REFLECTING ON POST-CALCULUS-REFORM

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This paper is written to consider the changes in the calculus in the first four years of the new millennium following the significant technological changes in the latter part of the twentieth century that gave rise to Calculus Reform. The analysis is based on a theoretical framework that distinguishes three different modes of mathematical thinking, which result in different worlds of mathematics, the first relates is based on our sensory experiences and is characterized by thought experiments, the second is based on our use of symbolism to carry out calculations and manipulations, the third relates to the building of formal theories based on definitions and proof.

Introduction

At each meeting of the International Congress of Mathematics Education every four years, it is customary to review the developments that have occurred since the previous conference. This time we are looking back over the first four years of the new millennium 2000-2004. In the development of the calculus and its teaching, this has been a period of consolidation of the technology that grew in the last quarter of the twentieth century and the development of new ways of teaching and learning to make use of that technology. At the same time, human individuals and societies are slower to change.

Cultural Change

As was eloquently expressed by R. L. Wilder in his classic book *Evolution of Mathematical Concepts* (1968), many successful cultural elements persist throughout periods of change because they are deeply ingrained and continue to work. The introduction of the metric system in Europe did not stop old units, such as the pound or pfund from continuing, now transmuted a little to be related to half a kilogram, likewise, in England the old imperial measures for a pint of milk or beer have survived the metric conversion and the old measure of a 'foot' has been translated to a 'metric foot' (30 centimetres).

The same is happening again in the calculus. When the new analysis of Weierstrass introduced epsilon-delta definitions, and the new set theory conceived of functions as sets of ordered pairs, it did not replace the old ideas of variables varying, nor has the introduction of graphic interfaces and symbol manipulators replaced traditional ideas that insist on a backbone of formal definition and proof. The calculus, like any other human endeavor is based on human experience and human beliefs.

Current Changes in the Calculus

The first years of the new millennium have seen a broad settling down of the changes that occurred in the frenetic developments of the 1980s and 1990s. As new technologies came on-stream, there was a rapid succession of new ideas for use in calculus and its teaching. First came **numeric algorithms**, at first on main-frames, then personal computers, then hand-held devices. These were soon supported by **graphic representations**, initially at a low (but acceptable) level of resolution, later better represented in high resolution on computers and reasonable resolutions on hand-helds. Computers were supported by new **enactive interfaces** that enabled us, with suitable software, to make intuitive choices and dynamically change information on-screen. In parallel came increasingly sophisticated **symbol-manipulation**. These are now all well established and becoming increasingly mature. In the new millennium, the internet in general, and broadband in particular, are leading to new forms of communication, to place information at one place on the web to be available to everyone around the world, to enable students, teachers and researchers to communicate in real time and share dynamic facilities for study.

A corporate enterprise can put its software support on the web, be it software, documentation, modules of work, Powerpoint overheads for teacher use, or means of communication between users. An individual can also open up a web-site offering free facilities, limited only by the imagination. A web-search will reveal a wide range of these. Creating materials for the calculus is an industry, growing in diverse ways that occur with human ingenuity.

But is this changing our culture? Clearly in terms of the way we work, things will never be the same. A laboratory with computers in it has a very different dynamic from a lecture theatre with a teacher in front and students in serried ranks. The tradition of each student being responsible for their own work and not cheating by copying from others is transformed to a corporate enterprise where we learn better if we share the insights of others and take the opportunity to work in groups to build a more comprehensive conception that we can improve by discussion with others.

Such changes in learning style are not restricted to the calculus alone (though it was the calculus reform movement that began to introduce these new practices). Therefore, although they are part of the overall picture, our main focus here is not the learning practices themselves, but the underlying subject matter and how we give it meaning through teaching and learning.

Calculus is the culmination of several strands of mathematical development. It uses numerical calculations, symbolic manipulations and graphical representations that are arise at the highest level in school mathematics and is a gateway to a huge range of avenues that follow. Our purpose it to consider this context and to build a framework that will enable us to take a broad view of the whole picture, in terms of the growth of knowledge, both in mathematics and in the development of a wide range of students with differing needs.

