Cognitive Units, Connections and Compression in Mathematical Thinking[1]

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This paper presents a theory of 'cognitive units' to describe mechanisms for compressing information into manageable mental units with powerful internal and external links. The first part of the paper addresses the fundamental ideas, relating them to other psychological concepts and to the neurophysiological activities that compress conceptual relationships into manageable cognitive structures. These include process-object duality, schema-object duality, cognitive networks, cognitive hierarchies and the varifocal theory of Skemp. We argue the necessity of having a more flexible view of the structures used in mathematical thinking to gain insight into why mathematics can be simple and powerful for some yet complex and difficult for others. In the second part of the paper we use the theory to analyze how different students cope with the standard contradiction proof that √2 is irrational and its generalization to the irrationality of √3. We identify three very different cognitive units that play fundamental roles and use them to reveal the contrast between the tight cognitive structure of those who grasp the meaning of the proof and the diffuse structure of those who do not.

1. Cognitive units and connections

Mathematical thinking is handled by the biological structure of the human brain. As a multi-processing system, complex decision-making is reduced to manageable levels by suppressing inessential detail and focusing attention on important information. We begin with a piece of cognitive structure that can be held in the focus of attention all at one time. This might be a symbol, a specific fact such as '3+4 is 7', a general fact such as 'the sum of two even numbers is even', a relationship such as '$\sin^2\theta + \cos^2\theta = 1$', a mental picture such as a unit circle centred on the origin, a step in an argument, a theorem such as 'a continuous function on a closed interval is bounded and attains its bounds', and so on.

These pieces of cognitive structure do not exist in a vacuum. Intimately connected with them are other ideas within the mind that may readily be called to the focus of attention. These vary from one individual to another. For some, the number sentence '3+4 is 7' may simply be rote-learned as an isolated verbal fact. More usually it is related to a larger structure of arithmetic knowledge, such as '3+4 is the same as 4+3', '7–4=3'

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or ‘3+4 is one less than two fours, so it is seven’. These conceptions are often so intimately connected that they seem to be different aspects of the same cognitive structure in the mind. Such a cognitive structure is called a ‘cognitive unit’ (Barnard & Tall, 1997, Barnard, 1999, Crowley & Tall, 1999, McGowen, 1998):

> A ***cognitive unit*** consists of a cognitive item that can be held in the focus of attention of an individual at one time, together with other ideas that can be immediately linked to it.

This terminology will be applied to the full range of possibilities, from a single idea with few links, to a structure with many linkages. Our main interest, however, is in the development of *rich* cognitive units that provide powerful ways of thinking. An example of this was expressed beautifully by a Polish mathematics educator who, when asked what ‘twelve’ meant to him, replied that it is ‘a cloud of facts that cluster around like butterflies.’ For him twelve was not a single symbol but a cluster of relationships, ‘twelve is six and six, it’s four threes, it’s fourteen minus two, and so on.’ He went on to explain that, for him, arithmetic is not just computation, it is a *choosing* from the cloud of facts at his disposal.

1.1 Previous uses of the term ‘cognitive unit’

Given that the term ‘cognitive unit’ is made up of two widely used words, it is not surprising to find that the phrase has been used with a range of meanings over the years. Baldwin used the term to describe an idea of Johann Hebart (1776-1831), who established psychology as a subject in its own right, separate from the parent disciplines of philosophy and medicine:

> By presentation or idea (Vorstellung) Herbart means, as German psychology always means, a *cognitive unit*, image, or idea; something presented to the mind, having objective character, not something felt or willed. (Baldwin, 1913, pp. 64, 65 (our italics))

The notion we intend to discuss in this paper therefore has some (but not all) of its origins in the very foundations of psychology in the early nineteenth century.

In the intervening years, the terms ‘cognitive’ and ‘unit’ have been joined with a variety of different meanings, often as an auxiliary idea rather than a generative concept. For instance, Gustav Fechner (1801–1887) founded the study of experimental psychology by developing indirect methods of measuring mental phenomena using three basic methods—differences that are just noticeable, right and wrong cases and average error. His most famous result, known as ‘Fechner’s law’, states that sensation is proportional to the logarithm of the stimulus (Fechner, 1860). In some instances (e.g. Starbuck, 1998), these measurements have been termed ‘cognitive units’—a quite different meaning from the Vorstellung of Herbart.

Other uses of the term refer to various structures being ‘units’ in a given cognitive context. Nunn (1994) asserts that the most basic ‘cognitive unit’ is not the nerve cell synapse, but the microtubular structure in the cells. Swenson (2001) describes ‘concept learning [as] the assembling of attributes into a cognitive unit, or getting “the idea” that attributes belong together.’ Luchjenbroers (1994) speaks of ‘the schema as the essential cognitive unit for information, organisation, integration and retrieval processes.' Gaines (1989) describes 'the basic cognitive unit normally considered—the individual as an anticipatory system, experiencing the world, perceiving it, acting upon it, modeling it to predict and explain it, interacting with it to experiment with and control it.' Heider (1946) used the term 'cognitive unit' to refer to a context consisting of two individuals, an object, and the relationships between them, enabling him to build what he termed a 'balance theory' of perception. Piero Scaruffi (in preparation) goes even further to observe that 'thanks to language, the entire mankind becomes one cognitive unit [turning] the minds of millions of individuals into gears at the service of one gigantic mind'. The theory of 'distributed cognition' (Hutchins,1995; Salomon, 1993) builds parallel cognitive models involving cognitive units at a range of levels from the neural level, the level of the individual, the coordination of the individual and a set of tools, up to a group of individuals interacting with each other and using a specific set of tools. What is interesting about all these examples is that they use the notion of 'cognitive unit' as a given in a particular context, rather than in the nature of a cognitive unit itself.

An exception to this general trend is Anderson's (1983) cognitive model of information processing, based on 'cognitive units' that are expressed as three types of data simulation—spatial images, textual data, and propositions. These three aspects of human thought happen to be easy to represent in a computational theory but they do not exhaust the types of cognitive unit that we wish to consider in the mind of the individual. Human thought processes are more diverse. For instance, the human senses of smell, taste, touch, balance, give us conscious thought experiences that are very different from the three given data types. Moreover, we would contend that the most important data type in mathematics, the written symbolism of arithmetic and algebra, behaves in a special way that does not fit into this classification. The symbol '2+3', for example, is not classifiable only as a spatial image, textual data or as a proposition or even as a useful combination of the three. As a spatial image, it may well link to the image of two objects and three objects or to the layout of the symbols themselves, but it is more than this. It is not simply text in the form 'two plus three', nor is it a proposition. It evokes a whole complex of mental ideas. The young child grows through a succession of counting techniques, such as 'count-all', 'count-both', 'count-on', leading to succinct 'known facts' such as 2+3=5. These known results then may become manipulable number *concepts*, that are used to derive related concepts such as '22+3=25'. Any useful theory of cognitive growth must do more than identify separate components such as images, texts and propositions; it must also model the way in which sophisticated thinking develops in the human mind.

1.2 Compression of knowledge

The manner in which various pieces of cognitive structure become related to give small but richly connected cognitive units will be called *compression*. This terminology is taken from the writings of Field's Medallist, William Thurston:

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics. (Thurston 1990, p. 847.)

This idea does not mean that the cognitive structure is physically compressed in the mind of the individual, although there is evidence from brain scanning that the repetition and routinization of processes reduces the range of brain areas involved in a given task (Edelman & Tononi, 2000, p. 51). What is essential is that the brain areas used become so intimately linked that referring to one will trigger others as part of what seems to the individual to be a single structure. In practice, quite distant parts of the brain are connected together in this way, so that cognitive units in the focus of attention need not link their activity to just one part of the brain. The opposite is true. The cognitive units in the focus of attention typically result from the simultaneous activity of a wide range of neuronal modules scattered throughout the brain.

Every neuron in the brain consists of a cell with frond-like dendrites that receive electrochemical charges from other neurons. When a threshold is reached in the build-up of the internal charge, the neuron fires a charge down a single pathway called the axon, which divides into many branches to connect to other neurons. The electrochemical messages can only pass in a single direction in a single neuron. However, neurons operate together in neuronal groups and the combined axonal links within and between groups allow a highly complex system of connections to be made.

