# RELATING THEORIES TO PRACTICE IN THE TEACHING OF MATHEMATICS

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***There is nothing as practical as a good theory. (Richard Skemp, 1989, p. 27.)***

*This paper reports the coming together of two major goals, the first to build a cognitive theory of mathematical development that has wide application at different stages of development and in different contexts, the second to address a particular practical problem in the classroom. This problem related to the teaching of vectors, which lies at the confluence of mathematics and physics and builds from practical contexts to theoretical mathematics. We seek to generate a coherent theory that is consonant with many aspects from the literature rather than aggregating disparate aspects of different theories. In the practical context we listened to the voices in the classroom, both teachers and students, seeking a practical solution that would make sense to the participants and be of direct value in both teaching and learning.*

## INTRODUCTION

This paper is a contribution to a discussion on "Different theoretical perspectives in research: From Teaching problems to Research Problems". Our purpose is to see how the development of a broad cognitive theory and a rich practical problem can be of mutual benefit. The specific problem considered is the teaching of vectors in the context of school physics and mathematics. The broader cognitive theory is the theory of three worlds of mathematics, which begins with the child's perception and action on the world to carry out thought experiments to develop an increasingly sophisticated conceptual-embodied world, a focus on actions that are symbolised to give a proceptual-symbolic world of arithmetic and algebra and beyond, and a long-term focus on properties that, for some, leads to a formal-axiomatic world of definitions and proof (Tall, 2004). The specific problem is the teaching of vectors in school with its embodiments in physics and mathematics developing into the symbolism of vectors in two dimensions (Watson[1], Spyrou & Tall, 2002). Here we focus on the relationship between the worlds of embodiment and symbolism.

The British culture is one of practical approaches to practical problems. The pragmatic solution to teaching vectors is to introduce them in practical situations in physics as forces, journeys, velocities, accelerations, and only later to study the mathematical theory in pure mathematics. The teaching of vectors has not gone well. It has followed the path of many other topics that students find difficult. The initial

[1] Anna Poynter published under the name Anna Watson before her recent marriage.

presentation has been made more and more practical and less and less dependent on mathematical theory. It shares a similar fate to other ‘difficult’ parts of mathematics, including fractions and algebra.

In the pragmatic culture of Britain, the teachers are professionals. They take their work seriously, work hard with long hours and relatively little time scheduled for analysis and reflection. Our experience (Poynter & Tall, 2005) of interviewing colleagues show that they are aware that students have difficulties, but their awareness relates more to an episodic memory of what didn’t work last year rather than a theory that attempts to explain *why* it went wrong and what strategies might be appropriate to make it go right. Where there are problems, the response it to try a new strategy the following year in an attempt to improve matters.

As an example, consider the case of adding two vectors geometrically. The students are told that a vector depends only on its magnitude and direction and not on the point at which the vector starts. Therefore vectors can be shifted around to start at any point and so, to add two vectors, it is simply a matter of moving the second to start at the point where the first one ends, to give a combined journey along the two vectors. All that is necessary is to draw the arrow from the start point of the first vector to the end point of the second to give the third side of the triangle, which is the sum.

The problem is that many students don’t seem to be able to cope with these instructions. Some ‘forget’ to draw the final side of the triangle to represent the result of the sum, others have difficulties when the vectors are in non-standard positions to start with, such as two vectors pointing into the same point, or two vectors that cross. Some find it difficult to cope when two vectors start at the same point, and draw the ‘result’ of the two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as the third side of the triangle, $\overrightarrow{BC}$.

Here we have a specific teaching problem that requires a solution. What theories are available to solve it? The science education theory of ‘alternative frameworks’ (Driver, 1981) suggests that that the students may have their own individual ways of conceptualising the concept of vector. However, it does not offer a theory of *how* to build a new uniform framework for free vector in a mathematical sense. Our goal is to study this problem not only in its own right to be meaningful to students and fellow teachers, but also within the goal of developing a wider theoretical framework.