A theoretical framework: calculus and three worlds of mathematics

In building a framework relevant to the development of mathematics in general and the calculus in particular, we have found it helpful to consider the different types of activity that underlie mathematical thinking. As children, we *perceive* and *act* on the world and by *reflection* build up increasingly sophisticated mathematical ideas. This occurs in two quite different ways. A focus our *perception* of real world objects leads us to categorise them, consider their properties and build more sophisticated conceptions through performing *thought experiments*. A focus on *actions* on objects, however, requires us to think about what we *do* rather than what we see. As actions occur in time, we can perform them, but it is far more difficult to think about them as a whole. The solution is to introduce *symbols* to carry out *processes* (counting, addition, differentiation, integration) and think of the symbols also as *concepts* that we can hold in our mind (such as number, sum, derivative, integral). The use of symbols dually as process and concept (termed *procepts* in **Gray & Tall, 1994**) plays a significant role in our analysis. Indeed, the conceptual understanding of visual representations and symbolic manipulations meets its zenith in the theory of calculus.

A further conceptual shift occurs from calculus to mathematical analysis that involves and even greater change in meaning. It involves a complete change in focus from visual graphs and symbolic calculus to a formal approach in which definitions are formulated as multi-quantified statements to build a systematic axiomatic theory based on formal definitions and formal proof.

This analysis leads the idea that mathematical thinking can be categorized into three significantly different worlds (figure 1):

- the *conceptual*-***embodied*** world of our physical perceptions that we build into mental conceptions through reflection and thought experiment;
- the ***symbolic***-*proceptual* world that begins with real-world actions, (e.g. counting) that are symbolized and considered as concepts (e.g. number) to lead successively to arithmetic, algebra and symbolic calculus. These symbols (called procepts) operate dually as processes (such as counting, addition, differentiation, integration) and concepts (number, sum, derivative, integral) to give precise quantitative information beyond that possible in the embodied world,
- the ***formal***-*axiomatic* world of axiomatic systems, definitions and formal proof.

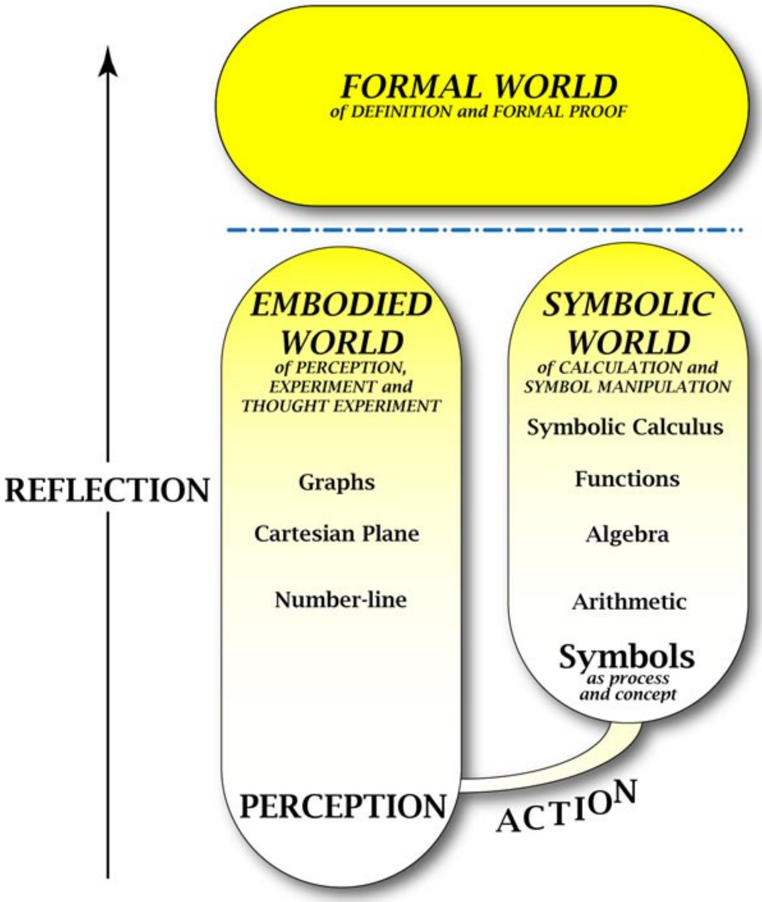

Figure 1:Three Worlds of Mathematics

This framework gives a new view of the structures underlying calculus and analysis. Whereas the expert looks down on calculus through the lens of formal proof, the student grows up to the ideas from embodied perceptions and actions. This leads to a significant cognitive gap between the worlds of embodiment and symbolism inhabited by the calculus, and the more sophisticated formal world of analysis used by professional pure mathematician.