This is achieved by neurons with two essentially distinct designs. One is a local neuron (or *interneuron*). These are densely packed together with short dendrites extending in all directions to a distance of about a tenth of a millimetre (roughly fifty times the size of the cell body). They are then able to operate together in consort as a neuronal group performing more complex tasks. The other type of neuron is the *projection neuron*, receiving pulses from dendrites about a millimetre in length, sending pulses down a long axon linking to another part of the brain. Projection neurons include connections between hemispheres, and to targets in the brainstem and spinal cord up to a metre in length (Freeman, 1999, pp. 52–54). They allow separate subsystems to work together to give a higher order of mental activity.

For instance, our visual system presents us with a meaningful dynamic picture of the outside world using a complex structure of regions that perform different tasks. There is a region of the cortex at the back of the brain which has a two dimensional representation of the sensations on the retina; a number of other regions identify colour, edges, orientation of edges, movement of edges, whole objects distinct from background and provide the ability to track the movement of these objects. Several subsystems doing different tasks are linked by projection neurons to yield our total perception of the external world.

(As mathematicians, we have been brought up using theories where mathematical systems have substructures (eg subgroups) and quotient structures (eg quotient groups), the latter found by identifying elements in the overall structure in a special way and operating on the identified collection as if it were a single element. Although the analogy is crude, we cannot help but note that not only does the brain have biological subsystems, it has biological quotient systems as well.)

In general, neuronal activity takes place with a background level of firing, which increases in frequency when a neuron is activated by others. The connections then become more sensitive to further firing from the same source and may stay in this state for several hours or even days. During this sensitive time, the neuron's threshold is lowered and it may fire even though it does not receive the same level of stimulation. This gives a 'recency' effect in which ideas used recently are more likely to be evoked. Repeated firings of the same neuronal configuration make the connection even more sensitive to a point where the slightest stimulation to one group will cause a simultaneous firing in a whole collection of linked neurons, forming a memory. This process is called *long-term potentiation* (see, for example, Carter, 1998, p. 159).

Long-term potentiation is the underlying process that leads to brain substructures operating in synchrony through cooperative firing within and between substructures. It is also the mechanism by which repeated actions become routinized so that they occur subconsciously and automatically. It underpins the capacity of the brain to form tightly connected structures that give rise to the conscious use of cognitive units with rich connections. Such units provide mental tools that carry not only a single piece of information in the focus of attention, but a subtle network of related ideas readily available for human thinking.

1.3 Different forms of memory

The process of long-term potentiation is the essential mechanism underlying the formation of human memory. However, as an evolutionary structure, the brain is more a serendipity collection of evolved units than a technically designed structure (Crick, 1994, p.10). Memory within the brain occurs in a number of qualitatively different ways that are interconnected and mutually supportive. For our purpose, it is useful to highlight several different categories. *Episodic memory* is the recall of previous events. These are not recorded in the brain as a sequential video, rather as a skeletal story, with specific perceptual stimulations (sights, sounds, smells, feelings) that are often embroidered when remembered over the years. In mathematics this might include 'what he told us to do', 'what she said', 'how he did it', together with emotions stirred at the time, such as pleasure, confusion, or fear. Episodic memory affects the learning of mathematics, both positively by remembering ideas emphasized during the lessons, but also emotionally in positive and negative ways by recalling feelings aroused at the time.

Semantic memory is detached from the original event when the ideas were first met to have a life of their own—the colour blue, the number three, the concept of justice. In mathematics, episodic memory is often recalled by students, attempting to remember an earlier mathematical activity, whereas semantic memory is required to provide a sound basis for conceptual links relating one mathematical idea to another.

Procedural memory provides the ability to carry out a sequence of activities without having to think consciously of every step. Procedural memory is a vital tool in utilizing the human brain to maximum effect. By routinizing sequential activities (through long-term potentiation), so that they are performed largely unconsciously, the load on conscious thought processes is alleviated, allowing more opportunity to think of other things. For instance, one may keep an overall problem-solving goal in mind, whilst carrying out routine procedures.

The laying down of procedures can be performed by rote-learning or by practicing skills based on meaningful relationships to automate performance. Skemp (1987) clearly distinguished between the two:

> This automatic performance of routine tasks must be clearly distinguished from the mechanical manipulation of meaningless symbols. (Skemp, 1987, p. 62)

Mechanical rote-learning can give a useful, if limited, automatic fluency; it can also lead to isolated ideas that may be prone to error. Focusing on meaningful relationships on the other hand can lead to a form of memory, that Davis et al (in preparation) have termed *explanative memory*. This is the capacity to recall information by reconstructing it consciously when desired. For instance, in learning the formulae for the trigonometric expressions such as sin(A±B), cos(A±B), it is only necessary to remember the formulae for sin(A+B) and cos(A+B) and deduce the other two using sin(–A)=–sinA and cos(–B)=cosB. Or one may remember that the area of a triangle is 'half base time height' by imagining the triangle placed in a rectangle of the same base and height, and 'seeing' the triangle having half the area of the rectangle. Explanative memory utilizes 'relational understanding' (Skemp, 1976) to support knowledge structures through conceptual relationships. From our perspective, we see explanative memory as building and using links between cognitive units.

Anderson (1983) formulated the distinction between 'declarative knowledge' and 'procedural knowledge' where the former is knowing 'what to do' and the latter knowing 'how to do it'. From this viewpoint, teaching and learning involves an initial declarative stage where the scene is set and success in learning is sought by the individual performing activities in a manner that becomes fluent, freeing conscious memory to attend to other problem-solving aspects. There is, however, a distinct gap between saying 'what to do' and the learner forming appropriate internal links to learn 'how to do it', in a meaningful sense. We consider this in terms of the quality of cognitive units formed by different individuals.

A cognitive unit can operate not only as a form of shorthand for a mental idea, a *rich* cognitive unit also *carries with it*—just beneath the surface—the structure of the idea, and is *operative* in the sense that the live connections within the structure are able to guide its manipulation. Such activities may set up new links that may in turn become increasingly strong so that new cognitive units may be formed, building a network of nested mental structures that may span several layers of thought.

Such a structure offers a manageable level of complexity in which the thought processes can concentrate on a small number of rich cognitive units at a time, yet link them or unpack them in supportive ways whenever necessary. This gives a cognitive structure that is both *simple* (because it only uses a small number of cognitive units to relate) but also *sophisticated* (because the rich cognitive units are formed as well-connected subsystems of ideas). Without compression, the quantity of information that is being used would grow and become increasingly complex to handle. This can occur with a relatively small number of disconnected procedural pieces of knowledge. But with compression of related ideas into cognitive units, the knowledge remains not only at a level which can be handled by the small human focus of attention, but it also allows those ideas to have a hierarchy of sophisticated structures within them.

These ideas are closely related to Skemp's (1979) 'varifocal theory' of cognitive concepts, where a concept may be conceived either as a global whole, or viewed under closer scrutiny to reveal more subtle levels of detail. He referred to this conceptual detail as the *interiority* of the concept. Cognitive units with a rich interiority (appropriate for the task in hand) give the individual flexible but easily used facilities for thinking.

In Skemp's examples of varifocal concepts, the nodes of the hierarchical classification are themselves concepts. For instance his original example was of a shopping list. At one level of detail are the items in the list, such as 'cheese' and at the next level each item has a variety of different possible choices of item ('cheddar', 'edam', 'gorgonzola').

Thurston's notion of compression is more flexible and has a looser structure. For instance, by referring to an idea both as 'it', and also as 'just one step in some other mental process', he implicitly refers to a flexibility and a duality between mental objects and mental processes. Such a broader structure seems to fit the thinking processes that we observe in mathematicians and students. In the next sections we consider different ways in which such compression can occur.

1.4 Process-object duality

Process-object duality is at the heart of several theories of mathematical development, for instance the *encapsulation* of process into object (Dubinsky, 1991) or the *reification* of process into object (Sfard, 1991). Unlike Skemp's theory that sees the schematic structure consisting of object-like concepts linked by properties and processes, these encapsulation theories describe how sequences of activities can become routinised into thinkable processes that are then in turn conceived as mental objects. This is described by Asiala et al (1997) as follows:

> According to APOS theory, an *action* is a transformation of mathematical objects that is performed by an individual according to some explicit algorithm and hence is seen by the subject as externally driven. When the individual reflects on the action and constructs an internal operation that performs the same transformation then we say that the action has been *interiorized* to a *process*. When it becomes necessary to perform actions on a process, the subject must *encapsulate* it to become a total entity, or an *object*. In many mathematical operations, it is necessary to de-encapsulate an object and work with the process from which it came. A *schema* is a coherent collection of processes, objects and

> previously constructed schemas, that is invoked to deal with a mathematical problem situation. (Asiala et al, 1997, p. 400.)

