A Theory of Mathematical Growth through Embodiment, Symbolism and Proof[1]

David Tall

Mathematics Education Research Centre
University of Warwick, UK
e-mail: David.Tall@warwick.ac.uk

This presentation considers the biological and mathematical mechanisms involved in the development from the child to a mathematician and theorizes how individuals grow in different ways over a life-time's experience. The theory is then used to respond to questions on the long-term teaching and learning of mathematics over the whole curriculum from child to adult.

INTRODUCTION

This paper is written in the context of the conference on *Mathematical Learning from Early Childhood to Adulthood*, which focuses on the *overall* growth of mathematical thinking in individuals. It is in two parts, the first presents a framework of long-term cognitive growth and the second uses this theory to address questions posed for discussion at the conference related to long-term learning.

THE COGNITIVE GROWTH OF MATHEMATICAL THOUGHT FROM CHILD TO ADULT

The overall framework

In studying mathematical learning from early childhood to adulthood we are involved with two very different frameworks. One is the coherence and structure of mathematics, the other is the biological development of the human mind. The brain has a number of essential aspects that enable us to build a highly sophisticated mathematical mind. First it has a complex parallel-processing structure that carries on many routine operations subconsciously but needs to focus on a small number of conscious items to be able to make coherent decisions. This in turn requires the complementary aspects of mental compression and connection:

- *Compression* of important ideas into thinkable concepts that can be held in the focus of attention.
- *Connections* between such thinkable concepts to can be built into dynamic *action-schemas* that connect successive actions

[1] Written for the *International Colloquium on Mathematical Learning from Early Childhood to Adulthood*, organised by the Centre de Recherche sur l'Enseignement des Mathématiques, Nivelles, Belgium, 5-7 July 2005.

in time, and more general *knowledge schemas* that connect ideas together in webs of relationships.

Compression may occur in a variety of ways:

- Action-schemas may be practised so that they can be performed automatically with little conscious effort, and imagined as a whole as thinkable processes.
- Processes may be further compressed into thinkable concepts, often by using a symbol to refer both to the process (eg 2+3 as addition) and to the concept (2+3 as sum). A symbol that can be used to switch between a do-able process and a think-able concept is called a *procept*[2].
- Concepts may be categorised and named so that the names can be held in the focus of attention to refer to the categories as thinkable concepts. This occurs in geometry where different figures are categorized to give hierarchies such as square, rectangle, parallelogram, quadrilateral, polygon, each with its own array of related properties. It also happens in arithmetic and algebra with concepts such as prime number, square number, irreducible polynomial, and so on.
- thought experiments can lead to connections between properties, in the form of 'if this property holds, then so does that property'. The results can be used as thinkable concepts to develop further structures.
- In formal mathematics, generative properties can be listed to define axiomatic systems that are named and considered as thinkable concepts in further hierarchies of mathematical theory.
- At higher levels, knowledge schemas such as 'whole number arithmetic', 'euclidean geometry', or 'category theory' may be named as thinkable concepts.

The general biological faculty that enables this to happen is the strengthening of links between neurons that prove successful and the suppression of others that are less relevant, building mental modules that operate together in consort in a process termed *long-term potentiation*. Such links may be helpful in new situations, but may also be subtly misleading, so that the cognitive development of mathematical ideas is by no means a simple logical development.

---

[2] See Gray and Tall (1994) for a technical discussion of the notion of procept.

The need to compress knowledge to fit into the focus of attention means not only making links that are important, but also suppressing conscious links to ideas that are less essential to decision-making. Thus real-world meaning, which is essential in the initial building of ideas may become less important as links between symbolic ideas become more important and the individual's thinking focuses more on mathematical activities and less on the original perceptual sources.

The three mental worlds of mathematics

The long-term construction of mathematical knowledge uses the power of the biological brain, with input through *perception*, output through *action* and the internal power of *reflection* to re-assemble ideas into useable mental structures. I hypothesise that mathematical thinking evolves through three linked mental worlds of mathematics, each with its own particular way of developing greater sophistication (Tall, 2004):

- An object-based *conceptual-embodied world* reflecting on the senses to observe, describe, define and deduce properties developing from thought experiment to Euclidean proof.
- An action-based *proceptual-symbolic world* that compresses action-schemas into thinkable concepts operating dually as process and concept (procept).
- A property-based *formal-axiomatic world* focused to build axiomatic systems based on formal definitions and set-theoretic proof.

Childhood: the origins of mathematical thought

Our mathematical growth is built on the same biological abilities that are part of our genetic inheritance at birth and develop through a life-time of experiences. We are not born as *tabula rasa* on which ideas may be written as on a blank slate; at birth we already have many built-in abilities that are in the process of maturation. For instance, a new-born child has a visual structure that not only builds a picture of the light falling on the retina, but also has a hierarchy of neuronal structures to detect colours, changes in colours, changes in shade, edges, to put this information together to distinguish objects from background, and to track the movement of these objects.

This gives us (and other species) the ability to observe one or more objects and to have a primitive sense of 'numerosity' already set in our cognitive structure. Children can track the movement of one, two and perhaps three objects, so that if two separate objects are seen being put behind a screen and the screen is removed to reveal only one object, a child of even a month old may stare longer and seem to express surprise.

Set-befores and met-befores

A mathematical concept (such as numerosity of small sets) which is with us at birth or soon after, I call a *set-before*, because it is set before our birth in our genes. As individuals meet new contexts, they build new ideas based on mental structure that they have at the time. A previous construction that is recalled to address a current situation is called a *met-before*. In practice the distinction between set-before and met-before is less important than the way in which both predispose us to think in new situations. For instance, scientific concepts such as force and momentum are subtle combinations of genetic and experiential origins that are deeply intertwined. In what follows, the term 'met-before' will be therefore be used to refer to either or both.