The student learning calculus has many earlier experiences that colour his or her conceptions. Early experiences of arithmetic tell us that symbols, such as 2+3, have a built-in process of addition, which makes them *operational*, leading to an *answer*. An algebraic symbol $2+3x$ is only *potential*, in that it represents an operation, but that operation cannot be carried out until x is known (**Tall, *et al*, 2001**). This causes a problem for many students beginning algebra who have difficulties with, even an aversion to, the manipulation of symbols that do not have 'an answer'. The beginnings of calculus reveal an even greater problem. A limit concept is *potentially infinite*. The process of reaching the limit not only goes on 'forever', it may not even have a finite procedure to carry out the operation. This leads to many conceptual problems, such as the belief that 'the limit is never attained' or $0 \cdot \dot{9} < 1$. However, when the students meet the rules for differentiation, they again have procedures that 'have an answer', yielding the formula for a derivative of a function built up from the derivatives of its constituent parts. It is therefore no wonder that students have difficulty with the limit concept and yet believe that they can 'do' the calculus. It is part of their expectations to be told *how* to do something and to be tested on being able to carry out the procedure.

Into this problematic area, the mathematician attempts to introduce another higher order concept 'from above', namely the idea of formal definitions and proper proof. Students are then faced with two seriously difficult mental constructions to perform. First the concepts are presented as *formal definitions* that need serious reconsideration in a context where the students' current beliefs are built from experiences that may not be consistent with the definitions they are asked to accept. Then, if they are given the 'full treatment', there is the huge problem of coordinating multi-quantified statements, which proves beyond the ability of the vast majority of students. To be able to change the conceptualization and teaching of calculus to make sense to students requires first an understanding both of the mathematical concepts as (seemingly) shared by the mathematical community, and the concepts as constructed by the individual building a personal knowledge structure.

Different meanings of mathematical concepts

In their theory of *Where Mathematics Comes From*, Lakoff and Nunez (2000) performed an 'idea analysis' of various formal mathematical concepts and made the claim that *all* mathematical ideas are built from fundamental human origins using links between ideas as metaphors. For example, they analyze the real numbers as a 'metaphorical blend' of three distinct conceptual metaphors:

- as naturally continuous space (a line),
- as decimals,
- as a set of distinct points.

Their categorisation is based on a top-down analysis of three distinct ways that mathematicians see the concept of number. Our analysis gives essentially the same categories developed in sequence by the typical learner:

- a conceptual embodiment as a line (in the embodied world),
- a symbolic representation as numbers (in the symbolic world),
- a definition as a complete ordered field. (in the formal world). (Figure 2.)

This categorization is a fundamental underpinning of the framework we are about to use to interpret the development of calculus. Historically, the Greeks saw a line as a conceptual entity that does not consist of a set of points, it is an entity in itself and it can have points located *on* it. The development of number symbolism and the duality of geometry and algebra through the Cartesian plane led to an environment appropriate for the calculus of Newton and Leibniz as a combination of conceptual embodiment and symbolic manipulation. The formal ideas of the Weiestrassian definition of limit and the various constructions of the real numbers as a complete ordered field are a subsequent development.