Sfard (1991) proposed a corresponding sequence, affirming that operational mathematics (use of processes) almost invariably *precedes* structural mathematics (use of objects). Dubinsky and his colleagues followed the process-object sequence for several years before the strict sequence was loosened:

> … although something like a procession can be discerned, it often appears more like a dialectic in which not only is there a partial development at one level, passage to the next level, returning to the previous and going back and forth, but also the development of each level influences both developments at higher and lower levels.
> (Czarnocha et al, 1999, p. 98.)

Gray & Tall (1994) saw the role of the *symbol* as being pivotal in switching between process and object. A symbol such as '3+4' could act as a pivot between a process (of addition) and the concept (of sum). This immense power—which is characteristic of symbolism in arithmetic, algebra and calculus—allows the thinker to switch between using the *symbol* as a *concept* to think about or as a *process* to calculate or manipulate to solve a problem. Gray & Tall (1991) initially formulated the notion of *procept* as a combination of process and concept evoked by a single symbol. This theory saw the notion of procept becoming richer (in interiority, to use Skemp's terminology) as different symbols and processes represented the same object, for instance, 6 as 3+3 or 5+1, or 2+4. The notion of 'procept' was later extended (Gray & Tall, 1994) to include all the triples of process-object-symbol that have the same object in a given cognitive context. For instance, 6 is a procept which embraces 3+3, 2+4, 5+1, and so on. Later in the development of the individual, it might come to embrace 12/2, $\sqrt{36}$, $3 \cdot 5 + 2 \cdot 5$. In our terminology, a procept is therefore a special case of a cognitive unit that grows with interiority as the cognitive structure of the individual gets more sophisticated. Furthermore, *proceptual thinking* involves the decomposition and recomposition of procepts, as in

$$\begin{aligned} 7+6 &= 7+3+3 \quad \text{(decomposing 6 into 3+3)} \\ &= 10+3 \quad \text{(composing 7+3 into 10)} \\ &= 13, \end{aligned}$$

together with any other appropriate strategy using known facts and counting. In this way, children who develop flexible proceptual methods have a built-in engine to derive new facts from old, more capable of dealing with place-value and the arithmetic of larger numbers, whilst those who remain with the security of counting have less appropriate cognitive structure to deal with more subtle arithmetic of larger numbers.

This is not to say that *all* cognitive development proceeds by encapsulating processes as mental objects. The brain has vast areas *designed* to deal with objects, including the visual cortex. Its structures are genetically laid down with substructures that we earlier described to perceive edges, orientation of edges, movement of edges, identification of objects given by edges, colour, depth perception, and so on. These physically distinct areas are linked together by the axons of projection neurons to produce a unified gestalt for vision that is a natural operation of the brain. Gray and Tall (2001) suggest that it is not that a process itself *becomes* an object, but that links are made between the process and existing object-like imagery to produce a total structure with both process and object-like aspects. For instance, number concepts can link both to visual imagery of sets with given numbers of objects, or to number symbols that are written, read, spoken, heard and manipulated mentally, 'as if' they are mental objects.

1.5 Schema-concept duality

The biological activity of the brain involves two different kinds of mental activity that are both referred to as schemas or schemes. One is a sequential action scheme that occurs in time and is stabilized by long-term potentiation, strengthening and coordinating cognitive links such as the 'see-grasp-suck' scheme in the young child. From such linear schemes, various alternatives may arise at different times, yielding a broader brain structure offering alternative links in a multi-connected schema. The process-object theories seem to focus more on the first of these, theorizing that sequential schemes are encapsulated or reified as mental objects. The second type of schema offers a more subtle way of building up mental concepts that can operate flexibly as cognitive units.

Skemp's varifocal theory was given a highly insightful gloss by a graduate student, Robert Zimmer, during a seminar in the 1970s. He made the perceptive suggestion that a 'concept' and a 'schema' are essentially the same thing. If one looked in detail at a concept it became a schema of connections and (if one could grasp a schema of connections as a single entity) then a schema could be considered globally as a concept.

The identification of schema and concept works only when the individual can comprehend the whole schema as a single cognitive unit. This has important implications in understanding students' difficulties in coping with large schematic structures. For instance, Crowley and Tall (1999) considered how the 'linear equation schema', for formulating and solving linear equations may represent the same idea in different forms:

- the equation $y = 3x+5$,
- the equation $3x–y = –5$,
- the equation $y–8 = 3(x–1)$,
- the graph of $y = 3x+5$ as a line,
- the line through (0,5) with slope 3,
- the line through the points (1,8), (0,5).

For some college algebra students these may all be compressed into a single cognitive unit, with the various representations just alternate ways of expressing the same thing. But it is also clear that there are students who see the structure as consisting of distinct ideas with procedures (that they may not be able to carry out) required to get from one thing to another. A student with such a diffuse view of linear equations may therefore have a partial schema for relating the various representations but not a global schema that easily sees them all as essentially the same cognitive unit.

Other evidence relating the notion of 'linear equation' as a tight cognitive unit or a loosely structured schema with diffuse links occurs with the concept of function and the student's mental links between representations as function-box, table, formula, graph, etc (DeMarois & Tall, 1999). The more successful readily link between the different representations, as if they are different aspects of essentially the same cognitive unit. The less successful tend to have fewer links between the facets and have greater difficulty in providing solutions to problems. In another study, McGowen & Tall (1999) trace the development of concept maps drawn by the students at different points in a mathematics course, to find that the more successful build ideas steadily, with old ideas being successively linked to newer ones. The less successful have no such stable structure, drawing each concept map anew with little relation to the maps drawn before.

Dubinsky and his colleagues theorized initially that mental objects are constructed by process-object encapsulation, but later extended their theory to incorporate schema-object encapsulation:

> As with encapsulated process, an object is created when a schema is *thematized* to become another kind of object which can also be *de-thematized* to obtain the original contents of the schema. (Asiala et al, 1997, p.400)

Such a cognitive structure can then act dually as schemas or as encapsulated cognitive units with rich interiority in the sense of Skemp.

1.6 Hierarchies and Webs

Hiebert and Carpenter (1992) proposed two alternative metaphors for cognitive structures—as *vertical hierarchies* or *webs*:

> When networks are structured like hierarchies, some representations subsume other representations, representations fit as details underneath or within more general representations. Generalisations are examples of overarching or umbrella representations, whereas special cases are examples of details. In the second metaphor a network may be structured like a spider's web. The junctures, or nodes, can be thought of as the pieces or represented information, and the threads between them as the connections or relationships.
> Hiebert & Carpenter (1992, p. 67.)

Skemp's formulation would allow the nodes in a web to be seen as varifocal hierarchies, thus allowing the two structures to be used together. Likewise a concept could be considered as a web of connected ideas, allowing both hierarchical and web-like structures to coexist in a single structure.

However, the notion of webs and nets are still simplified metaphors for a highly complex mental system. Greater subtlety is essential to be able to reflect on the way we think in mathematics. Consider for example, the statement:

$$\sin 60^\circ = \frac{\sqrt{3}}{2}.$$

This may be conceived by an individual as a cognitive unit. It may be linked to a picture such as that in figure 1.