The value of using a catchy term such as 'met-before', as opposed to a more technical term relating to prior knowledge, is that it can be used in conversation with learners to explore ideas that may have previously made sense in a particular context but is now causing problems. For example, subtraction is initially met as a physical 'take-away' that carries with it a met-before that 'you can't take away more than you have, because you can't have less than zero.' This met-before remains valid for taking away physical objects but no longer applies in contexts such as temperatures below zero, credits and debts, or the number line.

In building a curriculum, designers focus on the positive effect of met-befores, such as the way arithmetic of counting numbers generalises to fractions, decimals, and real numbers on the number line. However, there is far less emphasis on strategies for dealing with the negative effects of met-befores that no longer work in a new context.

As an example, consider the transition from arithmetic to algebra. Students learning arithmetic have met before the idea that every arithmetic expression has an answer: $2+3$ *is* 5, but algebraic expressions such as $2+3x$ do *not* have an answer. Children mystified by this may find that they are being asked to manipulate symbols representing processes that are not themselves thinkable objects. They may focus on the part they can calculate to add $2+3$ to get 5 and leave the x they do not understand to give the erroneous answer $5x$. Another possible met-before is the use of letters standing for numbers ($a=1, b=2, \ldots$) so that $30-x$ is seen as 24; another is the met-before of place value where 23 is two tens and a three, so $2x$ for $x=3$ may be seen as 23 and for $x=17$, $2x$ may be 217 or 37. There is also the use of letters to stand for units, as in 1m = 100 cm, which can lead to the reversal error in the 'Student-Professor' problem where the proportion of six students for each professor may be interpreted as 6S=1P, even when S and P are given to represent the *number* of students and professors. The proliferation of a variety of different met-befores makes the analysis of mathematical thinking very messy. However, the theory of met-befores itself gives such analysis a coherent overall framework.

Cognitive growth is revealed as a story of each individual born differently endowed with an underlying set-before structure and having a variety of experiences that construct met-befores used later to develop highly individual mental capacities. Some cling to the security of old ideas, finding it difficult to shift from the evident 'truths' suggested by their met-befores; others focus on the new ideas and see their relevance and power in new contexts to shift their focus of attention to a new way of working. There is thus a growing spectrum from those learning procedures to solve an increasingly complicated collection of problems and those that see the simple power of new ideas in new situations.

Taking note of the neural Darwinism of Edelman (1992), it is as if *each event in our lives modifies the fitness of our capacity to respond to new events, and long-term potentiation acts as an agent for natural selection of an increasingly sophisticated cognitive system.*

Compression of knowledge through focus on effect

The fundamental idea in powerful cognitive growth is the compression of ideas into thinkable concepts that can be connected together in increasingly flexible ways. This is facilitated an important parallel between compression in the worlds of embodiment and symbolism identified by Poynter (2004). When teaching the notion of vector she noted an essential shift of attention from the *action* of a hand translating an object on a table to the *effect* of that action. The effect can be seen in terms of a free vector representing only magnitude and direction of the action, and the combination of two shifts is simply the free vector that has the same effect as following one action after the other.

This idea proves to be generally applicable to the compression of an action into thinkable concept, always provided that the learner is aware of the precise effect to focus upon. For instance, in sharing an object or collection into 4 equal parts and taking 2 of them gives a different number of parts from sharing into 6 equal parts and taking 3; but in terms of the *quantity* produced, each operation has the same effect. It gives a *half* of the original. Thus compressing action into effect is an embodied way of representing the formal idea of equivalent fractions. It shows a way in which the natural development of human thinking can lead at a later stage to a fundamental formal idea.

The shift from action to effect is of central importance in symbolism. For instance the symbols $2n+2$ and $2(n+1)$ involve quite different sequences of actions ('double a number and add two', or 'add one and double the result') but they have the same effect. The notion of function relies on this idea, for two different procedures that give the same effect are regarded as the same function. More generally different symbols representing different actions but having the same effect are considered to be different ways of representing the same procept (Gray & Tall, 1994). All these instances exemplify the way in which the relation between action and effect gives parallel constructions in the worlds of conceptual embodiment and proceptual symbolism.

Successive shift in focus gives a steady compression of knowledge from step-by-step procedure, to the possible choice of several different procedures, to seeing the overall effect as a general process that can be carried out in various ways, to encapsulating it as a thinkable object through the use of symbols. While it may happen that learners at different stages of compression may be able to solve a particular problem, the manner of solution and the consequences of long-term development of learning can be very different, moving from rigid use of a single procedure through increasing flexibility to symbolic operations on thinkable concepts (figure 1).