## SOME EXISTING THEORIES

The embodied theory of Lakoff and his colleagues offers a viewpoint that encourages us to consider how students *embody* a concept such as vector. However, this theory takes a high-level view of mathematical concepts to perform a top-down *idea analysis* theorizing how such concepts have their origins in embodiment rather than a global view that integrates the genesis of the mathematical concepts with the actual conceptual development of the child. For instance, *Where Mathematics Comes From* (Lakoff & Núñez, 2000) includes references from mathematics education papers in its bibliography but makes no reference to them anywhere in the main text. We find the notion of ‘idea analysis’ formulated by Lakoff and Núñez to be a valuable technique, but prefer to use an analysis that relates to the cognitive development of the individual. For us, cognitive development builds from perception and action through reflection to higher theoretical conceptions. We use the term ‘embodiment’ first in the colloquial sense that a sophisticated concept may be ‘embodied’ physically (such as fractions represented as part of a physical whole or a vector as a physical transformation) after the manner of Skemp (1971) and later in the sense of conceptual mental embodiment using thought experiments. This sense relates to Bruner’s notions of enactive and iconic modes of operation as distinct from his symbolic mode, which we see in three distinct parts: language which underpins all increasingly sophisticated modes of thought, and the two increasingly sophisticated worlds of proceptual symbolism in arithmetic and algebra and the more advanced logical symbolism of axiomatic mathematics.

Focusing on the development from physical actions to mental conceptions, a relevant approach may be found in the APOS theory of Dubinsky (Dubinsky & MacDonald, 2001). Dubinsky theorizes that mathematical objects are constructed by reflective abstraction in a dialectic sequence **A-P-O-S**, beginning with **A**ctions that are perceived as external, interiorised into internal **P**rocesses, encapsulated as mental **O**bjects developing within a coherent mathematical **S**chema. The actions with which the theory begins may be physical or mental and, in the case of vector, we see transformations as actions on physical objects being routinized into thinkable processes and then encapsulated as mathematical objects in the form of free vectors. There is, however, a possible problem. Several papers in the literature show how students may routinize actions as processes but in several cases (including the notion of limit or of function) the further step to an object conception is less easily accomplished (e.g. Cottrill et al 1996, Dubinsky & Harel, 1992). This signals a possible problem in the shift from a procedural action to a conceptual mental object.

We considered Skemp’s (1976) theory of instrumental and relational understanding. It seemed evident that many students were learning instrumentally how to add vectors without any relational understanding. But what is the relational understanding that is necessary and how is it formulated? Likewise the theories of procedural and conceptual knowledge (Hiebert & Lefevre, 1986, Hiebert & Carpenter, 1992) suggest that the students may be learning procedurally and not conceptually. But here again, what is the conceptual structure and how are procedures and concepts related?

It is apparent that students learn based on their own experiences. They meet various practical examples of vectors, including vectors as journeys and vectors as forces. Many theories (e.g. Dienes 1960) suggest that students must experience *variance* in different examples and abstract the essential properties that are common while ignoring incidental properties that occur in some examples but do not generalise. In the case of vector, these incidental properties are coercive and lead to alternative frameworks that are difficult to shift.

We considered other frameworks, for example the framework of intuition and rigour that occurs in Skemp’s (1971) distinction between intuitive and reflective thinking or in Fischbein's (1987, 1993) tripartite system of intuitive, algorithmic and formal thinking. Indeed the latter theory is strongly related to our own development of three worlds of mathematics except that the three categories exist as separate aspects, as they did in the first design of the English National Curriculum where Concepts and Skills were put under separate headings.

Our inspiration for putting these elements together in an integrated manner arose from several theories that include both a global development of successive modes of operation (such as Piaget's stage theory or the enactive-iconic-symbolic modes of Bruner) and also a local sequence of concept formation within each of these modes. In particular, the SOLO taxonomy of Biggs and Collis (1982) made a significant step forward involving not only successive development of different modes (*sensori-motor, ikonic, concrete-symbolic*, *formal* and *post-formal*) but also local cycles of concept formation within each mode which were termed *uni-structural*, *multi-structural*, *relational*, *extended abstract*.

Pegg (2002) took a further step by noting how the Biggs and Collis cycle of concept formation operates in a similar sequence to the compression of process to concept, linking to the theory of Gray & Tall (1994) in which action-schemas such as counting (uni-structural) are developed into more compressed procedures such as count-all, count-on, count-on-from-larger (multi-structural), to the overall process of addition that may be implemented by different routes (relational), and the concepts of number and sum seen as mentally manipulable concepts (extended abstract).

