Introducing Three Worlds of Mathematics

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For several years I have been working with Eddie Gray and others on the ways in which we conceptualize different kinds of mathematical concept (Tall, 1995; Gray, Pitta, Pinto & Tall, 1999; Tall et al, 2000; Tall, Thomas, Davis, Gray, Simpson, 2000). Eddie and I were particularly interested in the distinction between objects formed in geometry (such as points, lines, circles, polyhedra) and concepts studied in arithmetic, algebra and symbolic calculus (numbers, algebraic expressions, limits). We concluded that the development of geometric concepts followed a natural growth of sophistication ably described by van Hiele (1986) (though subject to over-elaboration by others) in which objects were first perceived as whole gestalts, then roughly described, with language growing more sophisticated so that descriptions became definitions suitable for deduction and proof. However, numbers and algebra began through compressing the process of counting to the concept of number and grew in sophistication through the development of successive concepts where processes were symbolised and used dually as concepts (sum, product, exponent, algebraic expression as evaluation and manipulable concept, limit as potentially infinite process of approximation and finite concept of limit). We were also intrigued by the way in which experiences in elementary mathematics were reconceptualised from concepts that necessarily *had* properties to the formalism of advanced mathematics where specified *properties* are stated first as axioms and definitions, then other properties are deduced by formal proof.

In Gray and Tall (2002), we presented the idea that there were three (or possibly four) fundamentally different types of object, those that arise through empirical abstraction (in the sense of Piaget) by which is meant the study of *objects* to discover their properties, those that arise from what Piaget termed pseudo-empirical abstraction from focusing on *actions* (such as counting) that are symbolised and mentally compressed as *concepts* (such as number), and those that arise from the study of *properties* and the logical deductions that follow from these found in the modern formalist approach to mathematics. Piaget also formulated the notion of reflective abstraction (which is essentially a more sophisticated version of pseudo-empirical abstraction) in which the focus is on actions on mental objects which are routinized, then conceptualised as processes and considered as mental objects at a higher level.

Our possible 'fourth type of concept' described in that paper arose from distinguishing between abstract versions of pseudo-empirical abstraction (focusing on actions on mental objects) and abstract versions of empirical abstraction (focusing on the properties of abstract mental concepts). This could be considered as distinguishing between formal generalizations of arithmetic, algebra and symbolic calculus to give subjects like algebraic number theory, groups, rings, fields, vector spaces and analysis and formal

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generalizations of geometric objects to give non-euclidean geometries. However, because all types of formal mathematics involve specifying a system of formal axioms for a type of axiomatic structure and deducing the properties of that structure by formal proof, we settled on categorising all such formal under a single heading. Such a formulation was not set in stone. We were very open to suggestions and criticism to test and improve our ideas.

It was during the expansion of these ideas that I worked with Anna Poynter who, as Anna Watson before her recent marriage, was researching students' conceptualisation of vectors. Two papers (Watson 2001; Watson, Spirou and Tall, 2002) reveal the nature of our deliberations and are the first published indications of a developing theory of three mathematical worlds. Her study revealed two different kind of approaches to vectors in school: *geometric* with vectors as arrows representing various embodied concepts such as force, journey, velocity, acceleration, a *symbolic* approach based on calculation with matrices, and these were contrasted with a *formal* approach introduced at university level based on deduction from the axioms of a vector space. We realised that there were not only three distinct types of mathematical concept (geometric, symbolic and axiomatic), there were actually three very different types of cognitive development which inhabited three distinct mathematical worlds,

The first grows out of our *perceptions* of the world and consists of our thinking about things that we perceive and sense, not only in the physical world, but in our own mental world of meaning. By reflection and by the use of increasingly sophisticated language, we can focus on aspects of our sensory experience that enable us to envisage conceptions that no longer exist in the world outside, such as a 'line' that is 'perfectly straight'. I now term this world the 'conceptual-embodied world' or '*embodied* world' for short.