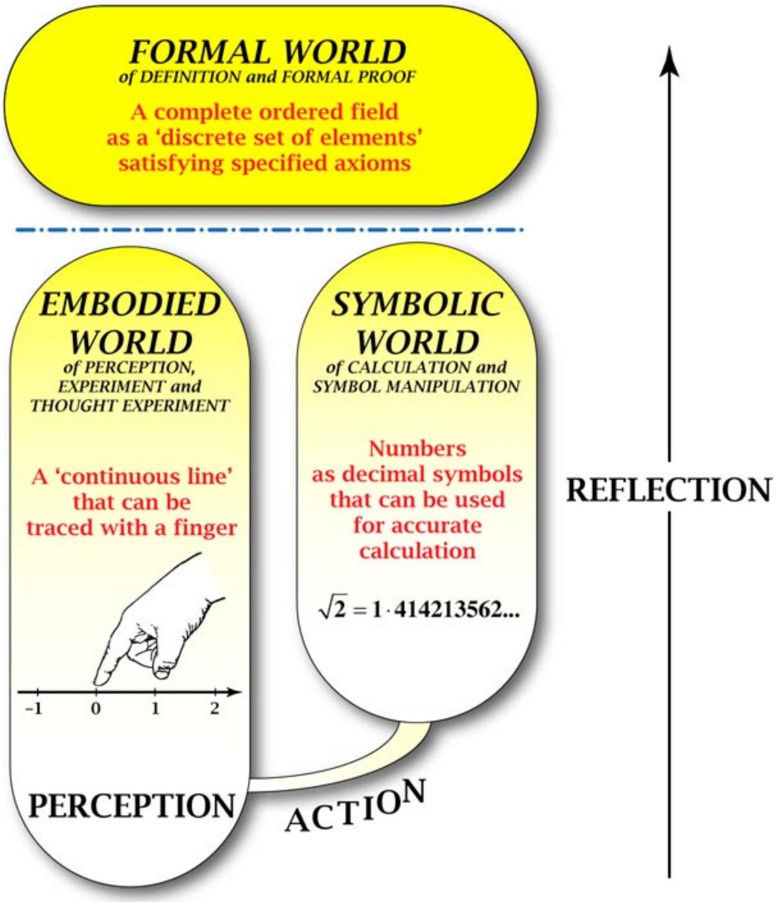

Figure 2: The real numbers in three different worlds

These various strands continue to co-exist today in the usual way that human society grows by adding new elements that enhance survival whilst maintaining old elements that continue to be useful. The calculus has geometric origins, symbolic calculations and manipulations, and more recent formal definitions and proof. As these all interact, the various meanings from different world-views build into a wide diversity of ideas in calculus and mathematical analysis. On looking more closely at each of the three worlds we uncover significant differences in the way that they view ideas.

Different warrants for truth

Each of the three worlds of mathematics has different ways of conceptualizing ideas and different warrants for truth (in the sense of Rodd, 2000). In the conceptual-embodied world, truth is based first on our fundamental human intuitions and later on more sophisticated thought experiments. Something is *true* if it is *seen* to be so. In the symbolic-proceptual world a formula is *true* because it can be *shown to be true* by *calculation* or symbolic *manipulation*. In the formal-axiomatic world, a theorem is *true* because it can be *proved* from the axioms and definitions (figure 3). In addition, each world increases in sophistication, as is evidenced by the van Hiele levels in Figure 4, where the first four stages move from recognition of objects, through description of some of the properties, then assembling appropriate properties to make definitions, then developing Euclidean proof referring to figures we build in our minds that have the required properties. Only at the fifth level, of formal proof, is the theory turned round where definitions *define* a concept regardless of any particular embodiment (Van Hiele, 1986).

