STUDENTS' EXPERIENCE OF EQUIVALENCE RELATIONS

A PHENOMENOGRAPHIC APPROACH

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This paper is based on a doctoral study in which we studied 'lay' students' understanding of equivalence relations. The study was conducted by holding individual task-based interviews. We report a conceptual gap between "the everyday functioning of intelligence and mathematics" as to equivalence relations.

INTRODUCTION

Equivalence relations are seen to play an important role in mathematics. They are present in our formal treatment of numbers, integers, fractions, algebraic expressions, vectors, to name but a few. Skemp (1971, p.173) asserted that an equivalence relation is "one of the ideas which helps to form a bridge between the everyday functioning of intelligence and mathematics" and goes on to recommend, "it will be useful to start with every day examples before defining it mathematically". (ibid, p.173). However, in meeting a practical example, we should ask what it is that the learner attends to. In this paper we adopt a phenomenographic approach to see how a range of learners make sense of a situation which is intended to lead to the notion of equivalence relation. We find that different students react in very different ways to the same situation and that the building of the concept of equivalence relation occurs in different ways.

LITERATURE

In a sequence of papers, Chin & Tall (2000, 2001, 2002) consider the cognitive growth of "equivalence relation" and "partition" in the first year of a university course in which students are given the definitions and are expected to operate in an increasingly "theorem-based" manner (ibid, 2000, p.2). These papers advance a "theory of development of formal thinking" involving "a significant shift" from "informal concepts" to "formal concepts". However, by giving the students the formal definition of an equivalence relation, the studies focus on how students interpret definitions that are presented to them, and find that many students do not build logically from the definitions, instead they relate the properties to other experiences which lead to misconceptions. For instance the transitive rule 'a~b, b~c implies a~c' may be mentally linked to the order relation '>' where the property 'a>b, b>c implies a~c' carries with it the implicit property that a, b and c are all different. A student with this view may find it difficult to deal with cases where some of these elements are the same, for instance, to deduce from this property that if a~b and b~a then a~a.

As a result of working with students already being exposed to the formal treatment of equivalence relations and partitions the focus of the papers inevitably turned to

Published in H.L. Chick and J.L. Vincent (eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, University of Melbourne, Melbourne, vol. 2, pp. 81-88.

students' understanding and usage of the formal concepts. Thus, in a sense, we furthered their study by investigating one of the two parts overlooked by their study, the part as to "informal" conception of equivalence relations and partitions. The other part, the shift from informal to formal, needs an independent investigation.

This paper is the result of a detaild phenomenographic analysis of twenty verbatim transcribed audio-taped interviews with students with varied background experience, building on experiences reported in Asghari (2004a, 2004b). The participants comprised four middle school students, four high school students, one first year politics students, one first year law student, six first year mathematics students, two second year physics students, one second year computer science student, and one postgraduate student in mathematics. None of them had any *formal* previous experience of equivalence relations nor of the related concepts usually used to formulate the definition. In a one-to-one phenomenographic interview each student was introduced to a set of tasks that were designed having the standard definition of equivalence relations in mind (see below). The interviews had a simple structure; the tasks (see below) were posed in order, but the timing of the interviews and questions were contingent on students' responses.

It is worth saying that such a varied range of interviewees reminds us of "pure" phenomenography in which "the concepts under study are mostly phenomena confronted by subjects in everyday life rather than course material in school" (Bowden, 2000, p.3). In contrast, Bowden favours what he calls 'developmental' phenomenography in which the concepts under scrutiny are confined to a formal educational setting and the purpose of the study is to help the subjects of the research, or others with the *similar educational background* to learn. But, as it can be seen when the concept involved is as basic as an equivalence relation, the line between pure and developmental phenomenography fades out. In other words, it was the nature of the concept(s) involved that justified our 'choice' of the sample of the interviewees from whom the data were collected.

The Tasks

First, each student was introduced to the definition of a 'visiting law' while they were told that their first task would be giving an example of a visiting law on the prepared grids. (See the result section)

> A country has ten cities. A mad dictator of the country has decided that he wants to introduce a strict law about visiting other people. He calls this 'the visiting law'.
>
> A visiting-city of the city, which you are in, is: A city where you are allowed to visit other people.
>
> A visiting law must obey two conditions to satisfy the mad dictator:
>
> 1. When you are in a particular city, you are allowed to visit other people in that city.
> 2. For each pair of cities, either their visiting-cities are identical or they mustn't have any visiting-cities in common.