This is not the same as the notion of 'embodiment' in authors such as Lakoff, who focuses on all kinds of embodiment, including *conceptual*—which refers to conceiving concepts in visuo-spatial ways—and *functional*, in terms of the (possibly unconscious) ways of operating using human abilities as biological individuals. Lakoff and his colleagues assert, in their own broad meaning, that *everything is embodied* (Lakoff & Johnson, 1999, Lakoff & Nunez 2000). This is fine to make a point (that mathematics arises from biological human activity) but a classification with only one class is hardly helpful to analyse the nature of mathematical cognition. Instead I focus more on the notion of conceptual embodiment, which relates to the way in which we build more sophisticated notions from sensory experiences.

By formulating the embodied world in this way, it includes not only our mental perceptions of real-world objects, but also our internal conceptions that involve visuospatial imagery. It therefore applies not only the conceptual development of Euclidean geometry but also other geometries that can be conceptually embodied such as non-Euclidean geometries that can be imagined visuo-spatially on surfaces other than flat Euclidean planes and any other mathematical concept that is conceived in visuo-spatial and other sensory ways.

The second world is the world of symbols that we use for calculation and manipulation in arithmetic, algebra, calculus and so on. These begin with *actions* (such as pointing and counting) that are encapsulated as concepts by using symbol that allow us to switch effortlessly from processes to *do* mathematics to concepts to *think* about. In collaboration with Eddie Gray, we had realised that symbols such as 3+2 in arithmetic had dual connotations as process (addition) and concept (sum). The phenomenon by which a symbol can enable us to switch fluently from processes to do and concepts to think about was enshrined in the formulation of the term 'procept' (Gray & Tall, 1994). This second world I call the 'proceptual-symbolic world' or simply the '*proceptual* world'.

The notion of procept builds initially on actions in the embodied world and the initial stages of counting and early arithmetic are largely embodied. But the focus on the properties of the symbols and the relationship between them moves away from the physical meaning to a symbolic activity in arithmetic. This becomes increasingly sophisticated, with the introduction of more sophisticated number concepts (fractions, negatives, rationals, irrationals, infinite decimals, complex numbers, vectors in two and three, then *n* dimensions, and so on) that enable us to calculate and manipulate symbols with great accuracy and precision. It moves on into generalised arithmetic and algebra through the manipulation of symbols to specify and solve equations, and to more general concepts in symbolic calculus and beyond.

This suggests that many symbolic concepts arise from natural embodiments, and lead on to more sophisticated symbolism. In fact, there are many occasions when individuals do not encapsulate a given process into a thinkable mental object and instead carry out the procedures in a routinized way based on repetition and interiorization of learned operations. This happens not only with students who fail, it can happen in a very successful way, in which familiar procedures are performed on symbols that do not have natural conceptual embodiments for the individual concerned. For instance, in solving cubic equations in the sixteenth Tartaglia and Cardano (in his Ars Magna, 1545) performed calculations which led to the square roots of negative numbers that happened to cancel and, in the end, give a genuine real solution. Such 'numbers' were initially devoid of any link to the individual's geometric imagination. Nowadays we are presented with complex numbers embodied as points in the plane. This shows that, over time, using symbol manipulation (which is, of course, functionally embodied because it is performed by imagining we are moving symbols around) can lead, in the end back to a meaningful conceptual embodiment.

The third world is based on *properties*, expressed in terms of formal definitions that are used as axioms to specify mathematical structures (such as 'group', 'field', 'vector space', 'topological space' and so on). This I have called the 'formal-axiomatic world' or '*formal* world', for short. It turns previous experiences on their heads, working not with familiar objects of experience, but with axioms that are carefully formulated to *define* mathematical structures in terms of specified properties. Other properties are then deduced by formal proof to build a sequence of theorems. Within the axiomatic system, new concepts can be defined and their properties deduced to build a coherent, logically deduced theory.