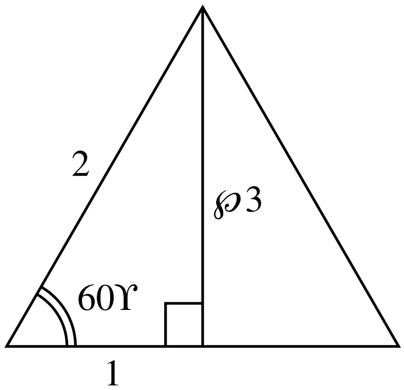

Figure 1: Relationships within an equilateral triangle

This, in turn, is related to many other ideas. ‘The angles in an equilateral triangle are all equal’, ‘the angles in a triangle add up to 180°’, ‘an angle in an equilateral triangle is 60°’, ‘the line joining the vertex to the midpoint of the base of an isosceles triangle meets it at right angles’, ‘if the side is two units, half a side is 1 unit’, ‘the theorem of Pythagoras’, ‘$a^2 + b^2 = c^2$’, ‘$b^2 = c^2 - a^2$’, ‘the square of √3 is 3’, ‘$1^2 + \left(\sqrt{3}\right)^2 = 2^2$’, ‘the sine of an angle is opposite over hypotenuse’, ‘the opposite is √3, the hypotenuse is 2’, etc. This single cognitive unit may link to definitions of the trigonometric functions, theorems about triangles, algebraic representation of a sum of squares, numerical facts about a specific triangle, and so on. It is possible to formulate these partly in terms of hierarchies, for instance, Pythagoras’ Theorem has ‘$1^2 + \left(\sqrt{3}\right)^2 = 2^2$’ as a special case and the definition of sine also includes this special case in terms of $\sqrt{3}/2$ as ‘opposite over hypotenuse’. The many links involved also relate to other ideas, for example, the definitions of trigonometric formulae relate to notions of similar triangles having sides in the same ratio, the trigonometric functions have relationships between them such as ‘$\sin^2\theta + \cos^2\theta = 1$’. The processes of the brain allow these ideas to become intimately connected in such a way that they are easily linked and manipulated.

Professional mathematicians build up highly subtle cognitive units packed with meaning. By focusing on commonly occurring properties which prove useful in making deductions, they build up a range of different theories based on generative systems of axioms: group theory, analysis, topology, algebraic number theory, probability theory, and so on. This involves a very special kind of reconstruction in which cognitive units, which began with a range of informal links, must be reconstructed to give formal cognitive units whose internal structure is based firmly on logical deduction.

There is also an aesthetic quality to the role of precision in this building of cognitive units and connections:

> Consider, for example, the concept of an abstract group. This is not just some vague notion that has something roughly to do with symmetry. On the contrary, the concept of a group captures the essence of the notion of symmetry and is connected in a precise way to the

> concept of an equivalence relation, which itself is a precise abstract formulation of the notion of 'sameness' with respect to a given property. Not only are the properties defining a group sufficiently general to be satisfied by a large variety of relatively concrete mathematical objects, but they are also sufficiently special to have lots of powerful consequences at the abstract level. Thus a group is a precisely defined concept which sits at a major junction in the mathematical network of relations. Indeed, one of the most beautiful features of mathematics is the way it allows such precision at even the deepest levels of abstraction. (Barnard, 1996)

The building of cognitive units and connections in such a manner requires a flexibility of mind that can switch between informal thinking (in which ideas can flow freely and suggest new links) and formal thinking (where the stated properties are deduced from the definitions). This involves a change in register in linguistic expression from 'speaking informally' allowing free association of ideas, to 'speaking strictly' using a logical form of speech which Alcock & Simpson (1999) call the 'rigour prefix'. When language is used with the rigour prefix, it employs terms in their formal defined manner and makes inferences only by logical deduction. By operating in this way a mathematician can build up huge theoretical constructions that can be managed easily by packing and unpacking subtly constructed cognitive units.

Informal experience of arithmetic and algebra involves various operations on symbols including addition and multiplication, which have certain observable properties. Formally appropriate properties can be isolated to give the definition of a formal construct, such as a 'commutative ring' (a set with two operations called addition and multiplication which satisfy certain rules). Another system that satisfies the rules of a commutative ring is the algebra of polynomials, and this turns out to be related to geometric ideas where sets of polynomials in several variables correspond to geometric curves. For example, the polynomial in two variables

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$$

has a set of zeros in the plane which is an ellipse. The correspondence between algebraic curves C in the plane and the sets $P(C)$ of polynomials having the points of these curves as zeros has a certain structure. For instance if f, g are both in $P(C)$ then $f+g$ is in $P(C)$, and if h is any other polynomial then the product of f and h is in $P(C)$. Inspired by this example (as well as by certain phenomena in the theory of algebraic numbers) a (non-empty) subset I of a commutative ring R that satisfies

If $x, y \in I$ and $z \in R$, then $x+y \in I$ and $xz \in I$

is called an *ideal*.

Over a period of time (months or years) working with the ideas of commutative rings and ideals, the developing student, or the expert, may build up many properties which open up further avenues of study. For example, it can be proved that the following three properties are equivalent:

(i) every ascending chain of ideals of R is eventually stationary,

(ii) every non-empty set of ideals of R has a maximal element,
(iii) every ideal of R is finitely generated.

A commutative ring satisfying any one of these conditions (and hence satisfying the other two!) is said to be *Noetherian*. In proving mathematical relationships involving Noetherian rings, the expert will use one or more of the above properties explicitly at various stages of a deduction. However, in thinking about properties of Noetherian rings, a mathematician's mind need not be cluttered with all the details of these three properties or with the details of the definitions of the component terms: 'ideal', 'ascending chain', 'eventually stationary', 'maximal element', 'finitely generated'. All of these may become cognitive units in the mind of the mathematician and the single word 'Noetherian' is a cognitive unit that is enough to conjure up an awareness of all these things together with a particular sense of 'finiteness' that the three properties embody. In this way, a Noetherian ring becomes a cognitive unit with a life of its own, having characteristic aspects that are as familiar to a ring theorist as are the many aspects of the number 7 to a primary school pupil.

1.7 Various ways of performing cognitive compression

The examples in the previous section show that cognitive compression can occur in a range of different ways. A common method of compression is to use words and symbols as tokens for complex ideas. In particular, words can be used in a way that allows a hierarchical structure to be conceived. For instance, a square is a special case of a rectangle, which is itself a parallelogram, which is a quadrilateral. The development of such a hierarchy of concepts is part of the cognitive growth of the individual. Geometry begins with the perception of shapes, their visual presentation in various orientations, the feel of their sides and corners, their verbal descriptions and the coordination of these perceptions and their communication through action, speech, writing, drawing, hearing and seeing. In the early stages a square and a rectangle may be seen as quite different entities, both having four right angles, but a square has *all* sides equal, whereas a rectangle has only *opposite* sides equal. The cognitive unit 'rectangle' develops greater sophistication and interiority, growing from meaning just a perceived figure, to a whole collection of rectangular figures (including squares) in any orientation. The realization that squares are special cases of the class of rectangles is an example of cognitive compression where one class of objects is subsumed (for certain purposes) within another.

Language supports the communication and refinement of such ideas, both between individuals and within the mind of the individual. It allows verbalized properties of perceptions to be used as a foundation for the development of cognitive units that are sophisticated mental idealizations. These include the notion of a point having position, but no size, or a line having arbitrary length and no thickness. These mental concepts go on to play their role in proving relationships in elementary Euclidean geometry and perhaps later in more abstract forms of geometry.

In arithmetic and algebra, cognitive units are often compressed through process-object encapsulation, leading to the use of a symbol as a pivot between concept and process. The process of counting is compressed via count-all, count-on and count-on-from-larger to the concept of sum. The child links practical activities with physical objects to symbolic representations of number and number operations. Dynamic algorithms with the symbols are rehearsed and routinised so that they may be performed in the background, taking up little focus of attention to provide a fairly automatic system of arithmetic to support further levels of problem-solving.

Compression can also occur in a number of other ways. For instance, when a collection of ideas or symbols is 'too big' to fit into the focus of attention, it can sometimes be 'chunked' to group into a single unit using some kind of alternative knowledge structure. The four digit number 1914 may be seen not just as a number, but as the year at the beginning of the first world war. This kind of associative link may be used to chunk numbers together into sub-units that can now be held in the limited short-term memory. Most individuals would find the 12-digit number 138234098743, impossible to remember on a single hearing. However, the twelve-digit number 246819141918 can be 'chunked' and remembered as the sequence '2, 4, 6, 8,' followed by the dates 1914–1918 of the First World War.