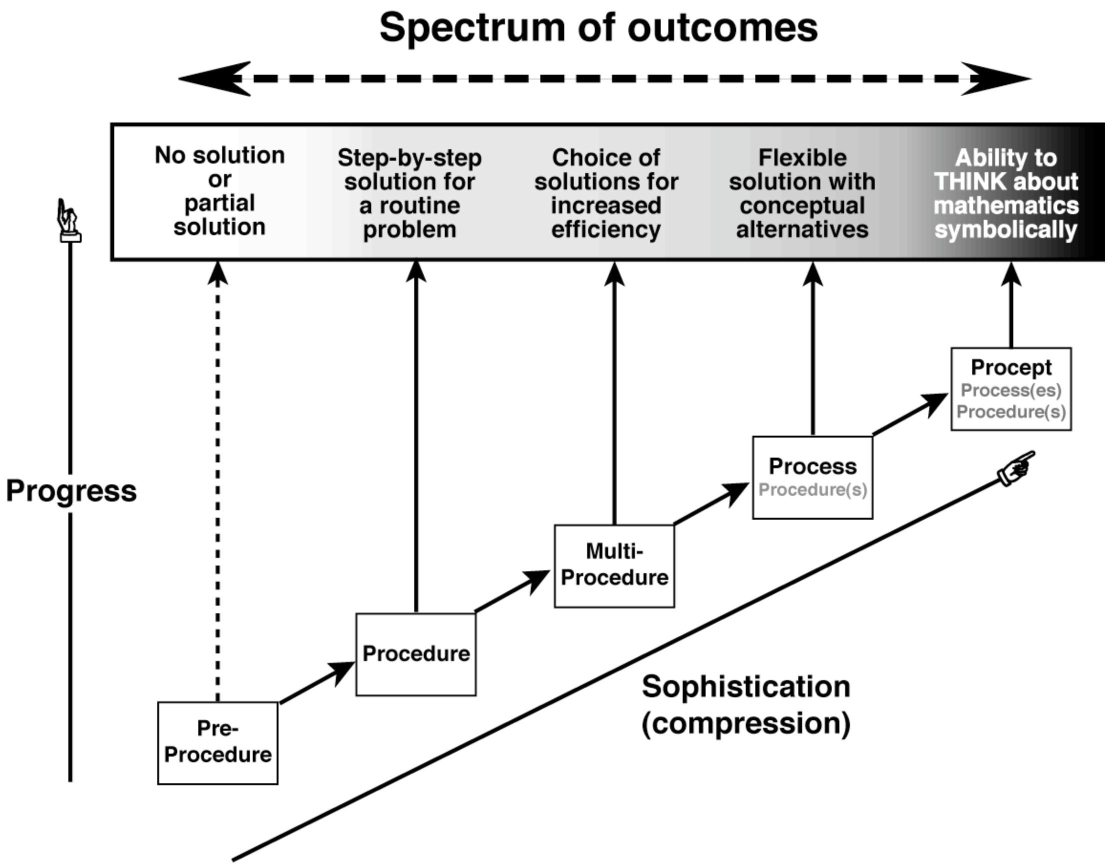

Figure 1: Spectrum of outcomes from increasing compression of symbolism (expanded from Gray, Pitta, Pinto & Tall, 1999, p.121)

Increasing complexity and simplicity

As the two worlds of conceptual-embodiment and proceptual-symbolism become more sophisticated, there are significant differences in the ways they operate. In particular, as each situation is replaced by a more sophisticated context, some embodiments can become more subtly complex while, in a very genuine sense, the mathematical meanings often remain simple.

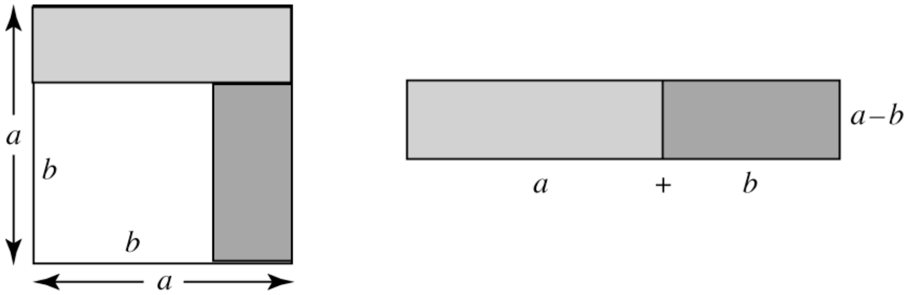

Figure 2: Take a square side *b* from a square side *a* and rearranging as $(a–b)\times(a+b)$

For instance. in measurement the product of two lengths is an area, which can be used to visualise algebraic formulae such as the difference of two squares; this can be 'seen' by picturing a square of side *a* and removing a smaller square of side *b* and rearranging what is left to 'see' $a^2 - b^2 = (a-b)(a+b)$. (figure 2).

However, this only works for *positive* values of *a* and *b* when *a* is bigger than *b*. If directed numbers are considered, there are a range of different looking pictures to be taken into account. Figure 3 shows one of several cases introduced by Percy Nunn (1914) teaching mathematics education as Director of the Institute of Education in London in the early part of the twentieth century.

If we move on from quadratic expressions to cubic expressions (such as the difference between two cubes) then we need to see it in three

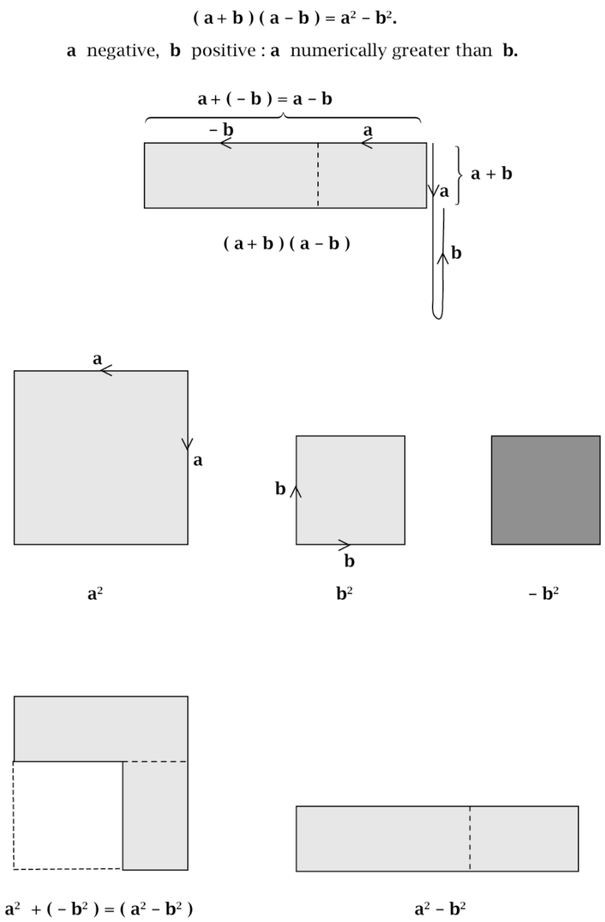

Figure 3: a directed number version

dimensions, and higher order expressions move into higher dimensions, accompanied by all the complications of a multipli-city of signs and sizes of the values. Meanwhile, algebraic expressions continue to work for all signs and sizes of values, based only on the coherent use of simple rules of algebra.