The formal world arises from a combination of embodied conceptions and symbolic manipulation, but the reverse can, and does, happen. Formal definitions and formal deductions can lead to special theorems called ‘structure theorems’. These show that a formal axiomatic system can be proved to have properties that give it a new, more sophisticated embodiment. For instance, an axiomatic group can be embodied through Cayley’s theorem as a subgroup of a group of permutations, returning formal group theory to the embodied idea of permuting elements of a set. A finite dimensional vector space is structurally isomorphic to a space of n-tuples, wherein two and three dimensional vector spaces over $\boldsymbol{R}$ are just two dimensional and three dimensional space, with higher dimensions easily computed symbolically but less easily imagined visually. The definition of a complete ordered field can be proved to be unique, and therefore embodied by the visual conception of a number line ‘completed’ by adding the irrational numbers to the real line.

However, even these structure theorems may be embodied in ways that have incidental properties that suggest theorems that turn out not to be true. For instance, the ‘completion’ of the rational numbers to add the irrationals to give the real number line is often conceived as being the ultimate destination, with the real numbers filling out the whole line, banishing the possibility of infinitesimal quantities on the line. Yet this conceptualization limits our imagination and is simply untrue in the formal world of mathematics. It is very simple, mathematically, to place the ordered field $\boldsymbol{R}$ in a larger ordered field (e.g. the field of rational functions consisting of quotients of polynomials in an indeterminate x) which can be mentally imagined as a more sophisticated line that can be magnified to ‘see’ infinitesimal quantities. (Tall, 2002a).

What is clear is that the idea of ‘three worlds of mathematics’ that arose in my discussions with Anna Poynter, and have been published in their application to vectors in Watson, Spirou and Tall (2002) and acknowledged in Watson (2002) have ramifications that need careful consideration and long term reflection. For this reason I limited my publications to *specific* examples of the theory as the details of the theory were debated and developed.

First, it became clear that each world grows in sophistication and individuals travel different paths through these worlds in their individual mathematical growth. As an individual travels through each world, various obstacles occur on the way that require earlier ideas to be reconsidered and reconstructed, so that the journey is not the same for each traveller. On the contrary, different individuals handle the various obstacles in different ways that lead to a variety of personal developments, some of which allow the individual to progress through increasing sophistication in a meaningful way and others which lead to alternative conceptions or even failure to continue.

The embodied world begins with things being true because they are *seen* to be true and progresses to a level where these truths are justified in general through a verbal

Euclidean proof. It even underpins the development of non-euclidean geometries in specific geometric models, for instance as geometries on elliptic or hyperbolic surfaces.

In the same way, the proceptual world of symbols develops through a sequence of different contexts that often require re-construction of strongly held beliefs based on previous experience. For instance, the shift from handling whole numbers to handling fractions requires the individual to change from a context in which each whole number is succeeded by a unique 'next' whole number, with no other numbers in between, to a context where there is no 'next' fraction, and any two fractions have an infinite number of fractions in between. Likewise, the shift from arithmetic to algebra involves a change from a situation where symbols of arithmetic such as 3+2 have built-in operations of calculation, whereas an algebraic symbol such as $3+x$ only has a potential operation that cannot be carried out unless x is given a specific value. In a paper with ten co-authors (Tall et al, 2000) we looked in more detail at the development of procepts from the 'operational' procepts of arithmetic to the 'potential' procepts of algebra and on to the 'potentially infinite' procept of limit. As learners attempt to cope with these changes, they respond in different ways which affect their future development.

The development of proof (Tall, 2002b) also reveals features that distinguish one world from another. In the initial stages of the embodied world, truth is established by performing an experiment to see if the expected result occurs. Truth is established because it is *seen* to be true. In the proceptual world, truth is established by calculating with numbers and manipulating algebraic symbols. In the formal world, truth is established by formal proof from the axioms.

