Reflections on Research and Teaching of Equations and Inequalities

David Tall

Mathematics Education Research Centre
University of Warwick, CV4 7AL, United Kingdom
<david.tall@warwick.ac.uk>

In reacting to this forum on 'Algebraic Equalities and Inequalities', I take a problem-solving approach, first, asking 'what is the problem?' then looking at the five presentations to see what can be synthesized from their various positions (acknowledging that they are here limited to very short summaries).

The 'problem', as initially formulated, focuses on the *algebraic* manipulation of equations and inequalities. Tsamir et al [TTT] focus mainly to this aspect by considering how a teacher might cope with errors that arise from the inappropriate use of earlier experiences in equations that produce errors with inequalities. This focus is broadened in the list of 'Key Questions', to encourage the consideration of different theoretical frameworks—and the use of technology—to see how research can improve teaching and learning. The other papers take the key questions in different directions. Boero & Bazzini [BB] and Sackur [S] consider broader issues, with a particular focus on the switch from algebraic to visual representations where an inequality $f(x) > g(x)$ is visualised by seeing where the graph of f is above the graph of g. Underlying both approaches are relationships between different representations (or semiotic registers, as described in the subtle theory of Duval).

Kieran [K] presents a different overall framework ('generational', 'transform-ational' and 'global meta-level') that may be described as a 'vertical' theory of development rather than a 'horizontal' theory of relationships between represent-ations. Finally, Dreyfus & Hoch [DH] broaden the context to the increasingly sophisticated structure of equations, from a procedure to undo an arithmetic calculation, to solving equations with xs on both sides, to more subtle cases of equations containing substructures and equations solved using specified rules.)

This brings me back to 'the problem'. What is it that this forum is really attempting to address? There seems to be an implicit understanding that we need to help students to understand and operate with equations and inequalities. *But for what purpose?* If the purpose is to solve a given equation or inequality, then a graphical picture may be appropriate. For instance, to 'see' what happens to the inequality $x^2 > x + c$ as c varies, a powerful visual representation is given by the quadratic $f(x) = x^2$ and a straight line $g(x) = x + c$ that moves up and down as c changes. However, if the problem is to enable the student to become fluent in meaningful manipulation of symbolism, then the activities with the graph may involve no symbolic manipulation whatever (particularly if the graph is drawn by computer). [S] considers the strengths and weaknesses of moving between different registers. These focus on different aspects, highlighting some, neglecting others. If an aspect is absent, then its variation does not figure in the link between representations. An example is the *evaluation* of a function by carrying out a procedure: $2(x+1)$ and $2x+2$ are different procedures in the symbolic register but are represented by precisely the same graph.

The focus of [BB] on graphs of functions as global dynamic entities uses the idea of 'grounding metaphors' of Lakoff & Nunez in a way that 'could also ensure a high level of the control of the solution process'. But *what* solution process? The visual enactive activity can give a powerful embodied sense of global relationships between functions as entities, but how does it relate to the meaningful manipulation of symbols? It emphasizes the strength of grounded metaphors but not the 'incidental properties' of Lakoff's theory, which may be usefully employed in a particular context but have the potential to be the sources of errors in new contexts.

It is my belief that the phenomenon of 'cognitive obstacles' arises precisely because the individual's subconscious links to incidental properties in earlier experiences are no longer appropriate in a new context. Rather than use the high sounding language of 'metaphor' for the recall of earlier experiences, I use the prosaic term 'met-before'. I hypothesise that it is precisely the met-befores in solving linear equations that causes problems in inequalities researched by [TTT]. Students taught to manipulate symbols in equations, will build personal constructions that work in their (possibly procedural) solutions of linear equations but operate as sub-conscious met-befores that cause misconceptions when applied to inequalities.

In a given context there are often several different approaches possible. [K] reveals a spectrum of responses to a problem that may be formulated as an inequality, including a physical representation, the use of tables, equations and inequalities. [DH] presents a compatible spectrum, with different emphases, numerical procedures to 'undo' equations, more subtle manipulation of expressions as mental entities, and seeing sub-structures of equations as mental entities in themselves. Some of these approaches may be more amenable to future development than others; in particular, theories of cognitive compression from process to manipulable mental entities (which are entirely absent from all the presentations) address the possibility that the construction of mentally manipulable entities is likely to be more productive for long-term development.

Later developments in the use of inequalities include the formal notion of limit, where the epsilon-delta method will certainly benefit from meaningful grounding of inequalities, but will also need to focus on the manipulation of symbols and the development of formal proof. Inequalities at a formal level involve axioms for order in a field F, for example, by specifying a subset P of F that has simple properties (if $a \in P$, then one and only one of these holds: $a \in P$, $-a \in P$ or $a = 0$; if $a, b \in P$ then $a+b, ab \in P$.) In this case $a > b$ is *defined* to be true when $a - b \in P$. This use of 'rules' is not a meaningless procedural activity but a meaningful formal approach that has the potential of giving new meanings. For instance, a structure theorem may be proved to show that every ordered field 'contains' the rational numbers and may also contain 'infinitesimals' that are elements in F which are smaller than any rational number. In this way intuitive concepts at one stage (infinitesimals as 'arbitrarily small' variable quantities) can be given a formal mathematical meaning.

An organisation such as PME needs to aim not only for local solutions to problems, but also for global views of long-term development. The papers in this forum present essential ingredients to contribute such a wider scheme.

When the 'problem' of equations and inequalities is seen in this way, a wider picture emerges. There are unspoken belief systems that get in the way of our deliberations. For instance, while several of the papers give examples of different individuals using different methods to solve the same problem, no one attempts to say whether one solution is potentially better or worse for long-term development. Differences are apparent in the success and failure in all the examples given. Do we need to look at different solutions for different kinds of needs? Rich embodiments have strengths that may be appropriate in some contexts (perhaps to solve an inequality in a specific problem) and misleading in others (where concepts of constructed that, if unresolved, become met-befores causing obstacles in later learning). Do all students follow through the same kind of Piagetian development or, does their journey through mathematics find them using methods that are more or less suited to long-term development that gives different kinds of possibilities for future development?

In addition to the horizontal framework of registers and the vertical framework of [K], I offer a third that relates to the algebraic spectrum of [DH]. A study of long-term development of symbolism in arithmetic and algebra (Tall *et al.*, 2000) led to a categorization of algebra (Tall & Thomas 2001) in three levels, which we termed 'evaluation algebra', 'manipulation algebra' and 'axiomatic algebra'. The first encompasses the idea of an expression, say $3+2x$ being used simply for evaluation, say in a spreadsheet or in a graph-drawing program. The second encompasses the idea of an expression as a thinkable entity to be manipulated. The third concentrates on the properties of the manipulation and leads to an axiomatic approach to algebra in terms of groups, rings, fields, ordered fields, vector spaces, etc. In what ways do the papers presented in this forum address problems both at a local level and also in producing a helpful global theory? Much of the discussion could involve evaluation algebra, [TTT] considers manipulation and [DH] looks from manipulation to axiomatic. Do we need one kind of algebra for some students and other kinds for others? Richard Skemp once said to me, 'there is nothing as practical as a good theory'. In our forum it would be practical to look for a global theory encompassing the local theory of equations and inequalities.

References

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