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A MISMATCH BETWEEN CURRICULUM DESIGN AND STUDENT LEARNING: THE CASE OF THE FUNCTION CONCEPT

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This paper presents an analysis of the function concept. Although it is seen by many as a fundamental building block of the mathematical curriculum (NCTM, 1989), in practice, few students grasp the full extent of the notion of function linking across its various different representations. The data presented here comes from a research project at Warwick University, drawing data from schools in Turkey. It analyses data from questionnaires in two studies, together with interviews from nine students representing a spectrum of performance from the small minority who have an overall grasp of the function concept to the majority of students giving limited responses that do not connect across the various modes of representation.

INTRODUCTION

In recent years the function concept has been suggested as a fundamental idea to underpin the whole curriculum. The introduction of the 'New Mathematics' in the sixties in the UK and around the world attempted to base mathematics teaching on set theory and modern mathematical structures. In the USA, the NCTM standards stated:

> The concept of function is an important unifying idea in mathematics. Functions, which are special correspondences between two sets, are common throughout the curriculum, In arithmetic, functions appear as the usual operations on numbers, where a pair of numbers corresponds to a single number, such as the sum of the pair; in algebra, functions are relationships between variables that represent numbers; in geometry, functions relate sets of points to their images under motions such as flips, slides, and turns; and in probability they relate events to their likelihoods. The function concept is also important because it is a mathematical relationship of many input-output situations found in the real world, including those that recently have arisen as a result of technological advances. An obvious example is the √2 key on a calculator.
>
> (National Council of Teachers of Mathematics, 1989, p. 154.)

The problem with this curriculum design based on overall structural principles is that it hasn't been shown to work. Even when students are given proper set-theoretic definitions of functions, they seem not to use them. In Britain, a pragmatic approach to the giving of basic definitions leads to students remembering their experiences of what they actually did in the classroom rather than focus on the definition itself (Bakar & Tall, 1992). We hypothesise that what students remember relates to what they *do that proves successful*, rather than what they are *told* to do. Students who are first told the definition of a function, but then go on to study particular kinds of function, such as the linear function $y = mx+c$, are likely to remember some of the properties that they studied, for instance, how to draw a function through two given points, or how to find the slope of the graph through two points, both of which require specific properties of linear functions rather than of the function concept itself.

FUNCTIONS IN THE TURKISH CURRICULUM

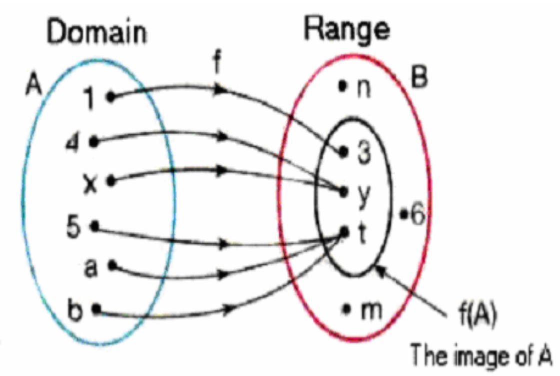
In the Turkish curriculum, the function concept is introduced in grade 1 in high school (15 year-old students). After the introduction of relations, equivalence and ordered relations, the function concept is given by the following definition:

> Definition: Let *A* and *B* be two non-empty sets. A relation from *f* from *A* to *B* is called a function if it assigns every element in *A* to a unique element in *B*.
>
> (Demiralp et al., 2000, our translation).

This formal definition is then rewritten in a more 'colloquial' manner:

> A function *f* defined from *A* to *B* assigns:
> 1. All elements in *A* to elements in *B*,
> 2. Every element in *A* to a unique element in *B*.

This colloquial definition is followed by a visual representation (figure 1), together with the introduction of notation as follows:



If $x \in A$ and $y \in B$ and if a function f from A to B assigns x to y then it is denoted by $f: A \to B$, $x \to y = f(x)$.

'$y = f(x)$' is read as '*y* is equal to *f* of *x*'.

Figure 1: Visual explanation for the definition of function

After this introduction, various examples of functions are given in different representational forms, such as sets of ordered pairs, set correspondence diagrams, tables and graphs. Towards the end of grade 1, students study parabolic functions; in grade 2, trigonometric functions and logarithmic functions; in grade 3, split-domain functions, absolute value functions, signum functions, integer value functions.