On the whole, as the mathematics becomes more sophisticated, more success-ful students tend to focus increasingly on the power of the symbolism than on the sensory meaning of the embodiments (Krutetskii, 1976; Gray et al (1998), Presmeg (1986)). These symbols too subtly change their cognitive meanings as the curriculum develops through arithmetic, algebra and symbolic calculus. Arithmetic invariably involves *operational* processes where symbols $3+4$, 5/9, … all have *answers* produced by carrying out an algorithm. In algebra, symbols such as $3+4x$ have *potential* processes that can only be carried out when the variable x is given a numerical value. The *potentially infinite* limit processes of sequences, series and the calculus usually tend to a limit without ever reaching it in a finite life-time. Each of these takes its toll of learners struggling to make sense of more sophisticated mathematics while those that embrace the new meanings find even greater power (Tall *et al.,* 2001).

Consider, for example, the symbol 2^3 which has a simple meaning as the repeated product of three twos ($x \times x \times x$); this extends naturally to x^3 as the product of three *x*s, or x^n as *n* *x*s:

$$(\underbrace{x \times x \times \ldots \times x}_{n \text{ times}}).$$

It leads naturally to the formula $x^{m+n} = x^m x^n$, valid for any real x and any whole numbers m and n. But the notation x^n cannot be thought of as 'n lots of x' if n is fractional or negative. The meaning no longer applies to $x^{1/2}$ as 'half an x multiplied together'. This can be a huge obstacle to many students struggling to make sense of a fractional or negative power. But for those willing to see what happens when the power law is used for $m = n = 1/2$ will find that $x^{1/2} \times x^{1/2} = x^{1/2+1/2} = x^1$, suggesting that $x^{1/2}$ can be given an entirely new meaning as the square root of x. Procepts such as $x^{1/2}$ have meaning, not because they have an operation that can be carried out, but because operating on them using the power law gives coherent results. Using them and building confidence in their coherence gives a sense of symbolic meaning quite separate from the original embodiment.

Reflection on properties leading to proof

As children develop cognitively, by reflecting on the properties of processes and objects encountered, they may build inferences such as 'if a triangle has two equal sides, then it will also have two equal angles', 'if two numbers are odd, then their sum is even', 'if I multiply $x+y$ and

$x-y$, then the product is x^2-y^2'. In this way a variety of means of proof develop in both the embodied world of objects (in particular of geometric figures) and the symbolic world of procepts dually representing process and concept.

The manner in which proof develops in the embodied world is well-represented by the geometric theory of van Hiele, starting from observations of global aspects of objects, to noting and describing specific properties, then using those properties in the form of definitions that can be used for deduction leading eventually to agreed conventions about congruent triangles used as a basis for Euclidean proof.

In the symbolic world, proof also starts by observing regularities. If I count 3 black balls and 2 white ones, I get the same result as counting the white ones first. When I do this with other numbers, the same thing happens. A change in focus from conceiving $3+2$ as a *specific* example to seeing it as a typical 'generic' example in its class, leads to a general principle of commutativity of addition. From here, generic examples in arithmetic can lead to generalised arithmetic using algebraic notation to give a symbolic representation of generality.

Throughout the embodied and symbolic worlds, definitions and deduction are based on experience of the properties of objects and of actions that are symbolised as procepts. A further shift becomes possible that focuses not on objects or on actions, but on *properties*.

The formal-axiomatic world expresses properties in general set-theoretic terms to turn mathematics on its head. Instead of building from perceptions and actions and teasing out their properties, we verbalise the definitions of mathematical structures by prescribing generative properties as *axioms* and deducing other properties by deductive proof.

This leads to very different methods of making arguments in the three worlds of mathematics. In the embodied world 3+2 is the same as 2+3 because I can *see* it is true. In the symbolic world 3+2 is the same as 2+3 because I can *calculate* it. In the formal world $x+y=y+x$ in a specific mathematical structure *because it is an axiom*.

Rodd (2000) explains these different ways of proof by looking at what *convinces* an individual of the truth of an observation, using the term 'warrant for truth' for 'that which secures knowledge'. In the embodied world, a warrant for truth occurs first through 'seeing' something is true, later developing more sophisticated warrants using agreed principles of deduction such as those arising in Euclidean geometry. In the proceptual world, a warrant for truth arises through calculating that a result is correct, or using the generalized arithmetic of algebraic manipulation to verify the required symbolic statement. In the formal world it is through specifying axioms and definitions set-theoretically and deducing theorems by formal proof.

The transition to the formal world requires a considerable change in approach in which the learner must build on the met-befores of embodiment and symbolism in elementary mathematics which need re-thinking to give the formal proof of axiomatic theories. According to Pinto (1998) some take a *natural* approach by performing thought experiments on imagery to *give meaning* to definitions and formal deduction, others take a more *formal* approach, *extract meaning* from definitions by working through given theorems until they make sense.

Weber (2004) studied students taught by analysis by a professor who first approached theorems in a logico-structural way, writing the beginning and end of the proof at the top and bottom of a column on the left and using a right-hand column as a scratch-pad to encourage students to think about the overall proof construction. Later he moved more quickly through the proofs focusing on the syntax in the left column and using the right hand column for working out detail. Then he moved on to topological ideas which he focused on the semantics by building on visual imagery in a more natural way. Students were likely to use different approaches depending on the context of the problems. In a topological question most used a natural approach, in questions on functions and on limits they used either a formal approach that gave logico-structural meaning to the deductions or a procedural approach which reproduced arguments learned by rote.