Those who seek meaningful learning may deprecate any use of rote-memorization by chunking, or by other associative methods. However, certain number facts prove notoriously difficult to memorize, such as which numbers multiply together to give 54 and which give 56. A meaningful method might be to link two known cognitive units ('five eights are forty' and 'two eights are sixteen') together to give 'seven eights are 40+16=56'. A less demanding method might be to remember the multiplication in the form '56 is 7 times 8' and link this associatively to the sequence '5678'. The former may be used by an individual who has developed a range of tightly organised number facts, the latter may be more easily remembered by a wider range of individuals, but is a totally isolated fact with little other apparent value.

Even so, many highly successful individuals use a range of techniques—both rote-learned and meaningful—to enable them to perform sophisticated computations. The physicist and Nobel Prizewinner, Richard Feynman, was shown by his colleague Hans Bethe how to assemble such a facility. He developed a whole arsenal of skills:

> I memorized a few logs and began to notice things. For instance, if somebody says, 'What is 28 squared?', you notice that the square root of 2 is 1.4 and 28 is 20 times 1.4, so the square of 28 must be around 400 times 2, or 800. If somebody comes along and wants to divide 1 by 1.73, you can tell them immediately that it's .577 because you notice that 1.73 is nearly the square root of 3, so 1/1.73 must be one-third of the square root of 3. And if it's 1/1.75, that's equal to the inverse of 7/4 and you've memorized the repeating decimals for sevenths: .571428… . (R. Feynman, 1985, p. 194.)

The great arithmeticians link numbers with a wide array of properties, as is evidenced by the remarkable story of Ramanujan on his deathbed:

> Hardy had gone out to Putney by taxi, as usual his chosen method of conveyance. He went into the room where Ramanujan was lying. Hardy, always inept about introducing a conversation, said, probably without a greeting, and certainly as his first remark: ' I thought the number of my taxi-cab was 1729. It seemed to me a rather dull number.' To which Ramanujan replied: 'No, Hardy! No, Hardy! It is a very interesting number. It is the smallest number expressible as the sum of two cubes in two different ways.'
> [$1729=1^3+12^3=9^3+10^3$.] (C. P. Snow, 1967, p. 37).

In this way the cognitive unit '1729' was an even richer concept in the mind of Ramanujan than in the highly sophisticated mathematical mind of G. H. Hardy.

1.8 Differing individual compression strategies

Although it is possible to formulate a range of possible compression strategies, individuals do not use them all to the same extent. This leads to different individuals processing mathematical ideas in ways that may have very different outcomes that may lead to success in some and failure for others. Gray and Tall (1994) noted that children who use the longest counting process—count-all (count one set, count the second, put them together and count them all)—could also remember certain 'known facts' such as 1+1 is 2 or 2+2 is 4. But *none* of these ever put together 'known facts' to obtain 'derived facts', such as '4+3 is 7' because '4+4 is 8', or '12+2 is 14' because '2+2 is 4'. Instead, they always computed arithmetic problems (whose answers were not immediately known facts) by *counting*. We hypothesize that the sums performed by these children are seen as counting processes and not as meaningful cognitive units with any interiority. The 'known facts' for them were isolated and not in a sufficiently rich compressed form to be mentally manipulated as cognitive units.

On the other hand, children who were able to *derive* new number facts from known related ones were able to perform arithmetic in a far more flexible and subtle manner which uses numbers as compressed cognitive units with a rich interiority. Gray and Tall proposed the following formulation:

> *Proceptual thinking* is characterised by the ability to compress stages in symbol manipulation to the point where symbols are viewed as objects which can be decomposed and recomposed in flexible ways.
>
> Flexible strategies used by the more able produce new known facts from old, giving a built-in feedback loop which acts as an autonomous knowledge generator. The least successful have only a procedure of counting which grows ever more lengthy as the problems grow more complex. In between these extremes, the less able who do attempt to derive facts from a limited range of known facts may end up following an inventive but tortuous route that succeeds only with the greatest effort. The high sense of risk generated may then lead to such a child falling back to the security of counting. We therefore hypothesize that what might be a continuous spectrum of performance tends to become a dichotomy in which those who begin to fail resort to counting. We believe that this bifurcation of strategy – between flexible use of number as object or process and fixation on procedural counting – is one of the most significant factors in the difference between success and failure. We call it the *proceptual divide*. (Gray and Tall,1994.)

Another bifurcation occurs in elementary algebra. Here an expression such as '2+3*x*' stands for a potential arithmetic operation such as 'add 2 to the product of three times whatever *x* is'. This can cause discomfort for students who feel that a problem 'must have an answer' as it does in arithmetic. They are therefore faced with manipulating expressions as mental objects that have only a potential, rather than an actual, internal process of evaluation. This can lead simply to procedural compression in which students learn to carry out a solution process by rote ('collect together like terms', 'get the numbers on one side and the variable on the other', 'simplify to get the solution', etc). Others are able to conceive the algebraic expressions as entities that can be manipulated. They may go on to conceive of the equation itself as a cognitive unit expressing a given relationship, with a 'solution process' as a cognitive unit that can be unpacked to give an efficient route to the solution.

Krutetskii (1976) studied this *curtailment of mathematical reasoning*, in which capable students would compress their solutions in a succinct and insightful manner.

> … mathematical abilities are abilities to use mathematical material to form generalized, curtailed, flexible and reversible associations and systems of them. These abilities are expressed in varying degrees in capable, average and incapable pupils. In some conditions these associations are performed 'on the spot' by capable pupils, with a minimal number of exercises. In incapable pupils, however, they are formed with extreme difficulty. For average pupils, a necessary condition for the gradual formation of these associations is a system of specially organized exercises and training. (Krutetskii, 1976, p. 352.)

1.9 Connections 'within' and 'between' cognitive units

We have seen that some individuals develop compressed cognitive units that can be manipulated as entities. Others have a knowledge structure that is more diffuse and therefore more difficult to manipulate. Skemp's varifocal theory suggests that we can make a distinction between cognitive structures that involve links 'inside' a given unit, and those that are 'between' units, depending on which level of cognitive structure is being considered. Of course, the structure is not physically 'inside' the unit. However, it is so intimately connected within the neuronal structure that it seems to be essentially 'part' of the same idea. The whole cognitive unit may then be conceived and manipulated as an entity and yet be unpacked to reveal more detailed structure 'within' it.

This suggests that two complementary factors are important in building a powerful thinking structure:

1) the ability to *compress* information to fit into cognitive units,
2) the ability to *make connections* within and between cognitive units so that other relevant information can be pulled in and out of the focus of attention at will.

Connections 'within' a cognitive unit can be carried around with that unit and be immediately available when required. An individual with compressed cognitive structures and relevant internal links will be able to make relationships between them far more efficiently than one who has a more diffuse cognitive structure.

We term those structures in the mind readily linked to the cognitive unit(s) in the focus of attention, the *intermediate working memory*. This includes the links ‘within’ a cognitive unit itself as well as links to other units that may be brought into the focus of attention in their turn. As we defined the notion of cognitive unit to include the item in the conscious focus of attention *together with other ideas that can be immediately linked to it*, then one might consider that the intermediate working memory is simply the totality of the cognitive units represented by current items in the focus of attention. However, the links need not be bi-directional, nor need they be of equal strength, so as new items are brought into the focus of attention, the intermediate working memory changes dynamically, opening up new connections and shutting off others. The tighter the connections, the more likely the intermediate working memory will retain links with previous ideas. If the cognitive units have little interiority and diffuse connections, then the attempt to solve a problem relating to them will pose greater cognitive stress, if it is soluble at all. In such a case, the working memory may move from one focus to another, perhaps losing track of various links on the way. This is consistent with the observation that better equipped minds suffer less cognitive strain and so develop even greater potential to become increasingly expert whilst those with more diffuse connections have greater cognitive strain and a greater possibility of failure. When students ask ‘how to do something without any long-winded explanations,’ they may not be just asking for rules without reasons. It may be that the ‘explanations’ involving links between rich cognitive units in the mind of the teacher are heard by some students as references to aspects of loosely connected structures in their minds which are too diffuse to be handled in their focus of attention. We believe that this difference in cognitive structure is a fundamental factor in the bifurcation between those who succeed in mathematics and those who fail.