THEORETICAL FRAMEWORK

Our theoretical framework is built on Thompson's notion of core concept of function:

> … the core concept of "function" is not represented by any of what are commonly called the multiple representations of function, but instead our making connections among

> representational activities produces a subjective sense of invariance…it may be wrongheaded to focus on graphs, expressions, or tables as representations of function. We should instead focus on them as representations of something that, from students' perspective, is representable, such as aspects of a specific situation.
>
> (Thompson, 1994, p. 39)

Thompson (1994) claims that if students do not realise that something remains the same as they move among different representations then they see each representation as a "topic" to be learned in isolation. The curriculum designers assume that students can conceptualise the core concept of function after studying various representations. Our aim is to reveal how students focus on the definitional properties for various aspects of functions.

METHODOLOGY

The data comes from the preliminary and the main phase of an EdD study (Akkoç, 2001; Akkoç, 2003). There are two main sources of data, a questionnaire given to different groups of students in the preliminary and main phase and a set of semi-structured interviews from the main study. The clinical interviews aim to understand the underlying thinking of an individual, to enter the individual's mind rather than take the written responses from a test, by asking questions like, "How do you do this?" and "Why?". Questions are student-centred, e.g. "What is your way of deciding whether a graph is a function or not?" (Ginsburg, 2000).

A second feature of a clinical interviewing technique is to determine the strength of conviction behind what the student says. As Ginsburg (2000) discusses, Piaget noted that children tend to say what they believe the adult wants to hear, so he used methods of "repetition" and "counter-suggestion" to gain insight into strength of conviction (Ginsburg, 2000). Therefore, phrases like, "It is not important to answer right or wrong. Try to tell me what is going on in your head" are repeated throughout the interview. If a student explains successfully why a given item is a function or not, s/he is asked a non-function item as a counter-suggestion. When a student gives a successful explanation, s/he is asked the same question from a different angle with a counter-suggestion to seek persistency in the responses. If a student seems to be reluctant, s/he is encouraged to say what comes into his/her mind, right or wrong.

PRELIMINARY STUDY

A preliminary study was conducted to investigate the coherence of students' responses in using the definition for various aspects of functions. Subjects came from three different grades (grade 1, 2, 3 of high school, equivalent to English Year 11, 12, 13) from three different high schools in Turkey, one public, one private and one selective. A hundred questionnaires were analyzed and, based on that analysis, eight students were chosen for individual interviews. These eight students represented a spectrum of performance in terms of the number of correct answers in the questionnaires. Generally speaking, both questionnaire and interview results indicated that students were more successful with set correspondence diagrams and sets of ordered pairs compared with graphs and expressions (Akkoç, 2001; Akkoç & Tall, 2002). However, in some specific cases, for example, for split-domain functions, students were remarkably successful in the questionnaire. The percentage of correct answers for split-domain functions (which Vinner, 1983, found to be problematic) is high (72%), despite low frequencies for other expressions. This probably relates to the fact that these functions are taught as a separate topic.

When asked for a definition, 40% of students referred to the colloquial definition:

Preliminary study	**Frequency**	**Percent**
Colloquial definition	40	40
Incomplete colloquial definition	13	13
Other responses	22	22
No Response	25	25
Total	100	100

Table 1. Responses to 'what is the definition of a function', preliminary study

MAIN STUDY

In the main study, 114 grade 3 students were administered questionnaires. These students were from three different subjects groups (40 students from Mathematics, 32 students from Turkish and Mathematics and 42 from Social subject groups) in two different schools (one private and one public school) in Turkey. When asked of the definition, only 10.5% of the students replied successfully as shown below. In part, the lower response rate may relate to the fact that the preliminary study included candidates from a school with highly selective entry standards.

Main Study	**Frequency**	**Percent**
Colloquial definition	12	10.5
Incomplete colloquial definition	15	13.2
Other responses	14	12.3
No Response	73	64
Total	114	100

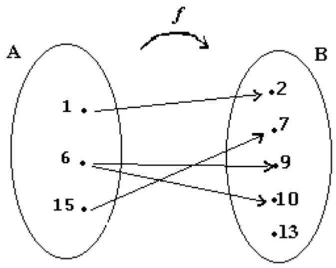
Table 2. Responses to 'what is the definition of a function', main study

In the main study, students were also asked the reasons behind their answers in the questionnaires. Results indicated that very few of them coherently used the definition focusing on the core concept of function.

Nine students were chosen for interview based on the main study questionnaire. Following Mason (1996), a theoretical sampling was selected on the basis of relevance to the research problem, to build certain characteristics which help to develop and test the theory. The preliminary study and questionnaires in the main study revealed that students were more successful with set correspondence diagrams and sets of ordered pairs compared to graphs and expressions. Therefore, in selecting students for interview, three deviant cases with more success with graphs and expressions were selected to test the theory.