Professional mathematics have a huge range of experience and techniques to construct and prove theorems, including embodied imagery, thought experiments, and a range of symbolic manipulation and logical deduction. However, every mathematician began life as a new-born child, who could not speak, and had only genetic set-befores to start their journeys of mathematical growth. Many experiences are encountered and met-befores constructed on the journey to become a fully-fledged mathematician.

THEORETICAL IMPLICATIONS

In this part of the paper, I give brief replies to the following questions suggested for the conference:

1. How do the notions learned at elementary school influence later learning?
2. What are the respective roles in the learning process of procedures and concepts? What is the meaning of the expressions "mental representation", "mental object", "mental image" and "mental model"? How do these mental entities unfold and relate to each other?
3. On which basis and following what criteria should one organise mathematical matters to induce a kind of natural learning? How to elaborate guidelines? How to determine necessary passage points.
4. What are the respective roles of intuition and rigor? How could the requirements concerning both aspects be modulated?
5. What are the respective roles of problem-solving and theoretical structuring?
6. What is the role of logic?
7. What about past attempts to grasp mathematical learning globally, in terms of matters and methods? How did they deal with the above questions? How did these attempts affect school practice.

1. How do the notions learned at elementary school influence later learning?

A contribution has been made to this question in terms of the notion of 'met-before', both in terms of the conceptions that have longer-term value and those which work in one context but not in another. In the apparent logical structure of a mathematics curriculum, the biological brain will bring previous experiences to interpret the situations that are presented which can lead to unforeseen difficulties that arise through apparently sensible approaches. For instance, in England, the Inspectorate has for many years encouraged all mathematics teaching to be based on practical activities. However, we now know of many instances in which ideas built in one context fail in another and may need significant re-thinking in new contexts. Practical experiences help build up a coherent overall picture, but may contain implicit elements that act as impediments in future learning.

My current belief is that there is a need to analyse the cognitive growth of ideas to help teachers and students to address inappropriate met-befores when they are likely to occur. This is a long-term strategy over years rather than over the duration of a particular lesson or a particular course. A current major concern in the UK is that students are learning necessary procedures to pass national examinations, yet seem to lack the flexibility to solve multi-step problems at university.

This would suggest the introduction of discussion concerning how compression is required to produce thinkable concepts, in part by giving meaningful embodiments to actions to focus on the effect and in part to the corresponding symbolic phenomenon where focus shifts from the steps of a procedure to its effect. This focuses on the simplicity of the desired effect as opposed to the complication of many possible actions. It also uses the language of met-befores to discuss why ideas that may have been perfectly sensible in one situation need rethinking in another.

While it may be considered (as did Skemp, 1971) that long-term learning needs to take account at any given time of the long-term use of a particular concept, in practice, the student at a given point of learning must learn in a way consistent with his or her current knowledge. Therefore the introduction of inevitable met-befores at a given point will result naturally in the need to address what is necessary when shifting to a new context. Thus re-organisation of knowledge is an important part of curriculum building. At present it is an idea that is almost totally absent from most curriculum frameworks.

2. What are the respective roles in the learning process of procedures and concepts? What is the meaning of the expressions "mental representation", "mental object", "mental image" and "mental model"? How do these mental entities unfold and relate to each other?

Here I address only the first part of the question. The second is a diversion into the meanings given to a range of terms from different theoretical positions and a discussion of their meanings and related theories would cause me to stray too far[3].

The terms *procedural and conceptual knowledge* have been widely used (eg Hiebert & Lefevre, 1986, Hiebert & Carpenter, 1992). In the theory of three worlds, these terms take on refined meanings. Conceptual knowledge relates to the forming of knowledge schemas that can be used

[3] The terms mental representations, objects, images, models have different meanings in different theories, and even in a single theory, variations in meaning can occur. For instance, I share the notion of concept image with Shlomo Vinner (Tall & Vinner, 1981), yet he and I have quite different meanings, with his original meaning being a distinction between mental pictures and concept definitions which are separate cells while mine has a cognitive biological meaning constructed in the brain where the concept definition (if it exists) is part of the concept image. This significant difference in meaning has had no effect in the shared used of the term in the mathematical education community who are largely unaware of it. The term 'object' is equally used in a range of different ways, for instance, Dieudonné (1992) used it to refer to an axiomatic mathematical structure, others use it to mean a cognitive entity that can be manipulated (eg. Sfard 1991, Dubinsky 1991), There is further discussion in *What is the Object of the Encapsulation of a Mathematical Process* (Tall et al. 2000). The term 'object' is polysemous—it has many meanings—so a debate to get a universally accepted meaning is essentially doomed to failure.

in a flexible manner. Procedural knowledge relates to step-by-step actions before they are condensed into overall processes and crystallised into thinkable procepts. This relates to APOS theory (Action-Process-Object-Schema) to build am overall conceptual structure (Cottrill *et al.* 1996).

Procedural knowledge occurs in both the embodied and the symbolic worlds, with the possibility of an insightful linkage between embodied focus on the effect of an action and the corresponding notion of symbolic process-object encapsulation (figure 3).

Furthermore, as various concepts are built into knowledge schemas, these may themselves be encapsulated into thinkable concepts in a manner that Skemp (1979) imagined in his *varifocal theory*, in which concepts at one level can be seen in more detail as a knowledge schema and vice-versa.

On the face of it, therefore, embodiment can support process-object encapsulation and shift the thinker from the routine doing of mathematical procedures to flexible thinking about mathematics.

This link, however, also has a weakness, as we have seen. Basing thinking on a *specific* embodiment may give rise to met-befores that may inhibit thinking in a different context. There is therefore an exquisite tension between embodiment as meaning and embodiment as cognitive obstacle in an inappropriate situation.