This opens up a vision of a cognitive development from embodied beginnings encompassing the SOLO sensori-motor and ikonic (a combination of Bruner's enactive and iconic modes) through successive encapsulations of actions as processes represented by symbols to symbolic manipulation of symbols as thinkable entities, relating the worlds of conceptual-embodiment and proceptual-symbolism.

## DEVELOPING A GENERAL THEORY THAT ALSO FITS THE PROBLEM

At this point, a single incident gave us a sudden insight into the relationship between embodiment and symbolic compression. The first-named author (Anna Poynter) was convinced that the problem arising from the complications of the examples of physics with their different meanings for journey, force, velocity, acceleration and so on, could be replaced by a much simpler framework in mathematics, if only (and this is a big if) the students could focus on the fundamental mathematical ideas. The problem was how to give a meaning to the notion of 'free vector' in a *mathematical* way that was meaningful and applied to all the other contexts in an overall coherent way.

The breakthrough came from a single comment of a student called Joshua. The students were performing a physical activity in which a triangle was being pushed around on a table to emulate the notion of 'action' on an object. Joshua explained that different actions can have the same '*effect*'. For example, he saw the combination of one translation followed by another as having the same effect as the single translation corresponding to the sum of the two vectors. He also observed that solving problems with velocities or accelerations is mathematically the same.

This single example led to a major theoretical development. In performing an action on objects, initially the action focuses on what to *do*, but abstraction (to coin a phrase of John Mason, 1989) is performed by 'a delicate shift of attention', to the *effect* of that action. Instead of saying that two actions are equivalent in a mathematical sense, one can focus on the embodied idea of having the same effect. At a stroke, this deals with the difficult compression from action to process to object formulated in APOS theory, by focusing attention on shifting from embodied action to effect.

In the case of a translation of an object on a table, what matters is not the path taken, but the change from the initial position to the final position. The change can be seen by focusing on *any* point on the object and seeing where it starts and ends. All such movements may be represented by an arrow from start point to end point and all arrows have the same magnitude and direction. In this way *any* arrow with given magnitude and direction can represent the translation, and the addition of two vectors can be performed by placing two such arrows nose to tail and replacing them by the equivalent arrow from the starting point of the first arrow to the end of the second. The *embodied world* of action has *a graphical mode of representation* that is more than a static picture: it represents the mental act of carrying out the transformations so that the learner can focus not just on the actions but on their *effect*.

This theory of compressing action via process to mental object by concentrating on the embodied effect of an action is widely applicable. It is a practical idea that can prove of value in the classroom, as well as bringing together a range of established theories developed over the last half century by Piaget, Bruner, Dienes, Biggs & Collis, Fischbein, Skemp, Dubinsky, Lakoff & Núñez and many others. In the following sections we give a brief outline of our empirical evidence from Poynter (2004a) which are summarized on the web (Poynter, 2004b).

## EMPIRICAL RESULTS

Poynter (2004a) compared the progress of two classes in the same school, Group A taught by the researcher using an embodied approach focusing on the effect of a translation, Group B taught in parallel using the standard text-book approach by a comparable teacher. The changes were monitored by a pre-test, post-test and delayed post-test, and a spectrum of students were selected for individual interviews. The tests studied the students' progress in developing through a cycle of concept construction in both graphic (embodied) and symbolic modes of representation.

In figure 1, two cycles of concept construction are involved. Stage 1 refers to the earlier cycle formulating the notion of a signed number in one dimension as journey or as a signed number. Stages 2, 3 and 4 are successive stages of encapsulation of the notion of free vector in two dimensions, starting from a graphical representation of an arrow as a journey represented symbolically as horizontal and vertical components, then focusing on the effect of the shifty as shifts with the same magnitude and

| Stage | Graphical | Symbolic |
|---|---|---|
| 0 | No response | No response |
| 1 | Journey in one dimension | A signed number |
| 2 | Arrow as a journey from A to B | Horizontal and vertical components |
| 3 | Shifts with same magnitude and direction | Column vector as relative shift |
| 4 | Free vector | Vector **u** as a manipulable symbol |