> The dictator asks different officials to come up with valid visiting laws, which obey both of these rules. In order to allow the dictator to compare the different laws, the officials are asked to represent their laws on a grid such as the one below.

After generating some examples (student-generated, ranging from one example to suggesting a way to generate an example) students were presented with the following three tasks:

> The mad dictator decides that the officials are using too much ink in drawing up theselaws. He decrees that, on each grid, the officials must give the least amount of information possible so that the dictator (who is an intelligent person and who knows the two rules) could deduce the whole of the official's visiting law. Looking at each of the examples you have created, what is the least amount of information you need to give to enable the dictator deduce the whole of your visiting law.
>
> The intersection task (we use this title and the title of the next task for the sake of this paper, we did not use them in interviews): One of the officials, for creating an example, uses other officials' examples: he takes two valid examples and put their common points in his own grid. Is the grid that he makes a valid example?
>
> The union task: Another official takes two valid examples and puts all of their points in his own grid. Is the grid that he makes a valid example?

Situation

In this section we informally outline the situation in terms of equivalence relations and related concepts. To do so, we shall use the eloquent, but still informal, account of equivalence relations given by Skemp (1971). To elaborate the idea, he starts with methods of sorting the elements of a parent set into sub-classes in which every object in the parent set belongs to one, and only one, subset (a *partition* of the parent set). He (ibid, p.174) considers two sorting methods: first, starting "with some characteristic properties, and form our sub-sets according to this"; and second, starting "with a particular matching procedure, and sort our set by putting all objects which match in this way into the same sub-set". The particularity of this matching procedure is in its "exactness", i.e. having an exact measure for the sameness; a necessity that if it is achieved, the matching procedure is called an *equivalence relation*. The exactness of the matching procedure also accounts for the *transitive* property. And the importance of the latter is that "any two elements of the same sub-set in a partition are connected by the equivalence relation" (ibid, p.175).

The matching procedure—as Skemp uses it—could shed some light on our task, where two cities are matched together if all their visiting-cities are the same, or two columns are matched together if they have the same status in each row (for a thorough analysis of the task see Asghari (2004a)).

RUSULTS

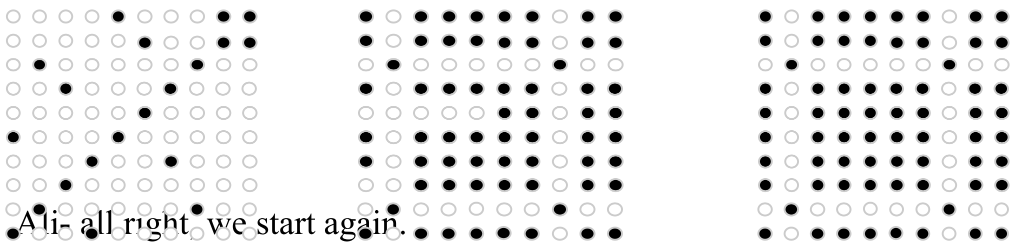
Our results concern the variation in students' focus of attention in this particular situation. We have determined three categories. The categories are characterized by experienced rather than those who experienced. In other words, as it is a phenomenographic slogan, we see the situation through students' eyes. As a result of this viewpoint, the very same student who at a certain time exemplified one of the categories, at another time could exemplify a different category. The categories are: *Matching procedure experience*, *Single-group experience* and *Multiple-group experience*.

Matching procedure Experience

In this category, the focus is on matching procedure; what students experienced and described is in terms of the elements involved, without resort to a group and/or groups of elements. Before giving an example, it is worth saying that somehow or other *the* defining properties of an equivalence relation determine an *exact* matching. So do the defining properties of a visiting law.

Now, let us have a look at what Ali (first year high school student) experienced when he was generating an example.

> Ali- I choose the very first things (points) haphazardly, and then I am going to correct the things that have been disturbed. (Have not been matched up)



> Ali- all right, we start again.