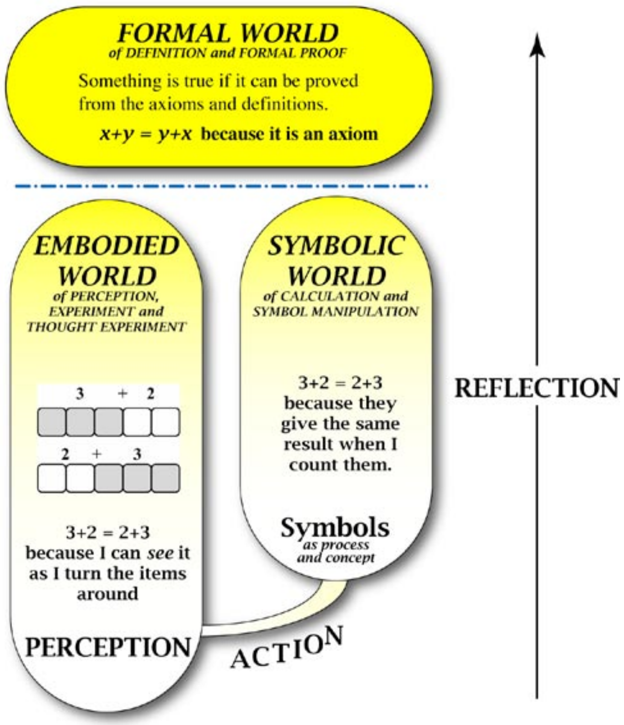

Figure 3: Different warrants for truth

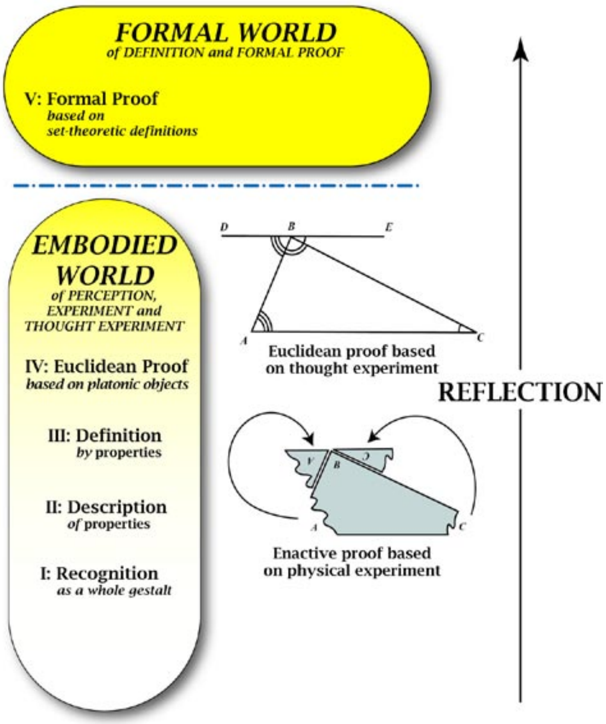

Figure 4: Stages of development in geometry, after van Hiele

A major problem that needs to be addressed is the precise role of proof in the calculus, particularly when this involves attempts to 'simplify' formal proofs to a level that students are hoped to understand, rather than build from the student's viewpoint to attempt to construct meaning. This suggests that the needs of the small minority who will go on to study analysis is being allowed to swamp the needs of the majority.

For example, much as we might like to believe it is different, professional engineers who design major structures never use 'mathematical proof' (Kent and Noss, 2000). Instead, they design a structure according to a professional Code of Practice. When the method fails (such as the building of the Millennium Bridge over the River Thames which oscillated dangerously in the wind), they go back to the drawing board and make a revised design. In this case the bridge oscillates longitudinally rather than laterally (as specified in the Code) as it resonates with pedestrians coping with a side wind. Their 'proof' that the bridge was now structurally stable was furnished by hundreds of engineers walking across the bridge to show that the solution 'worked'.

The essence of this argument, for the vast majority of calculus students, is that we do them a huge disservice by failing to cope with their needs. Most practitioners use two aspects of the calculus: thought experiments to think about the problem, and symbolic calculation and manipulation to provide a solution. Why is it necessary for these students to add further complications to the calculus that properly belong in formal analysis?

What difference does the computer make?

We first consider the three strands of mathematical thinking that blend together in a complementary fashion to give us the calculus and mathematical analysis. (Figure 5.)

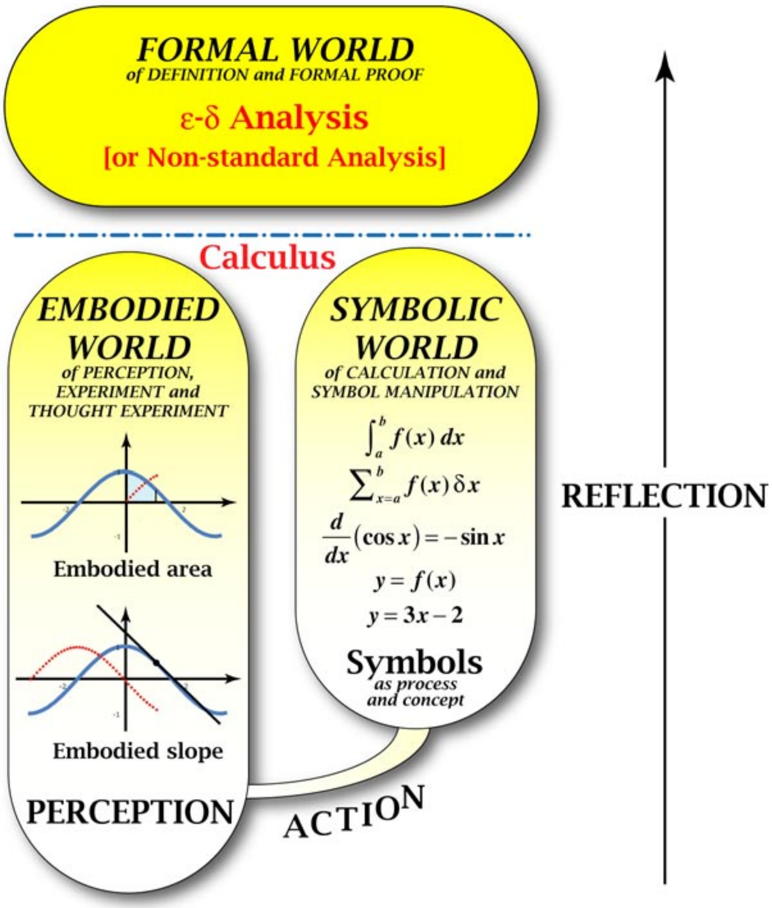

Figure 4: the conceptual structure of calculus and analysis

The introduction of the computer has radically changed the mathematical environment in which we can work. This works differently in the three worlds of mathematics. The *formal world* has the least benefit. Although symbol manipulation can be performed accurately (if idiosyncratically), with precise arithmetic of whole numbers and fractions, the arithmetic of real numbers can never be totally accurate, producing a mismatch between the finite machine, and the actual infinity of the theoretical limit process.