**Figure 1: Fundamental cycle of concept construction of free vector**

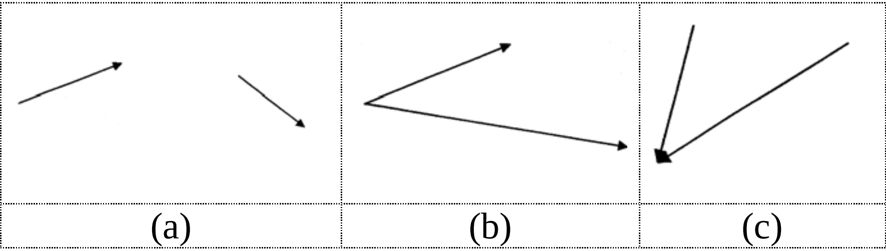
direction or as a column vector as a relative shift, then finally as a manipulable free vector that can be given a single symbol that can be operated upon. A similar cycle was formulated for the encapsulation of the process of adding two vectors to give the concept of sum, starting from addition of signed numbers in one dimension, then in two, where the arrows are seen, for example, as one journey following another then focusing on the effect to see the sum of two vectors as the single vector with the same effect and finally as free vectors added as mental entities.

Poynter (2004a) focused on several aspects of the desired change that could be tested. Here we consider three of them. It was hypothesised that students, who encapsulate the process of translation as a free vector, are able to focus on the *effect* of the action rather than the action itself. This should enable them to add together free vectors geometrically even if the vectors are in 'singular' (non-generic) positions, such as vectors that meet in a point or which cross over each other. It should enable them to use the concept of vector in other contexts, e.g. as journey or force. In the case of a journey, it should allow the student to recognise that the sum of free vectors is commutative. (As a journey, the equation $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{BC}+\overrightarrow{AB}$ does not make sense, because $\overrightarrow{AB}+\overrightarrow{BC}$ traces from $A$ to $B$ to $C$ but, $\overrightarrow{BC}+\overrightarrow{AB}$ first represents a journey from $B$ to $C$ and requires a jump from $C$ to $B$ before continuing. As free vectors, $\mathbf{u}=\overrightarrow{AB}$ and $\mathbf{v}=\overrightarrow{BC}$, we have $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.)

It was hypothesised that experimental students would be more able to:

1. add vectors in singular (non-generic) cases
2. use the concept of vector in other contexts (eg as journey or as force)
3. use the commutative property for addition.

Students were asked to add two vectors in three different examples:




2) In each case add the two vectors together

3) If there is any other way you could have done any of the additions of the two vectors in Q2 show it.

**Figure 2: questions that could be considered singular**

When we asked other teachers what they felt students would find difficult, we encountered differences between the responses of a colleague who taught physics and two others who taught mathematics. As mathematicians, we saw part (a) to be in a general position, because it only required the right-hand arrow to be pulled across to the end of the left-hand arrow to add as free vectors; (b) evoked the idea of a parallelogram of forces; (c) was considered singular because it was known to cause problems with some students embodying it as two fingers pressing together to give resultant zero.

All teachers considered part (c) would cause difficulties. However, they differed markedly in their interpretations of parts (a) and (b). The physics teacher considered that the students would see the sum of vectors either as a combination of journeys, one after another, or as a sum of forces. For her, (a) was problematic because it does not fit either model, but (b) would invoke a simple application of the parallelogram law. As an alternative some students might measure and add the separate horizontal and vertical components. The two mathematics teachers considered that students would be more likely to solve the problems by moving the vectors 'nose to tail' with the alternative possibility of measuring and adding components. One of them considered that students might see part (a) as journeys and connect across the gap, and in part (b) might use the triangle law in preference to the parallelogram law. The other sensed that (b) could cause a problem because 'they have to disrupt a diagram' to shift the vectors nose to tail—an implicit acknowledgement of the singular difficulty of the problem—and part (c) would again involve shifting vectors nose to tail although she acknowledged that some students might do this but not draw the resultant (which intimated again that they see the sum as a combination of journeys rather than of free vectors).

The performance on the three questions assigning an overall graphical level to each student is given in Table 1.

| Graphical stage | Group A (Experimental) (N=17) | | | Group B (Control) (N=17) | | |
|---|---|---|---|---|---|---|
| | Pre-test | Post-test | Delayed | Pre-test | Post-test | Delayed |
| 4 | 0 | 1 | 12 | 2 | 0 | 7 |
| 3 | 1 | 9 | 4 | 1 | 10 | 3 |
| 2 | 4 | 6 | 1 | 1 | 3 | 2 |
| 1 | 4 | 1 | 0 | 4 | 1 | 0 |
| 0 | 8 | 0 | 0 | 9 | 3 | 5 |

**Table 1: Graphical responses to the singular questions**

Using the t-test on the numbers of students in the stages reveals that there is a significant improvement in the experimental students from pre-test to delayed post-test ($p < 0.01$) but not in the control students.