So he paired up city 1 with all the other cities, one-by-one; when two focal columns find something in common, he matched them up, and when they have been already matched or they have nothing in common, he left them as they were. Then he did the same process on city 2 and paired it up and matched it up (if necessary) with all the other city *after* city 2, and so on. The result of this long process was the middle figure above. Then he continued:

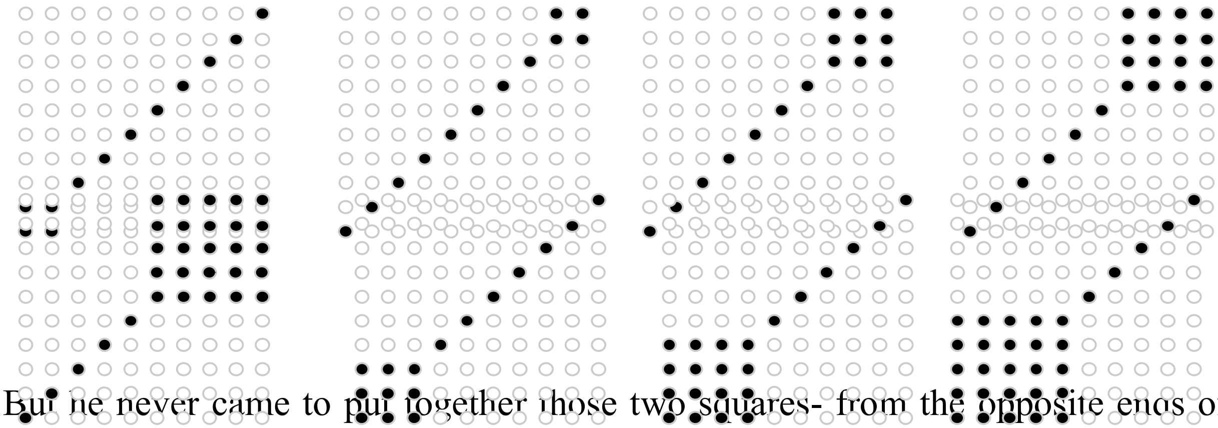
> Ali- Now, we are checking from start; *it is going to be full*.

And he did so. Eventually the process ended with the right figure above. As it can be seen, experiencing the situation only as a matching procedure Ali could not see which elements would be related to and grouped with each other.

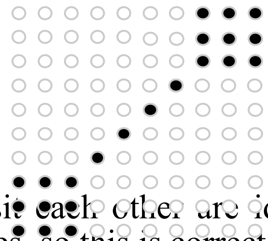
Single-group experience

In this category, focus is on only one single "group" while all the other elements that do not fall into that group are treated as individuals. The elements in the focal group *in one way or another* are related to each other while all other elements are in the background as individual elements. *Each* student in the present study could exemplify this category. However, we have chosen one which at the same time could exemplify different aspects of this category.

Kord (middle school student) has subsequently (sic) generated the following figures:



But he never came to put together those two squares- from the opposite ends of the grid- at his will; even later, when he put two of them together on demand of the union task he checked the status of the new figure by focusing on one square after the other; while he checked whether the square that he has been focusing on has been properly packed, he unpacks the other square and treated its elements on a par with all other individual elements.

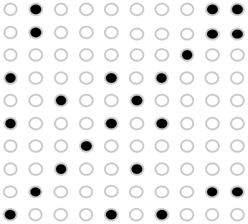


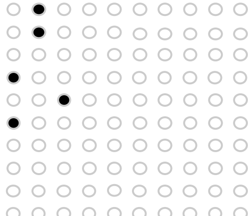
> Kord- … those that can visit each other are identical and they have no commonality with *other* cities, so this is correct (this is an example).

Let us give *a few* other examples from outside our data. Somewhere in his *informal* account of equivalence relation and partition, Skemp[1] (ibid, p.174) asked us to imagine that "we are standing on the pavement in London, and in a hurry to get to the station, then we may divide {passing objects} simply into the sub-sets {taxis} and {everything else}". Doing so, we probably could not remember when we went sightseeing in London we divided the very parent set into the sub-sets {double-decker buses designed for tourists} and {everything else}. And still in both situations we do not think of the other passing objects around the world. Given this, it seems in the most *practical* and/or *everyday* situation we, ourselves, could exemplify our second category, single-group experience!

Multiple-Group Experience

In this category, "disjoint groups" are experienced; the groups have no elements in common and the elements of each group are related to each other in one way or another. There are only three students who exemplify this category. Let us follow the youngest one (Hess, middle school student) when He was dealing with giving the least amount of information for the following figure on the left, which then was abbreviated to the figure on the right: ("abbreviated" is the way that Hess describes the figure with the least amount of information)





Hess- for example, one, five and seven make a group (it is the first time that he uses the word "group") with each other, so I only draw five and seven, It doesn't need (to do something) for five and seven, then I see two, nine and ten make a group with each other, I do for two these, it doesn't need for nine and ten; three and six make a group too, four nothing, *it make a group for itself*, for five, one, no five has been done(suddenly shift to the third category); *how many groups are they*? It's been finished, eight, it's been finished; that's it.