The symbolic world, on the other hand, benefits greatly because the computer allows us to tackle far more complex problems, performing calculations and symbol manipulations at a level of accuracy that would be difficult or impossible by hand. Indeed, computer algebra systems, such as *Mathematica*, *Maple*, *Derive*, *MathCad*, *MatLab*, take us into the realms of advanced calculus in three dimensions in an environment for applications and solutions of real-world problems using embodiment and symbolism rather than the formal world of epsilon-delta analysis.

In addition to drawing graphs, the computer benefits the *embodied world* in a more subtle way by providing an enactive interface, such as a mouse or touch-screen, that allows the user to control and experiment with visual representations.

At the beginning of the calculus, this can give embodied meanings for the idea that if the graph is 'locally straight' (meaning that under high magnification the graph *looks* like a straight line). This revolutionizes the beginnings of calculus, making the notion of limit an *implicit* embodied concept rather than an *explicit* formal definition. Instead of a formal introduction to the limit concept as a potentially infinite process, the slope of a graph can be conceived *visually* zooming in on the graph until one can *see* the slope of the graph as it magnifies to 'look straight'. One can then perform a thought experiment of *looking* along the graph to *see* the slope changing, to *see* the derivative.

With the computer giving an accurate picture, stepping along the graph of $\cos x$, one can *see* the graph of the slope function is '*minus* $\sin x$'—because it is *visibly* the graph of $\sin x$ upside down. One could then use the standard angle formulae for $\sin(A+B)$ to see how this might come about, as

$$\begin{aligned}\frac{\cos(x+h)-\cos(x-h)}{2h} &= \frac{\cos x\cos h-\sin x\sin h-(\cos x\cos h+\sin x\sin h)}{2h}\\ &= \frac{-\sin x\sin h}{h}\\ &= -\sin x\frac{\sin h}{h}\end{aligned}$$

For small *h*, this approximates to $-\sin x$ which fits the visual evidence.

Is this a proof? "No!" says the mathematician, because the same argument applied to $f(x)=|x|$ calculates derivative at the origin as zero, where it has different left and right derivatives. However, as an embodied proof, we say, "Yes!" It *is* a proof, *if we know that the graph is 'locally straight'*. In this case, the method of calculating the derivative is the same as the one-sided limit. In the embodied world we can *see* the graph of $\cos x$ is locally straight, so it *has* a derivative and the only problem is to *calculate* it.

In the same way, in the embodied world, one can *see* the area under a graph. The area will have a numerical value, so the main problem is not to prove it exists, but to *calculate it* as accurately as possible symbolically. The problem of 'existence' of area is a problem for the mathematical analyst studying the Riemann or Lebesgue integral. This problem lies in the formal world (of which the student as yet has no knowledge) and will mean nothing to the vast majority studying calculus who do not move on, not only to study, but also to *understand*, mathematical analysis. A combination of embodiment and symbolism can easily provide the cognitive foundation for the calculus with meaningful embodiments of formalism (**Tall, 2003**).

Has this happened in practice? In some contexts, perhaps, in other contexts, no. All our attempts to move into a new paradigm continue to be coloured by our pre-computer experiences, and the language of mathematics that includes successive layers of meaning laid down over generations. Speaking at a meeting of the *International Conference on Technology in College Mathematics Teaching*, Kenneth Hoffman pointed out the flaw of the attempt to reform calculus alone by saying that 'you will not maximise a system by maximising one of the components'. The search to see calculus taking up an appropriate role in the new technological era is a complex issue. To *see* the nature of the problem, we need to stand outside the system and see it as a whole.

Papers to be presented at ICME-10

Of the papers submitted to the Topic Subgroup, a selection of those considered appropriate for discussion are available on the website at

http://www.icme-organisers.dk/tsg12

Given the very limited time for discussion, a subset of these have been invited to give a brief presentation to outline their main ideas at the meeting.

The changes in the calculus that have occurred include both:

- mathematical (to use the new technology to advantage, in mathematics, in its applications, and to improve the teaching of mathematics to students)

and

- cognitive (to reflect on how we think about the calculus, and how the new technology changes our conceptions, to improve the learning of mathematics by students)

After the presentation of the current paper, the focus of attention will turn to the work in France, which is the country that has given us both the work of Bourbaki in formal mathematics and Brousseau in mathematics education. The French School focuses on giving students conceptual understanding of the formal mathematics within a didactic framework. **Isabelle Bloch and Maggy Schneider** will present aspects of the Francophone theoretical framework. In particular they will consider heuristic work in what we call the embodied and symbolic worlds as a preliminary stage of growth to lead students 'to cope not only with intuitions and mental objects, but also with pragmatic ones'.