2. Cognitive Units in Mathematical Proof

Mathematical proof requires more than procedural ability to carry out familiar processes. In addition to sequential procedures of calculation or symbol manipulation, mathematical proof often requires the synthesis of several cognitive links to derive a new *synthetic* connection. For example, in the standard proof of the irrationality of $\sqrt{2}$, the step from ‘a^2 is even’ to ‘a is even’ requires a synthesis of other cognitive units, for instance ‘a is either even or odd’ and ‘if a were odd, then a^2 would be odd.’ These synthetic links constitute an essential difference between procedural computations or manipulations in arithmetic or algebra and the more sophisticated thinking processes involved in mathematical proof.

Manipulating statements in mathematical proof involves a range of operations from constructing negations and converses to recognizing implications and contradictions. In order to do this successfully, not only do the components of a statement have to be meaningful, but the statement itself needs to be seen as a cognitive unit both seen in terms of the internal relationships of its component parts and also as an entity in itself that can become an ingredient in subsequent theorems.

Students often say that they can follow proofs when the lecturer goes through them in class, but they are unable to construct proofs for themselves when required to do so for homework. One explanation of this phenomenon (Barnard, 2000) has to do with the shifting of focus through the different layers of detail in the cognitive units manipulated: as statements, as statements within statements, expressions within statements, symbols within expressions, and so forth. In a proof given in a lecture, the lecturer implicitly specifies the level of items that are to be the primary objects of thought at any stage. For example, in a proof by induction on n of a statement $P(n)$, the strategy as to when $P(n)$ is to be thought of as a compressed item within the statement, '$P(n)$ implies $P(n+1)$', or when it is to be unpacked for a finer grained manipulation, is provided by the lecturer.

> It is this focus shift of compression and expansion that often lies at the heart of the difficulty when students try to construct proofs for themselves. It is a bit like knowing when and how to change gear while driving. When students ask the seemingly bizarre question, 'How do you do proofs?', they may simply be reacting to a predicament similar to that of trying to drive without awareness of the existence of gears. (Barnard, 2000, p.421)

2.1 A theoretical analysis of the traditional proof that √2 is irrational

As an example of formal proof in practice, we consider the standard proof that $\sqrt{2}$ is irrational and analyse it in terms of cognitive units. Of all the pieces of cognitive structure involved in the proof of the theorem, three cognitive units will prove to play pivotal roles in our analysis. These operate on quite different levels. They are:

- (the strategy of) proof by contradiction,
- (the statement that) $\sqrt{2}$ is irrational,
- (the deduction that) a^2 even implies a even.

The first of these is a strategy that may be considered in three main phases:

(i) assume the statement is false,
(ii) from this statement use deduction to establish a contradiction,
(iii) deduce the statement cannot be false, so it must be true.

This is represented diagrammatically in figure 2.

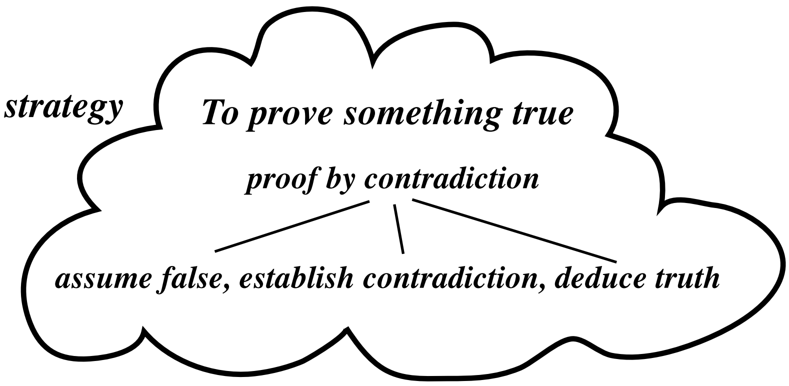

Figure 2: Strategy to prove a statement by contradiction

The proof starts by assuming that √2 is not a rational number. This is a cognitive unit that can be expressed in a variety of ways, that √2 is irrational, that √2 is not a fraction, that √2 cannot be written as the quotient of two integers, and so on. The statement needed for the proof by contradiction requires something more: that √2 is not expressible as a quotient of two integers that can then be cancelled down until numerator and denominator have no common factors. In particular, they cannot both be even.

As the proof unfolds, the general strategy should remain in the mental background, controlling the sequence of the proof. (Figure 3.)

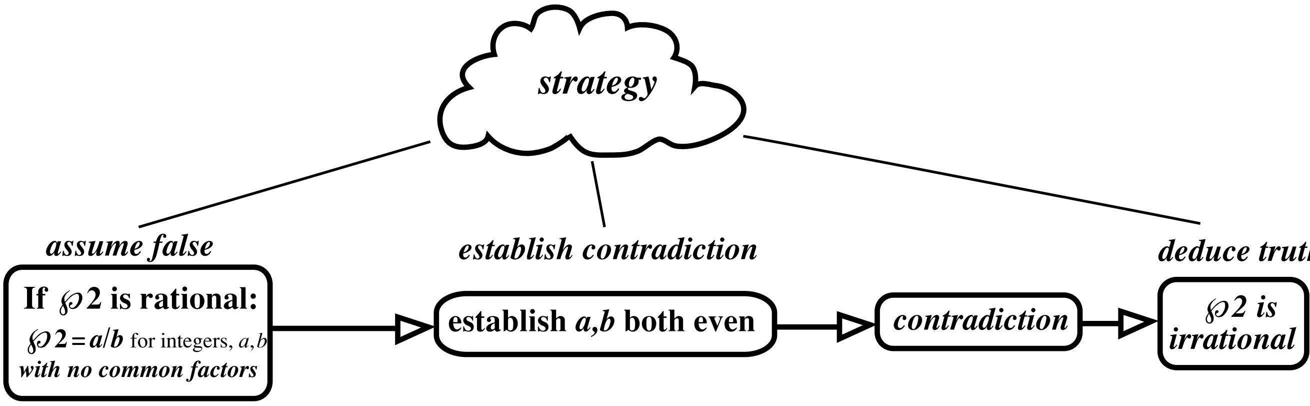

Figure 3: Unpacking the proof strategy for √2 irrational

The central part, to establish the contradiction requires the assumption that $\sqrt{2}=a/b$ which may be simplified to give $a^2=2b^2$ and so a^2 is even. From this the deduction is made that a is also even. This step is more subtle than at first sight because it involves a further contradiction proof (if a were odd, then a^2 would be odd, which it is not). Thus the traditional proof of the irrationality of √2, often given as the prototypical first example of the strategy of proof by contradiction actually contains a second proof by contradiction nested within it. For this reason the statement 'a^2 even implies a even' is singled out as the third cognitive unit to be studied in greater detail.

After analysis of the empirical data from interviews about this proof, we move on from conversation about the irrationality of √2 to the irrationality of √3. We hypothesise that the 'recency effect' of ideas just considered for √2 are likely to feature in the proof for √3, and that these can have a *negative* effect in obscuring the general argument.

2.2 Data collection

Twenty students were chosen to represent a spectrum of performance at different stages of familiarity with proof by contradiction:

Ten students who had not previously been shown the proof:

(i) six 15/16 year old students in a mixed comprehensive school taking GCSE mathematics (three male, three female),

(ii) four 16/17 year olds in a boys' independent school taking A-level mathematics (four male), *Ten students who had met the proof before:*

(iii) six first year university mathematics students, (three male, three female),

(iv) four second year university mathematics students (two male, two female).

Initially students were asked if they knew how to prove that $\sqrt{2}$ is irrational. If they did, then they were invited to give a proof. If not, or if their explanation broke down, they were shown the next step in a proof in the following sequence and asked to explain it and move on from there.

(i) *Suppose $\sqrt{2}$ is not irrational.*

(ii) *Then $\sqrt{2}$ is of the form a/b, where a, b are whole numbers with no common factors.*

(iii) *This implies that $a^2 = 2b^2$,*

(iv) *and hence that a^2 is even.*

(v) *Therefore a is even.*

(vi) *Thus $a=2c$, for some integer c.*

(vii) *It follows that $b^2 = 2c^2$,*

(viii) *giving that b^2,*

(ix) *and hence also b, is even.*

(x) *The conclusion that a and b are both even contradicts the initial assumption that a and b have no common factors.*

(xi) *Therefore $\sqrt{2}$ is irrational.*

Each student was then asked to suggest a proof for the irrationality of $\sqrt{3}$.