This tension between different embodiments and their meanings leads

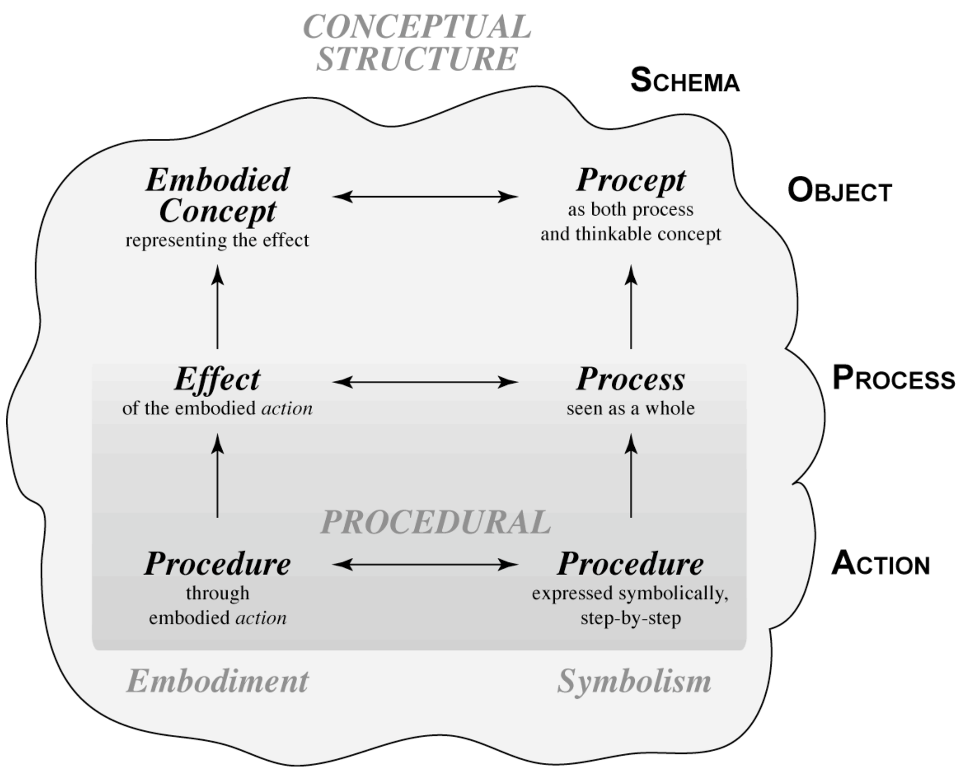

Figure 3: Procedural knowledge as part of conceptual knowledge

to a spectrum of performance in which some children cling to inessential aspects of a particular embodiment and fail to make flexible sense of the essential mathematical symbolism. Gray & Tall (1994) formulated the 'proceptual divide' between those individuals who move on to flexible manipulation of numbers as process and concept and those who remain with inflexible procedures. Gray *et al*. (1998) reveal how this bifurcation continues as the more successful make sense of flexible relationships between symbols (perhaps with continuing links to embodiment), while the less successful continue to be imprisoned in limited embodiment and inflexible procedures.

This gives a mechanism underlying the spectrum of approaches noted by Krutetskii (1976) and Presmeg (1986) between those who think symbolically, geometrically, or a harmonic blend of the two. The research evidence suggests that most able students tend to focus more on symbolism than on visualization, just as the mathematical community tends to value the symbolic in examinations and gives less emphasis to the visual.

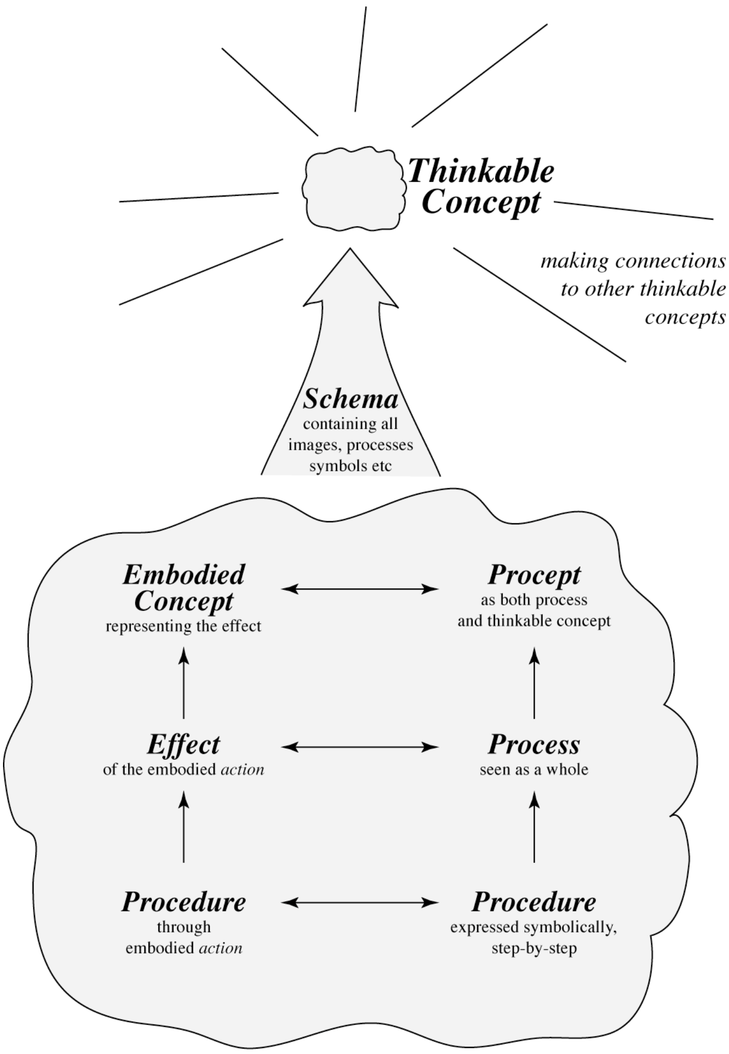

Figure 4: compressing a proceptual knowledge schema as a thinkable concept

However, significant mathematics is also created through a powerful combination of embodiment and symbolism leading to formal proof. A shining example is the recent Abel prize awarded to Atiyah and Singer (1962) for their Index Theorem yielding a formal proof relating topological and analytic aspects of a problem in n-dimensional space.