The second session has three items relating to the beginnings of the calculus in relation to the fundamental understandings of real numbers (**Leviatan** and **Barthel**), the concept of derivative in a computer environment (**Giraldo and Carvalho**) and a broad teaching conception of the calculus building on traditional college calculus, taking into account the new technology (**Helfgott**). The fundamental understanding of the real numbers addresses the way in which decimal representations on the computer have been used in Israel to attempt to address the known cognitive problems relating to the limit concept. Giraldo and Carvalho present research from Brazil, which uses the cognitive theory of concept images and an embodied approach to calculus via local straightness to underpin a computer-based approach which considers the finite limitations of calculation to encourage students to construct the formal meanings of the calculus. In between these two presentations, Michel Helfgott will present his views as a teacher on teaching the calculus within the present USA calculus context, maintaining traditional views but introducing computer technology where it seems appropriate.

The third session has three presentations. **Erhan Bingolbali** presents an empirical study in an English university which reveals broad differences in meaning between the development of mechanical engineering students and mathematics students in the concept of derivative, one focusing more on rate of change, the other on aspects related to the tangent. Again we see our different cultures causing us to focus on different aspects of a complex theory. **Shestopalov and Gachkov** from Sweden then take us into the realms of the new teaching of the calculus and computational mathematics in which 'hands-on' sessions could add substantial understanding in the introduction of […] mathematical concepts'. The article also outlines the structural organisation of such a course focusing on 'hands-on' experiences rather than definition-theorem-proof aspects of a formal mathematics course. It is followed by a paper of **Bloch and Ghedamsi** based on a Francophone perspective in Tunisia, which focuses on the complexity of the limit concept in the changing environment from secondary school to university. This is framed in a range of theories including process-object encapsulation (characteristic of the proceptual-symbolic world used here) and the semiotic registers of Duval (which offer a deeper human analysis of different graphic, numeric and verbal representations) that relate to the different world views represented in this paper.

The fourth session includes two papers focussing on later aspects of the calculus. **Arslan and Chaachoua** from France consider the teaching of differential equations, questioning the dominance of algebraic approaches and relating them to numeric and graphical representations to give three distinct representations: numeric and algebraic (successive levels in the symbolic world in our terms) and graphic (which is part of the embodied world). **Ralha, Hirst and Vaz** present a Portuguese study looking at the use of *Mathematica* in producing and conceptualizing functions of two variables. It uses the symbolic interface of *Mathematica* to produce three-dimensional graphs and encourages students to share ideas about what they see, to focus on misconceptions that may arise and to construct a more meaningful relationship between symbolic and visual meanings. This lives in more advanced aspects of the symbolic and embodied worlds rather than shifting to the formal world.

The final part of the closing session is devoted to a synthesis and discussion of relevant questions raised in TSG 12 and the way they have been treated. The aim is to focus on future orientations in the practices and research in calculus learning and teaching.

In addition to those papers given as short presentations, there are several others that are on our web-site and highly pertinent to our discussion:

Floris gives a Francophone Swiss view using pocket computers to study numerical sequences having noted that approaches using only a symbolic approach led to a domination of symbolic aspects over numerical ones. The aim is to give an environment in which the numerical limiting process can provide experiences for the formal limit definition. The paper discusses the strengths and difficulties of such an approach.

Subbotin, Hill and Bilotskii, on the other hand, focus on the formal mathematical viewpoint, by revealing that the various definitions of 'elementary function' (which are the foundation of symbolic calculus) are often unclear or in need of greater precision, and suggest a careful analysis to produce a satisfactory definition.

De Ting Wu formulates practical ways in which a Computer Algebra System can be used in a modern Calculus course, focusing on the techniques and algorithms involved.

Prabhu, Porter and Czarnocha present a thoughtful combination of theory and empirical study focusing on historical methods of integration and testing their ideas out in the classroom.

Tarp produces an imaginative personal vision of the calculus in 'a sceptical fairy-tale study' that strikes out in new directions, quite different from traditional concerns.