For instance the statement 3+4 = 4+3 is true in the early stages of the embodied world because it can be *seen* that if two sets of 3 and 4 are rearranged, the total remains the same. In the early stages of the proceptual world, it is true because the same answer is obtained whichever way it is calculated. In algebra, the statement $a+b=b+a$ is assumed to be true from earlier experiences with embodiment and calculation. In the formal world of axiomatic theories, $a+b=b+a$ is stated to be true as an axiom.

Building theories

Even though a theory of 'three worlds' has been developing for nearly two years, it is still under construction. A theory in progress is a particularly delicate creation. Theories do not appear fully formed. There is a period of exploration and incubation that precedes the eventual formulation. In the case of the theory presented here, although it is moving towards a stable form, it is still in the process of being filled out and refined. I say this, not to excuse errors of omission or obfuscation, but to be honest about the current stage of development and to encourage others to reflect on the ideas to help in filling them out. We build theories by reflecting on our experience and, because we have different experiences, we naturally produce different theories or different aspects of a theory that can be made stronger by refinement.

Theory-building needs to nurture surprising insights and allow them to grow; aggressive criticism comes later. Mason, Burton and Stacey (1982) talked about three different kinds of specialization in the book *Thinking Mathematically*; these are *random* specialization (in which one tries a few examples almost at random to get an idea of what is going on), *systematic* specialization (in which one considers particular cases in a more organized way to build a theory) and *artful* specialization (in which one chooses cunning cases that test the theory).

The development of the theory of 'three worlds of mathematics' has gone beyond random specialization and on to systematic specialization in three areas: calculus (Tall, 2003), proof (Tall, 2002b) and vectors (Watson, Spirou & Tall, 2002).

In the article 'Three worlds of mathematic and the imaginary sphere', Matthew Inglis (2002) has published an artful specialization to test the theory of 'three worlds of mathematics' based on the evidence of these three systematic specialisations. It is a criticism of a growing theory which can usefully test the theory and either destroy it, or, as I shall shortly show, make it stronger.

First I must give credit to Anna Poynter, who is the only person who has published an article with me on 'three worlds of mathematics' and who was instrumental in inspiring my own ideas on this topic. Inglis refers to 'Gray and Tall's three worlds', however this is factually incorrect. The paper with Eddie Gray to which Inglis refers speaks of 'three forms of mathematical concept' but *not* of 'three worlds of mathematics'. I credit Eddie with all the inspiration for the notion of 'procept' and for maintaining my productivity in research in a wide range of areas over many years by his support and insight. Indeed, I am certain that, without his continuing wisdom and inspiration I would not have developed my ideas anywhere near getting close to the idea of 'three worlds'. However, the actual origins of the 'three worlds of mathematics' were in Watson (2002) and Watson, Spirou and Tall (2002) which were substantially written before any other papers on the topic.

The conceptual leap from three forms of concept to three worlds of mathematics may seem simple, but in practise it has proved to be both profound and daunting. It is one thing to have an insight into three different kinds of mathematical concept formed in different contexts; it is a much greater leap to claim that there are (at least) three distinct worlds of mathematics, each with a different mode of development and (in the formulation of Melissa Rodd (2000)) each with a different kind of warrant for mathematical truth.

Matthew's beautiful example of the imaginary sphere rightly challenges the published papers which referred to specializations in which such considerations are not discussed. However, within the wider theoretical framework, where his contribution is much appreciated, it is clearly an example arising out of the proceptual world of symbolic manipulation in which the algebra of spheres $x^2 + y^2 = r^2$ is applied to the case where $r^2 = -1$. In this context, the conceptually embodied meaning of spheres described algebraically no longer applies. This is no different from the case of complex numbers which, at their inception, had no conceptually embodied meaning. Even when complex numbers were visualized as points in the plane—as early as Wallis in his 1685 book on algebra, and diverse authors such as Wessel in 1797, Gauss in his 1799 doctoral thesis and Argand in 1806 re-invented complex numbers as points in the plane—it was still possible for De Morgan (1831) to state that the imaginary expression $\sqrt{(-a)}$ and the negative expression $-a$ indicated 'some inconsistency or absurdity [...] since $0-a$ is as inconceivable as $\sqrt{(-a)}$.'