Building on embodiment and symbolism leads to different kinds of mathematical minds, some maintain fundamental links with embodiment and translate it into formalism, others continue to use the symbolism of arithmetic and algebra for more subtle purposes, and others move firmly into a formal world of definition and axiomatic deduction for public dissemination of ideas, though they may continue to use a private world of thought experiment and embodiment to inspire their formal theorems.

3. On which basis and following what criteria should one organise mathematical matters to induce a kind of natural learning? How to elaborate guidelines? How to determine necessary passage points.

The theoretical perspective presented in this article reveals a natural evolution of ideas making sense of met-befores that can enhance or inhibit the formation of later concepts. It also reveals a phenomenon in which different individuals develop different ways of addressing their problems in learning. Some remain limited to rigid embodiments and rote-learning of procedures in increasingly complicated confusion while others benefit from successive compressions of knowledge that make it simpler and more comprehensive.

I theorise that the idea of focusing on the effect of actions can be used to explain to teachers and learners how to develop more sophisticated ideas but it cannot be expected that *all* the population will benefit from any one style of teaching. Broad guidelines outlining these ideas may be helpful for both teacher and student, though the institutionalization of testing structures can have both positive and negative effects.

My own personal view is that the growth of the individual child needs a mentor as teacher to guide the child to focus on essential connections, which requires the teacher to understand not only the mathematical structure but also the role of previously constructed met-befores that need to be addressed in learning new topic areas.

The King’s College Project (Askew et al. 1996) studying teaching styles categorised teachers into those who taught by *transmission* in which the teacher gave information to the class, those who taught by *discovery learning*, giving children contexts where they could construct ideas for themselves, and those who were *connectionist*, actively helping children to construct important connections between ideas. It was no surprise to me that the most successful teachers by far were the connectionist teachers.

4. What are the respective roles of intuition and rigor? How could the requirements concerning both aspects be modulated?

The theory presented reveals intuition not just as a basic human faculty, but as a growing cognitive facility based on previously constructed met-befores. Our earlier discussion on warrants for truth reveals that thought experiments give a warrant for truth before the rigor of formal deduction is considered. There is a therefore a mismatch between the views of mathematicians who inhabit the formal world and the learner who is building upon embodiment and symbolism in contexts that grow increasingly more sophisticated.

Ideas of proof begin in the worlds of embodiment and symbolic operations where warrants for truth relate to what can be seen and what can be calculated. In such contexts it is possible to work with the world of the child to focus on explicit properties to see that certain other proerties must follow.

As we saw earlier, at different stages of development, different warrants for truth are likely to convince. The idea that addition is independent of order is evident in the early stages of the embodied world where re-ordering a collection of 9 objects and 2 objects gives the same number as 9 objects and 2 objects. It is self-evident because *it can be seen* to be true. It is less evident in the initial stages of counting where counting on 2 after 9 is fairly simple but counting on 9 after 2 is a much longer procedure. It becomes more evident when experience reveals that addition gives the same answer irrespective of the order of addition. Now 9+2 is the same as 2+9 because this always happens whenever any two numbers are added. It is true *by calculation*. Later, in formal mathematics, $x+y$ equals $y+x$ because it is assumed *as an axiom*.

Ideas of proof are therefore very different for children at different stages of development. The level of detail required in an argument must both respect the child's developing conceptions while at the same time seeking increasing clarity and precision.

Even in the formal world of mathematicians, proofs are rarely *strictly* rigorous. Mathematical proof works in a context where some truths are well-accepted and taken for granted, while important logical turning points are given greater attention. In *Foundations of Mathematics*, Stewart & Tall (1976) introduced the notion of *contextual proof* to students, so that as the mathematics became more complex, the focus of attention changed to the more important ideas. For instance, in building up the notion of complete ordered field, starting with the axioms of a field we needed to quote all the relevant axioms to deduce the truth of a new statement, but when the order axioms were introduced, arithmetic properties became part of the context and the proof only explicitly focused on the order axioms, then in the theory of analysis, the arithmetic and order properties were accepted contextually and the focus of attention shifted to the completeness property.

We believe that such a distinction between what has become contextually acceptable in a given context and what needs to be carefully documented is an art that needs to be made explicit to students.

5. What are the respective roles of problem-solving and theoretical structuring?

First we need to say what we mean by 'problem-solving'. In my own case I see this as a development of a strategy for solving problems in general, not simply a harder kind of exercise that occurs at the end of a structured approach. I taught a course I called 'problem-solving' for over twenty years, based on the book *Thinking Mathematically* by Mason, Burton and Stacey (1982). At the end of the course students rarely felt that they had learnt any new mathematics but they considered they knew a great deal more about how to formulate problems and to attack them in new situations. In particular, their *attitude* to mathematics changed in highly positive ways (Yusof & Tall, 1999).

My personal view is that a teacher as mentor can do a great deal by adopting a connectionist viewpoint to help each learner to address a problem by building on current knowledge. Such connectionism has a problem-solving sense of adventure to make new connections and to use existing connections in innovative ways. What *is* the problem? Can I make sense of it? Can I look at it in a more flexible way? This is a world away from the transmission of procedures that work in a familiar context but fail in slightly different situations.

6. What is the role of logic?

Logic as an abstract concept belongs in the formal axiomatic world, not the worlds of embodiment and symbolism. However, as in the development of conceptions of proof, the search for clarity and precision is a worthy enterprise in all education. The role of logic in the long-term curriculum needs to be rethought in the light of the different warrants for truth in different contexts, as part of a spiralling curriculum of increasing sophistication. Different individuals may respond to it in different ways.