In essence it was expected that the ten students who had not seen the proof before would be in the state of building the proof structure for the first time. Our purpose was to guide them through the proof step-by-step, seeing which steps occur more naturally and which are less likely to occur without prompting. On the other hand, the ten university students had all seen the proof previously. We were interested in the manner in which they recalled the proof (if at all), to see if the whole proof were readily available as a complete cognitive unit, or whether the student was only able to recall a collection of ideas, some of which were readily linked, but others were forgotten and may, or may not, be (re)constructed with input from the interviewer.

2.3 A preliminary global analysis of responses

We consider the way in which the students build up the proof in sequence with particular emphasis on the three cognitive units selected earlier for special consideration. First, note that *none* of the ten school students and *only two* of the ten university students gave a proof unaided. Both successful university students were in the first year and had experienced the proof in the previous few weeks. We follow one of them, Lucinda, through the steps she took. She quickly explained the proof in outline being able to unpack any step further when requested. We would therefore say that she had the statement and proof in her mind as a cognitive unit, with the strategy of proof by contradiction immediately available to guide her deduction.

Several other students were also aware of the overall shape of the proof, without being able to fully unpack it satisfactorily. Steve, a first year student, gave an outline in his own terms:

> I'd take the case where I assumed it was a rational and fiddle around with the numbers, squaring, and try to show that … if it was rational then you'd get the two ratios *a* and *b* both being even so they could be subdivided further, which we'd assumed earlier on couldn't be true so our assumption it was rational can't be true.

However, he was not able to cope with several parts of the proof that followed. There are several clues that his knowledge is less tightly structured. In the given quotation, he begins by assuming √2 to be rational, rather than rational *in lowest terms*. He wrongly used the word *ratios* for *a* and *b* when he presumably meant whole numbers. He then refers to them being *subdivided further* if they were both even, rather than cancelling common factors. Nor does he mention the concept of 'being in lowest terms'. However, his statement suggests he has at least an informal notion of the *process* of cancellation and his statement 'could be subdivided further, which we'd assumed earlier on couldn't be true' suggests an awareness of proof by contradiction. Yet his lack of precision in word and thought seems to prevent him from unpacking the proof in formal detail. We analyse his difficulties in greater detail as we consider student responses to successive steps in the proof.

2.4 Detailed analysis of the opening steps of the proof

Our detailed analysis of the responses refers to the steps formulated in section 2.2.

(i) the notion of proof by contradiction

Of the eighteen students unable to give a full proof, several recalled seeing the proof before, often giving episodic memories of how the proof was explained to them. For instance, Steve, the first year student mentioned in the previous section, could remember being involved with the proof on four different occasions. He was shown it at school, questioned about it in a university interview, and experienced its proof on at least two occasions in recent lectures. Another first year university student, Melvin, also remembered seeing the proof before but could not remember any details. Even when shown the proof, he only followed it with difficulty, step by step.

Some of those who would have seen the proof before made links that were quite unexpected. Beth, a second year university joint honours student, would almost certainly have been shown the proof, but did not remember it. Instead she made a link with a current course of study involving rationals and irrationals:

> Interviewer: If you were to go about proving it [√2] was irrational, how would you start? Any idea?
> Beth: Well, maybe I'd start that √4 is rational, √1 is rational as well.
> Interviewer: √1, did you say?
> Beth: √1, which is 1. …
> Interviewer: OK, yeah.

> Beth: … And there's always an irrational number. It wouldn't prove that $\sqrt{2}$ is an irrational number, but there's always an irrational number between two rationals.
> Interviewer: Yeah.
> Beth: But … What else would I do? Yes I guess I would do – take … –- but that wouldn't prove it anyway, because there's infinitely many irrational numbers between two rational numbers.

Her link is not to the proof as she may have seen it, but to the recently encountered ideas in her current study, recalling that between any two rationals there is an irrational. She thought about it for a time and realised it was not appropriate for the question.

(ii) translation from verbal to algebraic

The translation of the verbal statement '$\sqrt{2}$ is rational' to the algebraic form '$\sqrt{2}=a/b$ (in lowest terms)' is not an obvious starting point for the uninitiated. *It was not suggested spontaneously by any of those who had not seen the proof before*. University student Melvin referred to the process explained in a recent lecture of 'dividing top and bottom by two until one of them is odd'. He seemed happier with the *process* of cancelling common factors than thinking of the expression 'a/b in lowest terms' as a cognitive unit with an operative meaning.

(iii) a routinised algebraic manipulation

The routine manipulation from $\sqrt{2}=a/b$ to $a^2=2b^2$ was carried out successfully by six out of the ten university students who had seen it before. Lucinda compressed the whole operation in a single step and was also able to give more detail on request. The other five successful students referred explicitly to being shown how to do it. The four university students who had difficulty, all attempted to use episodic memory to recall how it was done, rather than attack the problem algebraically. University student Steve—who had begun the proof by stating the general strategy for proof by contradiction—referred to an episode in a recent lecture saying that the lecturer 'did some fancy algebra which I couldn't actually reproduce.'

When asked to write down the proof, he wrote

$$\left(\frac{a}{b}\right)^2 = 2,$$

followed immediately (in one step) by

$$a^2=4b^2,$$

saying, 'I think that's what he did, but he did it in one step whereas normally I would've taken two.' When asked to fill in the details in his own way, he obtained the correct result $a^2=2b^2$. Similarly, student Melvin said, 'I remember him saying to prove that a is even' but could not remember how. In both of these cases, some episodic memory remains but has not been translated into a meaningful semantic form.

***(iv) identifying that a^2* is even**

The link from '$a^2=2b^2$' to 'a^2 is even' was not spontaneously suggested by *any* student who had not seen the proof before. When it was suggested by the interviewer, however, they *all* readily accepted its truth. This 100% switch from not knowing to accepting deserves further analysis.

All of these students have an informal notion of what it means for a whole number m to be even. However, they recognize even numbers from the list 2, 4, 6, 8, … or as a number ending in 0, 2, 4, 6, 8, long before they learn the algebraic property that an even number is of the form $m=2n$ where n is a whole number. More subtly, the equation '$a^2=2b^2$' is not between simple whole numbers or single symbols, but involves the *squares* of numbers. If the students do not see the sub-expressions simply as certain whole numbers (which requires the expressions a^2 and b^2 to be seen as cognitive units), then this complexity may obscure the implicit detail. This step of the proof is easy for students to accept, but those who had not seen it before required guidance to evoke it for themselves.

2.5. The embedded proof by contradiction

(v) proving that if a^2 is even then a is even.

This central argument in the proof proved to be the most subtle in the whole proof. A correct justification was produced on request by three of the ten school students, and only two of the ten university students. Various strategies were offered, including:

(a) *Correct justification* involving a sequence of appropriate connections, along the lines 'a is either even or odd', but 'a odd implies a^2 is odd', and as 'a^2 is *not* odd', this implies 'a must be even.' Such students were able to relate the various cognitive units in a flexible manner.

(b) *Believing it to be true (often with strong conviction), providing some argument, but unable to formulate a general proof*

Such justification as given might be:

- *Repeating the statement with emphasis*, such as, 'an even number square has got to have a square root that is even' or 'well, it just sort of *is* [even].'
- *Inconclusive reasoning*, offering related statements, justified or otherwise, which did not help further the argument, such as, 'If you could say that a^2 had a factor of 4, then that [a even] would definitely be true.' Here the links inside the cognitive unit 'a^2 is even implies a is even' appear to be associative rather than causal.
- *Erroneous reasoning*, using incorrect links. An example, which occurred more than once, was the claim that if a^2 is an integer multiple of 2, then a is an integer multiple of $\sqrt{2}$.
- *Empirical verification*, giving specific numerical cases and asserting that this showed a universal pattern with no exceptions.

(c) *Unable to respond without help.*

Responses in category (a) employ the implication 'a^2 is even implies a is even' as a genuine cognitive unit that can be unpacked to reveal the logical implication inside. In contrast, those under (b) treat the same statement as an 'informal cognitive unit'. It can be manipulated as a single entity that is operated on 'from outside' (used as a step in a proof), but its internal links are, to varying degrees, fragmented or associative.

Of the two university students giving an initial correct response, Lucinda asserted 'if a^2 is even then a would have to be even' with great conviction. When asked why, she replied somewhat disdainfully:

> 'well if it were odd, then a^2 would be odd.'

