\{a, b, c\}$ and the relation ~ be defined where $a\sim b$, $b\sim a$, $a\sim a$, $b\sim b$, but no other relations hold. Is this an **equivalence relation**? If not, say why?

The simple answer is that it is not an equivalence relation because the reflexive law does not hold for $c \sim c$. However, a quarter of respondents (68 out of 277) asserted that this is not an equivalence relation not because of the reflexive law, but *because the transitive law fails*. In this case the transitive law is actually true, so there must be some subtle reason why this mistake was made. Typical responses included:

> No because $a \sim b$, $b \sim a \not\Rightarrow a \sim a$.
> need 3 elements for transitivity to hold.

and

> $a \sim a$ hence reflexive
> $a \sim b$ & $b \sim a$ hence symmetric
> However not transitive as c is not involved. ∴ not

Both of these students, and others in interview, asserted that the transitive law needs three *different* elements, *a, b, c* (Chin & Tall, 2002). The difficulties students meet from their encounter with relations, and three types of relations (function, order relation, equivalence relation) are in part bound up by the very different underlying embodiments that they bring to bear when they perform thought experiments to solve problems.

Reflections

The discussion presented here has shown how prior experiences (met-befores) from the worlds of embodiment and symbolism subtly affect formal meanings. Professional mathematicians depend on embodiment and symbolism to inspire their choice of theorems to prove and use symbolic manipulations in their proofs. Conversely, structure theorems can lead back to embodiment and symbolism. However, students who are learning about formal proof bring their prior experience to bear in different ways. Some build on their embodied experience in a natural manner to give meaning to the formal theory and some of these are successful. Some find their embodied ideas conflicting with the formal theory and find it difficult to make sense of the formalism. Others take a formal logico-structural approach and work successfully at the formalism, but others fail to complete such a formal programme, focusing on the procedures that are given in the definitions without making sense of the full impact of the formal theory.

This presents teachers of mathematics at university level with choices how to help students make sense of formal proof. Simply presenting the theory in a logical order and hoping the students will make sense of it will work for some. But to make sense of formalism requires students to gain some insight into how they think and to help them realise how their prior knowledge—which worked in perfectly well in previous contexts—may need re-thinking in the new context. Learning about the subtle coercive effects of met-befores gives a powerful new meaning to Ausubel's famous dictum from the opening pages of his monumental book on psychology (1968):

> "The most important single factor influencing learning is what the learner already knows. Ascertain this, and teach him accordingly."

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