A Challenge for the Future Conceptualisation of Calculus

Our challenge in this working group is to think very carefully where the teaching of mathematics is at the moment and where it is going. At this meeting we will have a wonderful range of presentations looking at certain aspects of the calculus. Our challenge is not the local description of calculus teaching of a particular topic, or the continuation of cultural elements that worked in the past and are being continued in the present, but a fundamental rethink of the global context of calculus in a new technological age. In our view, this should cater for growing students each developing their own cognitive path to satisfy a range of different needs. It should focus, as a goal, on a version (or versions) of calculus that is appropriate in the wider scheme of things, be it for the student who is going to be a pure mathematician, an engineer, or someone who studies the course purely for personal interest.

All students build their new conceptions on their cognitive experiences, initially based on embodiment and symbolism. Embodiment involves practical interaction with the world outside and thought experiments that constitute the inner embodied world. The symbolic world enables us to formulate and solve problems with great precision. A calculus course needs to build on these experiences and produce a course of value to each student in their own personal development. Let us hope that the working group can rise to the challenge and address these broader issues.

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TSG12-Calculus Programme
ICME-10 Copenhagen July 2004

Session 1

David Tall and Juan Pablo Meja Ramos: *Reflecting on Post-Calculus Reform.* (This paper.)
http://www.icme-organisers.dk/tsg12/papers/tall-mejia-tsg12.pdf

Bloch, I. & Schneider, M.: *Various Milieu for the Concept of Limit: From determination of magnitudes to a Graphic Milieu allowing Proof.*
http://www.icme-organisers.dk/tsg12/papers/bloch-schneider-tsg12.pdf

Session 2

Leviatan, T.: *Introducing Real Numbers: When and How?*
http://www.icme-organisers.dk/tsg12/papers/leviatan-tsg12.pdf

Barthel, L.: *A computerized interactive approach to real numbers and decimal expansions.*
http://www.icme-organisers.dk/tsg12/papers/barthel-tsg12.pdf

Helfgott, M.: *Five Guidelines in the Teaching of First-year Calculus.*
http://www.icme-organisers.dk/tsg12/papers/helfgott-tsg12.pdf

Giraldo, V. and Carvalho. L. M.: *The Role of Computational Descriptions and Conflicts in the Teaching and Learning of the concept of Derivative.*
http://www.icme-organisers.dk/tsg12/papers/giraldo-carvalho-tsg12.pdf

Session 3

Bingolbali, E.: *The Calculus of Engineering and Mathematics Undergraduates.*
http://www.icme-organisers.dk/tsg12/papers/bingolbali-tsg12.pdf

Yury Shestopalov and Igor Gachkov: *Teaching Computational Mathematics in the Real-time Mode.*
http://www.icme-organisers.dk/tsg12/papers/shestopalov-tsg12.pdf

Isabelle Bloch and Imène Ghedamsi: *The Teaching of Calculus at the Transition Between Upper Secondary School and University: Factors of Rupture. A Study Concerning the Notion of Limit.*
http://www.icme-organisers.dk/tsg12/papers/bloch-ghedamsi-tsg12.pdf

Session 4

Salahattin Arslan: *Reflections on the Teaching of Differential Equations: What Effects of a Teaching to Algebraic Dominance?*
http://www.icme-organisers.dk/tsg12/papers/arslan-tsg12.pdf

Elfrida Ralha, Keith Hirst, and Olga Vaz: *A Portuguese Study on Learning Concepts and Proofs: Multivariable Calculus and Mathematica.*
http://www.icme-organisers.dk/tsg12/papers/rahla-et-al-tsg12.pdf

Panel Discussion: David Smith, Michael Thomas, De Ting Wu.

Accompanying papers

Ruhal Floris: *Some Didactical Variables for the Study of Numerical Sequences using a Mathematical Pocket Computer.*
http://www.icme-organisers.dk/tsg12/papers/floris-tsg12.pdf

Igor Subbotin, Milla Hill, Nikolai Bilotskii: *An Algorithmic Approach to Elementary Functions.*
http://www.icme-organisers.dk/tsg12/papers/subbotin-tsg12.pdf

De Ting Wu: *CAS and the teaching of calculus.*
http://www.icme-organisers.dk/tsg12/papers/de.ting.wu-tsg12.pdf

V. Prabhu, J, Porter, B. Czarnocha: *Research into Learning Calculus: History of Mathematics and Mathematical Analysis.*
http://www.icme-organisers.dk/tsg12/papers/prabhu-et-al-tsg12.pdf

Alan Tarp: *Per-number calculus: A Postmodern Sceptical Fairy-Tale Study.*
http://www.icme-organisers.dk/tsg12/papers/tarp-tsg12.pdf