The imaginary sphere is part of the natural process of extending the manipulations of symbols that have meaning in the proceptual world to a situation where the corresponding link to the embodied world no longer holds. It is parallel to the idea of using the square root of a negative number as a manipulable symbol before it has a conceptual embodiment. The fact that neither Matthew nor I can 'see' an embodiment, just as Cardan and Tartaglia could not 'see' a conceptual embodiment of $\sqrt{-1}$ is not a denial of the distinction between the conceptual-embodied world and the proceptual-symbolic world. It is an affirmation that developments in the latter can operate independently of the former. The proceptual world is *not* just an extension of conceptual embodiment, it has properties of its own which work (in a functionally embodied manner if you wish) in a way that need not have immediate counterparts in the embodied world.

A word about words

As I work with many colleagues developing the notion of 'three worlds of mathematics' one of the major considerations is not only to develop the ideas but to communicate them in ways which make sense to others. The term 'embodiment' has caused problems here, particularly since Lakoff has written so much on the subject. As I have explained, my own notion of 'embodiment' relates to how we consciously embody concepts in visuo-spatial ways, which, corresponds to ways in which we embody an abstract concept by giving it a familiar concrete referent. This corresponds closely to Lakoff's discussion on 'conceptual embodiment'. In arithmetic and algebra there are functionally embodied elements that involve 'moving symbols' around, 'cross-multiplying' by moving b and d in $\frac{a}{b}=\frac{c}{d}$ across to get $ad=bc$, and so on. But as we perform manipulations in algebra we soon do it in a world of its own which no longer relates step-by-step with operations in the problem context. We model a situation algebraically, solve the algebraic problem by concentrating on the symbols and their manipulation, and then return to the original situation to interpret the solution. Thus the links in symbolic manipulation usually operate in a world which temporarily suspends the conscious correspondence with the physical world. This relates to my distinction between the embodied and proceptual worlds.

In a draft of a book that I am writing on mathematical growth (available in its partially complete state for private study and comment on the web at davidtall.com/mathematical-growth, but not yet ready for publication), the chapter on 'Language and Three Worlds of Mathematics' begins with the famous quotation from *Alice through the Looking Glass*:

> 'When I use a word', Humpty Dumpty said, in a rather scornful tone, 'it means just what I choose it to mean – neither more nor less.' 'The question is,' said Alice, 'whether you can make words mean so many different things.' 'The question is,' said Humpty Dumpty, 'which is to be the master – that's all.'
> (Carroll, 1872)

In building a new theory, words which have familiar meanings may be used in different ways by different individuals, so it is first important to give such words appropriate meanings in the new context. There are at least two different ways of doing this. One is to invent totally new words. I have taken this route on a small number of occasions, for instance, the notion of *procept* (Gray & Tall, 1994) which is not currently a word you will find in any dictionary as it has only recently been introduced. Another is to put old words to new uses. This is particularly valuable when one puts together two words, each with a clear meaning, in an unusual juxtaposition, such as the notion of *relational understanding* (Skemp, 1976) or *concept image* (Tall & Vinner, 1981). Then the new phrase is sufficiently novel to allow the researcher putting the idea forward to begin with experiences that others may share and put them together in a new way that is relevant to the new theory.

Interestingly, neither 'relational understanding', nor 'concept image' is currently used with the original meaning that was first coined. The notion of *concept image* in the paper I wrote with Shlomo Vinner in 1981 has become the classical reference for the idea. However, Shlomo arrived to work with me for a few weeks in 1980, bringing with him a paper on the notion of 'concept image' in the context of geometry (Vinner & Hershkowitz, 1980). I immediately grasped what I thought was the *sense* of his idea and applied it to a collection of data I had collected but lacked the tool to enable an insightful analysis. In doing so, the phrase used in the 1981 paper said that the concept image is 'the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties *and processes*' (my italics). The added words 'and processes' shifted the idea from a visuo-spatial concept to include *all* the cognitive structure that is related to the concept in question.