7. What about past attempts to grasp mathematical learning globally, in terms of matters and methods? How did they deal with the above questions? How did these attempts affect school practice?

This is a big question that I can only deal with briefly. Essentially the curricula of earlier generations arose in part because of the need to educate a particular population for a certain purpose and then, with the beginnings of international activities from the mathematical community in the early twentieth century, to teach the mathematics that was conceived by the mathematical community to be essential. This reached its zenith in the 'New Mathematics' of the sixties and seventies, based on the premise that if children are taught the full truth of mathematics in fundamental terms then proper learning will occur.

The New Mathematics did not prove to have the desired universal effect, certainly not with the school population as a whole. The parallel development of constructivist mathematics brought about an alternative child-centred approach in which the child constructed his own knowledge. The 'Math Wars' in the USA are a sign of the great controversy still raging between math-centred and child-centred mathematics. A new synthesis is required that takes into account the rich structure of mathematics that is available to our culture, but at the same time, attends to the diverse ways in which the child's thinking grows in different individuals. My perception is that we need to see the coherence and simplicity of mathematics as a whole, but we need also to look at how different learners build these ideas over time so that hard-earned simplicity is constructed and re-constructed in ways appropriate for each individual.

References

Askew, M., Brown, M., Rhodes, V., Johnson, D., & Wiliam, D. (1997). *Effective Teachers of Numeracy*, Final Report of a study carried out for the Teacher Training Agency 1995–96 by the School of Education, King's College. London: King's College.

Atiyah, M. F. & Singer, I. M. (1963). The Index of Elliptic Operators on Compact Manifolds. *Bull. Amer. Math. Soc.* **69**, 322-433.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K. & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a co-ordinated process schema. *Journal of Mathematical Behavior*, *15*, 167–192.

Dieudonne, J. (1992) (translated J. Dales). *Mathematics-The music of reason*. New York: Springer.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. O. Tall (Ed.), *Advanced Mathematical Thinking*, (pp. 95–123). Dordrecht: Kluwer.

Edelman, G. M. (1992). *Bright Air, Brilliant Fire*, NY: Basic Books, reprinted Penguin 1994.

Gray, E. M. & Tall, D. O. (1994). Duality, Ambiguity and Flexibility: A Proceptual View of Simple Arithmetic, *The Journal for Research in Mathematics Education*, *26*, 115–141.

Gray, E. M., Pitta, D., Pinto, M. M. F., Tall, D. O. (1999), Knowledge Construction and diverging thinking in elementary and advanced mathematics, *Educational Studies in Mathematics*, 38 (1–3), 111–133.

Hiebert, J. & Carpenter, T. P. (1992). Learning and Teaching with Understanding. In D. Grouws, (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 65–97). New York: MacMillan.

Hiebert, J., & Lefevre, P. (1986). Conceptual and Procedural Knowledge in Mathematics: An Introductory Analysis. In Hiebert (Ed.), *Conceptual and procedural Knowledge: The Case for Mathematics*, (pp. 1–27). Hillsdale. N.J.: Erlbaum.

Krutetskii, V.A. (1976). *The Psychology of Mathematical Abilities in Schoolchildren* (Trans, J. Teller.,ed. J. Kilpatrick & I. Wirzup). Chicago IL: University of Chicago.

Mason, J. (1989). Mathematical Abstraction Seen as a Delicate Shift of Attention, *For the Learning of Mathematics*, 9 (2), 2–8.

Mason, J., Burton, L. & Stacey, K. (1982), *Thinking Mathematically,* Addison Wesley, London.

Nunn, T. P. (1914). *The Teaching of Algebra (including Trigonometry)*. Longmans: London.

Pinto, M. M. F. (1998). *Students' Understanding of Real Analysis*. PhD Thesis, Warwick University.

Poynter, A. (2004). Effect as a pivot between actions and symbols: the case of vector. *Unpublished PhD thesis*, University of Warwick. http://www.annapoynter.net.

Presmeg, N.C. (1986). Visualisation and mathematical giftedness, *Educational Studies in Mathematics*, 17, 297–311.

Rodd, M. M. (2000). On mathematical warrants, *Mathematical Thinking and Learning*, 2 (3), 221–244.

Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on processes and objects as different sides of the same coin, *Educational Studies in Mathematics*, ***22***, 1–36.

Skemp, R. R. (1971). *The Psychology of Learning Mathematics*, London: Penguin.

Skemp, R. R. (1979). *Intelligence, Learning, and Action*: *A foundation for theory and practice in education*, Chichester: John Wiley.

Stewart, I. N. & Tall, D. O, (1977). *Foundations*, Oxford University Press.

Tall, D. O. & Vinner S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity, *Educational Studies in Mathematics*, **12** 151–169.

Tall, D. O., Thomas, M. O. J., Davis, G., Gray, E. M., Simpson, A. (2000). What is the object of the encapsulation of a process? *Journal of Mathematical Behavior*, 18 (2), 1–19.

Tall, D. O., Gray, E. M., Ali, M. b., Crowley, L., DeMarois, P., McGowen, M., Pitta, D., Pinto, M. M. F., Thomas, M. O. J., Yusof, Y. b. (2001). Symbols and the Bifurcation between Procedural and Conceptual Thinking. *Canadian Journal of Science, Mathematics and Technology Education* 1, 81–104.

Tall, D. O. (2004). Thinking through three worlds of mathematics. *Proceedings of the 28th Conference of PME*, Bergen, Norway, 158–161.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: a case study of one professor's lectures and proofs in an introductory real analysis course, *Journal of Mathematical Behavior*, 23 115–133.

Yusof, Y. b. M. & Tall, D. O. (1999). Changing Attitudes to University Mathematics through Problem-Solving, *Educational Studies in Mathematics*, 37, 67–82.