In the main interviews, students were asked to decide whether the given representations are functions or not. After each question follow-up questions were asked to seek reasons behind the replies using clinical interviewing techniques discussed in the methodology section.

Set correspondence diagram



Set of ordered pairs

$A = \{1, 2, 3, 4\}, \quad f : A \to R,$

$f = \{(1,1), (1,2), (3,3), (4,3)\}$

Expressions

(a) $f : R \to R, \quad f(x) = \begin{cases} 1, & if \quad x^2 - 2x + 1 > 0 \\ 0, & if \quad x^2 - 2x + 1 = 0 \\ -1, & if \quad x^2 - 2x + 1 < 0 \end{cases}$

(b) $y = 5$

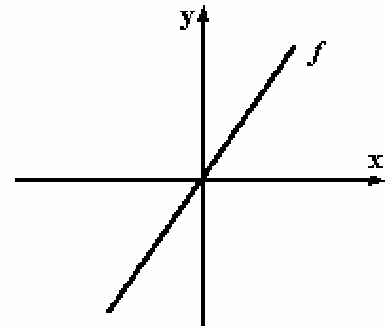
(c) $y = 5$ (for $x \leq 2$)

(d) $y = 5$ (for all values of x)

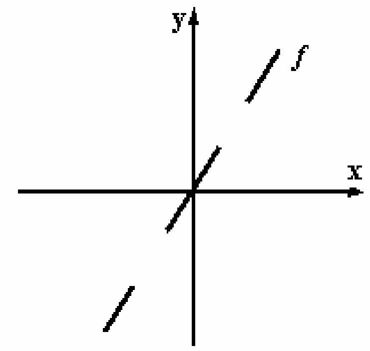
(e) $f : R \to R, \quad f(x) = \sin x - 2$

Graphs

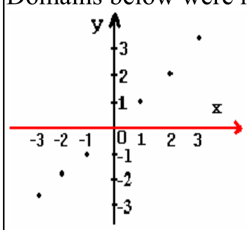
$f : R \to R$



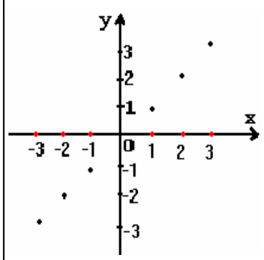
$f : R \to R$



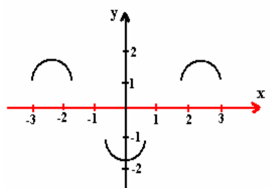
Domains below were marked in red.



(Domain is R)

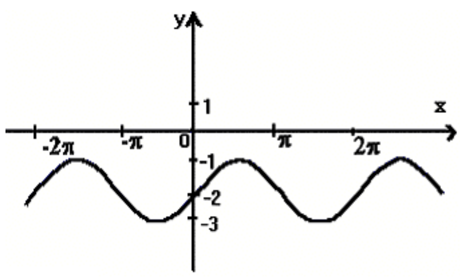


(Domain includes -3, -2, -1, 0, 1, 2, 3)



(Domain is R)

$f : R \to R$



$f : R \to R$

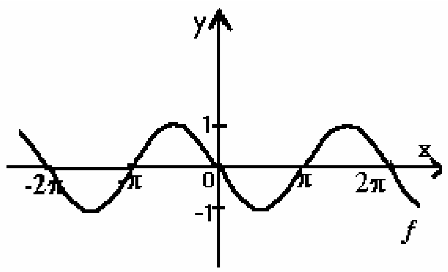


Figure 3: The questions used in the clinical interviews

ANALYSIS OF DATA

A grid was prepared to compare students' success using the definition for various representations of functions, with the entries shaded so that the darker shades represented a more complete form of the core concept of function. Therefore, the spectrum of shading reveals a categorization of students.