Similarly in his original (1976) paper on instrumental understanding and relational understanding, Skemp introduced a meaning that was quite different from the original uses of the terms coined by his Stieg Melin-Olsen in a more social context. Skemp found the distinction between two forms of understanding to be so helpful that he spent several years trying out these ideas on audiences before he published his famous article on the idea. In fact, he never *defined* instrumental or relational understanding (not even in the original paper, if you read it closely). He *talked* about them and *wrote* about them, but he was always interested in how others understood their meaning. I can still see his knowing smile as we discussed relational understanding in the early seventies when I had the privilege of being his last doctoral student.

In his much-loved book on *The Psychology of Understanding Mathematics,* Skemp (1971) asserted that it is not possible to *define* higher order concepts, what is necessary is for the individual to encounter examples of the concept to construct the higher order meaning. Thus in developing new higher order concepts, he suggests it is necessary to start from a variety of good examples. I did this for the notion of different worlds of mathematics by looking at examples in very different contexts.

In building a new theory, it is usually the *total structur*e that one is trying to create, including all the relevant concept imagery, not just a distilled definition. The formulation of a subtle cognitive theory therefore cannot be reduced to a short formal definition alone. It needs to be established at a level where it applies to different contexts and links need to be made between ideas in a manner that is relevant to the overall theoretical perspective.

In my publications I have used the term 'embodiment' with a meaning that I believe is consistent with the colloquial notion of 'giving a body' to an abstract idea. This includes all cases of conceptions in visuo-spatial terms, not only those which arise from perception of actual objects. As a result of Matthew's intervention, I have found that many individuals interpret my writing to refer only to perceptual embodiment. I have therefore moved to using a two-word definition to alert the reader to this fact and now use the name conceptual embodiment' at least in the initial stages to alert the reader of my intended meaning.

Matthew closes his criticism with another reference to the use of words, by referring to the 'frustrating habit' of Gray and Tall in using the words 'object' and 'concept' interchangeably. We don't do that. We use both words in clear contexts to express appropriate ideas. Mathematicians (such as Dieudonné 1992) freely use the word 'object' for the things we talk about in formal mathematics, such as a 'group' or a 'matrix' or a 'topological space'. In colloquial language, however we speak of counting processes and number *concepts*, not number *objects*. For instance we use the term 'concept of number' and certainly not 'object of number'. We also speak of 'fraction concept' rather than 'fraction object'. When the term 'concept' is used in this context, it therefore has the meaning that mathematicians consider to be an (abstract) object. However, it is not an object in the sense of a physical thing that we can perceive in the world. To overcome this difficulty, I freely use the term 'concept' when I speak about numbers, fractions, algebraic expressions and so on, in a manner which fits with common useage but, at the same time is in a context where the symbol refers dually to process or concept (as a mental object).

English is a language where we intentionally use the richness of diverse meanings to express rich ideas which may be ambiguous if isolated as individual words but are intended to be made clear by the context. It is fortunate that we have an English language with 500,000 words to speak in flexible and suggestive ways. In a language such as Chinese with 5000 regular characters, individual characters have little meaning in isolation, but rich meaning in combination.

1n building a theory of different worlds of mathematics, therefore, I cannot begin by stating definitions and proving theorems. I have to begin with ideas that I test out by trying out formulations to see if they make sense to others and to test the ideas in several different contexts (calculus, vectors, proof) to see if they have a useful practical meaning. In the examples I chose, it is clear from the response of many readers that the idea I expressed in terms of ‘embodiment’ have been made in a context that has over-emphasized the relationship with perception of the physical world and at the expense of the longer-term conceptual embodiment of mental concepts as visuo-spatial concepts.

I thank Matthew for his attention to the limitations in my initial examples and will continue to work at a broad theory of the growth of mathematical concepts that gives insight into the nature of mathematical growth.

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