Similar results testing the responses to questions in different contexts and questions involving the commutative law are shown in tables 2 and 3.

| Graphical stage | Group A (Experimental) (N=17) | | | Group B (Control) (N=17) | | |
|---|---|---|---|---|---|---|
| | Pre-test | Post-test | Delayed | Pre-test | Post-test | Delayed |
| 4 | 0 | 0 | 8 | 0 | 0 | 2 |
| 3 | 0 | 9 | 3 | 2 | 3 | 5 |
| 2 | 1 | 2 | 2 | 0 | 3 | 3 |
| 1 | 1 | 5 | 4 | 0 | 2 | 3 |
| 0 | 15 | 1 | 0 | 15 | 9 | 4 |

**Table 2: Graphical responses to questions set in different contexts**

The change is again statistically significant from pre-test to delayed post-test ($p<0.01$) using a t-test.

| Graphical stage | Group A (Experimental) | | | Group B (Control) | | |
|---|---|---|---|---|---|---|
| | Pre-test | Post-test | Delayed | Pre-test | Post-test | Delayed |
| TOTAL | 0 | 7 | 12 | 4 | 6 | 5 |

**Table 3: Responses using the commutative law of addition**

In this case the change is from a significant difference in favour of Group B on the pre-test ($p<0.05$ using a $\chi^2$-test) to a significant difference in favour of Group A ($p<0.05$ using a $\chi^2$-test). Further details may be found on the web (Poynter, 2004a, b).

What is clearly important here is not the statistical significance, but the evident changes which can be *seen* not only to improve the situation for Group A from pre-test to post-test, but more importantly to *increase* the level of success by the delayed post-test. There is a clear difference in the long-term effect of the experimental teaching programme.

## BROADER THEORETICAL ASPECTS

The theory reveals a parallel between focusing on the effect of embodied actions and the compression of symbolism from procedure to process to object has the potential to be simple to describe and implement with teachers and students. The theory has proved to be a practical theory, in that the idea of focusing on the *effect* of an action in the case of vector has proved to be not only successful with students, as in the experiment described, but also in subsequent discussion with other teachers (Poynter 2004b). All that is necessary to have appropriate activities and to mentor the participants to focus on the effects of carefully designed actions.

This applies in a variety of areas, not only in representing vectors dually as transformations and as free vectors, but also in other areas where symbols represent a process being encapsulated into a concept. For instance the process of counting is compressed to the concept of number by focusing on the *effect* of counting in terms of the last number spoken in the counting schema. Likewise, the process of sharing and the concept of fraction, in which, say, sharing something into 4 equal parts and taking 3 of them has the same effect as sharing into 8 equal parts and taking 6. This corresponds symbolically to having equivalent fractions ($^3/_4$ or $^6/_8$). Likewise different algebraic procedures having the same effect gives an alternative way of looking at the idea of equivalent algebraic expressions. Other processes in mathematics, such as the concept of function, also result from a focus on the *effect* of an input-output action, rather than on the particular sequence of actions to carry out the process, revealing the wide range of topics in mathematics that benefit from this theoretical analysis.

This research into a single classroom problem has therefore stimulated developments in the relationship between embodiment and (proceptual) symbolism as part of a wider general theory of the cognitive development of three worlds of mathematics (embodied, symbolic and formal), (Watson, Spyrou & Tall, 2003, Tall, 2004). This theory, in turn, also builds on earlier work that theorizes three distinct kinds of mathematical object: "One is an *embodied object*, as in geometry and graphs that begin with physical foundations and steadily develop more abstract mental pictures through the subtle hierarchical use of language. The second is the *symbolic procept* which acts seamlessly to switch from an often unconscious 'process to carry out' using an appropriate algorithm to a 'mental concept to manipulate'. The third is an *axiomatic concept* in advanced mathematical thinking where verbal/symbolical axioms are used as a basis for a logically constructed theory" (Gray & Tall, 2001).

In this way, looking at how a particular teaching problem benefits from different theories can be fruitful, not only in addressing the teaching problem in a way that makes practical sense to pupils and teachers, but also in analysing and synthesising aspects of a range of theories to produce a practical theory.

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