And then to explain that why this abbreviated figure uniquely determines the original figure he added:

Hess- there is only one case, *when we draw the diagonal, the groups are determined; and when the groups were determined there is only one case*.

Now, let us enjoy the great extent of the *operability* of this new idea:

After examining different arguments for the intersection problem he decided to work on the abbreviated figures, since "*their abbreviations are themselves"* and by using them *"our way would be simpler"*, he suggested.

Hess- suppose we have an abbreviation, suppose I am deleting certain points, even randomly, it still remain an abbreviation; *they have been divided into some groups that have no intersection with each other, certain different groups are created...* so if two abbreviations have intersection the intersection is some kind of abbreviation… (In other words) the remained figure is again the abbreviation of another figure.

Reflexivity, Symmetry and Transitivity

Looking at the above categories, one may wonder what about reflexivity, symmetry and transitivity, three properties that constitute our normative conception of equivalence relation. Let us start with reflexivity that is "quite unnatural" (according to one of the anonymous referees of one of our other paper) and at odds with the everyday functioning of intelligence. Though in our case "the artificiality of the situation seems to have given a natural flavour to it" (according to the very same referee), there is still much more to it, in particular when we consider that matching means matching *two* things. As we have discussed it elsewhere (Asghari, 2004a), we leave it here.

As opposed to reflexivity, symmetry seems to be the most *natural* properties of a matching procedure; simply two things are matched together. To see how natural it is, let us recall the example given in matching procedure category where Ali matched up all *possible* pairs to guarantee examplehood of his figure; however, not quite all possible pairs! Taking symmetry of the matching procedure for granted, he only needed to match forty-five pairs of cities not ninety pairs, as he did so. Saying all these, we should also add that the ways that our students experienced the geometrical symmetry of each example (see any one of the above examples) or the more algebraic form of symmetry (if (a, b) then (b, a)), and the ways that all these different form of symmetry are related to our categories need a more lengthy paper. Thus let us, as we did to reflexivity, leave symmetry here and turn our attention to transitivity. To do so, let us start by recalling Skemp's words:

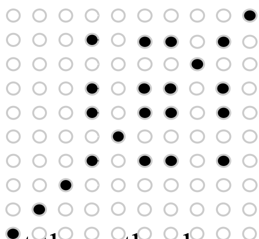
> The importance of the transitive property is that *any* two elements of the same sub-set in a partition are connected by the equivalence relation".

Given this, it seems that the transitive property is what that makes the vague phrase used in the second (and third) category clear; where we said that "the elements in the focal group *in one way or another* are related to each other". However, what our students experienced in each single group (of related elements) was *F-transitivity* rather than transitivity; where the following property is what we mean by F-transitivity:

If two objects are equivalent to a third, then they are also mutually equivalent (Freudenthal 1966, p.17).

As a matter of fact, Freudenthal (ibid, p.17) defined equivalence relation by reflexivity and what we call F-transitivity. However, he himself did not call it F-transitivity (he called it *transitivity*). The term F-transitivity is due to Bob Burn (personal communication).

Let us now come back to our data. Hess is about to explain why the following figure that he has just generated is an example of a visiting law.



> Hess- I am going to show that those that have commonality with four are equal to it.

And he did so. And shortly after that, while generalizing his argument he added:

> Hess- For each column we check that those that are equal to it, those that must be equal to it, are they equal to it or not.

CONCLUSION

Our data suggest that by the standard (and mathematical) treatment of equivalence relation and partition in which we jump from the former to the latter and vice versa, we ignore a *gap* in *everyday* experience of the subject, i.e. single-group experience.

And if for some purposes we do form our focal single-group by a certain matching procedure, it is likely the experience of F-transitivity (not transitivity) that saves us from matching all possible pairs; however, *logically* both amount to the same thing.

Even though we used the above tasks only as research devises, let us end by quoting a lecturer (in one of the top five ranked universities in the UK) who used them for teaching purposes in a class composed of fifteen prospective teachers. Doing so, we should also add that the potential pedagogical implication of this task needs further research.

> The students worked in groups to try to invent new visiting laws. They quickly discovered that just the diagonal and the whole grid were valid laws... one group produced a generic visiting law where each identical equivalence class was coloured the same. They independently 'discovered' the notion of equivalence classes (although they didn't use this terminology of course) and came up with the two main theorems I had on the next seminar's lesson plan.

End notes

1- Skemp himself used this example to illustrate that characteristic properties do not necessarily have to have a characteristic property.

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