		Ali	**A**hmet	**A**ysel	**A**rif	**B**elma	**B**elgin	**C**em	**D**eniz	**D**emet
SET-CORRESPONDENCE DIAGRAMS		CD	CD	CD	CD	CD	CD	CDW	EBF	EBF
SETS OF ORDERED PAIRS		CDW CD	CD SD	CD	CD SD	CD	---	CDW	EBF	Oth
GRAPHS	Straight line	EBF CD	VLT CD	EBF CD	DRC	Oth	Oth	EBF	---	---
	Straight line in three pieces	CD	CD SD	CD	CD	EBF	---	EBF	EBF	EBF
	Points graph (D=R)	CD	CDW	CD	EBF	Oth	CDW	Oth	EBF	EBF
	Points graph (D=points)	CD	CD	CD VLT	CD	EBF	CDW	Oth	EBF	EBF
	Smiley graph (D=R)	CD	VLT SD	CD	CD	EBF	EBF	EBF	EBF	EBF
	$f(x) = -\sin x$ graph	EBF	VLT CD SD	EBF	EBF	EBF	EBF	EBF	EBF	EBF
	$f(x) = \sin x - 2$ graph	CD	CD VLT SD	CD	Oth	EBF	EBF	EBF	EBF	EBF
EXPRESSIONS	Signum function	EBF GR SD	EBF GR VLT	EBF WGr	EBF	EBF	DRC	EBF	Oth	Oth
	$y = 5$	Gr CF	Gr CF	Gr CF	SD	WGr	EBF	Oth	---	WGr
	$y = 5$ (for $x \le 2$)	CF GR	Gr SD	GR	CD	WGr	EBF	Oth	---	Oth
	$y = 5$ (for all values of x)	CD Gr	CF CD	CD Gr	CD	Gr	DRC	Oth	---	WGr
	$f: R \to R$ $f(x) = \sin x - 2$	CD	EBF CD	CD	Oth	---	EBF	---	EBF	WGr

Table 3: A grid for a summary of students' responses. Abbrevations: CD: Colloquial Definition; CDW: Colloquial definition wrongly used; EBF: Example-Based Focus; SD: Set Diagram; CF: Constant function; VLT: Vertical Line Test; Gr: Graph; WGr: Wrong graph; Oth: Other; ---: No Response.

Students from each category are named starting with a different letter, A, B, C, D. Grey colours spread across all aspects of functions for four students (Ali, Aysel, Ahmet, Arif) as seen in table 3. These four students are considered in the first category. They could focus on the definitional properties not only for the set-correspondence diagrams and the sets of ordered pairs but also for the graphs and expressions. In the second category are two students (Belma and Belgin) who could focus on the definitional properties for set-correspondence diagrams and sets of ordered pairs but not for graphs or expressions. They gave different responses for graphs and expressions. It should be mentioned that Aysel, Arif and Belma were selected for the interviews as deviant cases since they were more successful with graphs and expressions in the questionnaire. However, as seen in the grid above they responded differently in the interviews. They were more successful with set correspondence diagrams and sets of ordered pairs. In the third category is one student (Cem) who could focus on the definitional properties but could not check the definitional properties correctly. In the fourth category are two students (Deniz and Demet) who could not focus on the definitional properties for any aspect of the function concept. In other words, they gave different explanations that did not act as a coherent whole.

Some of the responses focused on various aspects of functions to use the colloquial definition. For instance, for the graph of $f: R \to R, \quad f(x) = \sin x - 2$, Ahmet used the colloquial definition then applied the vertical line test to the graph and then drew the set correspondence diagram to check the properties of the definition. These kinds of responses are labelled as CD-VLT-SD. Some students drew the graph of the given function to decide. For instance, Ali, Ahmet, Aysel first drew the graph of y=5, then considered it as a constant function (these are labelled as Gr-CF).

When students do not refer to the colloquial definition, they heavily relied on the previous examples they have experienced before. These kind of responses were labelled as EBF (Example-based focus). As seen in the grid above, less successful students tend to respond in that way. For instance, Belma, Cem and Deniz did not consider the graph of $f: R \to R, \quad f(x) = \sin x - 2$ as a function since the graph passes through the y-axis only. Demet did not consider this graph as a function since it is below the x-axis.

DISCUSSION

One of the difficulties of learning mathematics encountered by students is that the logical development of mathematics is not the same as the cognitive development of students. Skemp (1971) makes the distinction between logical and psychological developments. He recommended that we should teach the process of mathematical thinking rather than the product of mathematical thought. This dichotomy is exemplified by the case of function concept. Formally functions are special relations which have special definitional properties. However, the results indicate that very few students could coherently focus on the definitional properties of the function concept for various aspects of functions. Both preliminary and main studies revealed a spectrum of performance of students on different aspects of function in which a minority of students focus on the core concept of function while most students coped with a range of disconnected contexts. This reveals a mismatch between the curriculum design and students' cognitive structures. While the curriculum developers designed the course so that the function is a foundational concept and an organizing principle, most of the students do not focus on the essential function properties and they did not act as a organising principle. Instead many students focus on the individual properties of each representation without connecting them together.

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