This is a type (a) compressed response, typical of an expert mathematician, highlighting the essential reason without any embroidery. For her, the total statement that '$\sqrt{2}$ is irrational' was a cognitive unit with tight logical links that she could summon with conciseness and precision. The connections are *explanative*.

Type (b) responses were more varied. Students asserted that if a^2 is even then a is even, and may well be go on to complete the proof. In this case the statement can be used as a unit as a step in the proof, but it cannot be unpacked to reveal the inner detail.

Second year university student Beth announced that 'if a^2 is even then a is even,' and was about to move on to the next step when she was challenged to give a reason. She continued as follows:

Beth:	If a is even, obviously a^2 is even.
Interviewer:	Yeah.
Beth:	But it goes this way [pointing from the written statement 'a is even' to 'a^2 is even']. But it must also go this way [from 'a^2 is even' to 'a is even']. But isn't there maybe also an even number whose square of it is odd. No I'm … (*Pause*) OK I'll start from the beginning again. (*Softly reads steps (i) to (v) from the sheet*) And we [indistinct] here like you said, doing it the wrong—the other way round. Which would imply if – if a is even, a^2 is even. Does that imply that when a^2 is even a is even?

She was unable to continue.

Anthea, an able but not overly confident first year university mathematics student, was also unable to give a satisfactory proof of 'a^2 is even implies a is even'. She realized that a^2 is even but b^2 need not be, but was unable to respond to the interviewer's cues, ending up with an using examples:

Interviewer:	Right. So we've got down to this situation where a^2 is even.
Anthea:	Yes — but b^2 might not be.
Interviewer:	Yes, so far. We don't know anything yet. (*Pause*) What about a? (*Long pause*) … a^2 is even.
Anthea:	Yeah.
Interviewer:	Does that tell us anything about a?
Anthea:	(*Pause*) Yes, a is even.
Interviewer:	Why is that?
Anthea:	Because the root of an even number is even if it can be a whole number. I thought, you know, 2 times 2, 4 times 4, 6 times 6. Just kind of logical.

Anthea was a willing student who worked consistently throughout the course; she achieved a first class honours degree. This shows that students who initially struggle may work to achieve a high level of performance in the examination.

Several responses of type (b) showed an awareness that the cognitive units 'a is even' and 'a^2 is even' are intimately linked and *coexist* in the focus of attention. In this context it is not that one statement implies the other, but that both happen *at the same time*. On the negative side, the cognitive unit 'a^2 is even' seems to have a stronger natural link to 'a is even', rather than to 'a is not odd', thus failing to evoke the alternative hypothesis and failing to give a proof without further guidance. Colin, a second year mathematics student, looked at the equation $a^2 = 2b^2$ and said:

Colin:	In that case, a^2 has got to be even … which means that a is even.
Interviewer:	Why does it mean that a is even?
Colin:	Because two even numbers multiplied always make an even number.

The interviewer focused on this, reiterated his argument, and then continued:

Interviewer:	Can you see that what we want here is the converse of that?
Colin:	Yes.
Interviewer:	Right, OK. Have you any idea why you … said 'even times even is even' as the reason?
Colin:	That was the easiest way of explaining it.
Interviewer:	Yes, but you now realize it wasn't explaining it.
Colin:	Yeah.
Interviewer:	So why do you think you went to that?
Colin:	I don't know. It was a way of explaining it. Well, I thought it was a way of explaining it.

After further discussion, Colin turned to *empirical verification*, saying:

Colin:	Because you can start off saying two squared as four Both those are even. Four squared, six squared and go on from there. And all those are even.

University student Steve asserted the authority of the lecturer with strong conviction, saying, 'the root of an even number is even—he just assumed it.' The conversation continued as follows:

Interviewer:	So the root of six is even.
Steve:	Good point. [five seconds pause]
Interviewer:	If a number is not even, what is it?
Steve:	It's odd.
Interviewer:	So you've got a choice of odd or even, does that help you?
Steve:	Yeah, I see, it's got to be rational, I think, so … a rational root is either … odd or even and if the square is even, then the rational root is even. Is that clear?
Interviewer:	Uh, well …
Steve:	So what I'm thinking is the root of four, Four's even and two's even, root of sixteen equals four, … 's even. I can't remember any other simple squares in my head that are even …

He used loose language ('rational' instead of 'integer') and *inconclusive reasoning* ('if the square is even then the rational root is even'). He ends by seeking *empirical verification* with specific numbers. At this point the interviewer moved on, sensing that any further investigation into his mental links would only provoke more confusion.

Students in category (c) (who could not proceed initially) were given the suggestion that ‘a number could be even or odd’. On some occasions this led to a satisfactory response of type (a), often via the link ‘an odd number squared is odd.’ Other responses were of type (b). However, none of these responses were in the subheading *empirical verification*. It is as if the *general* prompt usually linked to a general argument, rather than to the use of *specific* numbers.

2.6 Analysis of the final stages of the proof

(vi) From ‘a is even’ to ‘a=2c for a whole number c’

The translation from the verbal statement ‘a is even’ to the algebraic statement ‘a=2c’ was either suggested by students or accepted as self-evident. Melvin, however, had a faulty recollection of what to do:

Interviewer:	If you know that a is even, how can you write a? How do you write down that a is an even number ?
Melvin:	If you put a 2, … you put an a in front of it, like 4a ... I don’t know, I’m sorry. I can’t remember.

Here he seems to be using episodic memory of what he was told, rather than an explanative memory of relationships.

(vii)–(ix) The chance to repeat earlier arguments

Having concluded that ‘a=2c for a whole number c’, the next steps of the proof usually proceeded easily. A typical response might be ‘oh, it’s the same as before.’ No student had any difficulty with the procedural steps substituting ‘a=2c’ into ‘a^2=2b^2’ and simplifying ‘4c^2 = 2b^2’ to get b^2= 2c^2. Students invariably saw that this situation was similar to the earlier case for a, and immediately asserted that b is also even. The initial part of this activity to ‘substitute and simplify’ is a natural procedure from elementary algebra. The use of the just-established principle ‘a^2 is even implies a is even’ had now become a working cognitive unit for the time being. This ready use is consistent with the ‘recency’ property of neuronal circuits.

(x)–(xi) establishing the contradiction

At this stage, some students new to the proof did not recall that a/b was assumed in lowest terms, and so did not see that ‘a and b both even’ gives a contradiction. Student Colin remained silent for 45 seconds, until reminded: ‘we cancelled out until we had no common factors.’ He immediately responded:

> ‘Oh, right, … that can’t be the case because if they are both even numbers, then they will have common factors, like two.’

All ten who had seen the proof before immediately grasped the contradiction, including those who had mis-remembered the detail of earlier steps.

2.7 Generalising the proof to the irrationality of $\sqrt{3}$

Following the proof that $\sqrt{2}$ is irrational, the students were asked to prove the irrationality of $\sqrt{3}$. They all began by supposing that $\sqrt{3}$ was equal to a fraction a/b in its lowest terms. A typical remark was 'I presume you start in the same way', consistent with evoking a recent mental resonance. Translating this to $a^2=3b^2$, they *all* evoked the link with a being 'even or odd' and were unable to proceed further. One of the university students wondered whether the 'evenness' in the earlier case might be related to the 2 under the square root sign, but did not spontaneously suggest that $\sqrt{3}$ could be handled by thinking about 'divisibity by 3'. Divisibility by 3 does not have any natural linguistic terms corresponding to the use of 'even' or 'odd' for divisibility by 2. The recency of the language formulation in terms of 'even/odd' in the proof for $\sqrt{2}$ seems to be so much stronger than the link to 'divisible or not by 2' that it impedes the link to the corresponding mathematical equivalent for $\sqrt{3}$.

The equation $a^2=3b^2$ was sometimes linked to 'the oddness of a^2', possibly because the 3 is odd. However, it does not *immediately* follow that a^2 is odd, because in the first place b might be *even*. The fact that if a/b is in its lowest terms then b could *not* be even would follow by yet another contradiction argument. However, establishing that b must be odd, just shows that a^2 is odd, and so fails to give the required contradiction.

A prompt that the equation $a^2=3b^2$ tells something more than the 'oddness of a' sometimes evoked divisibility by 3, but this led to a further sticking point in attempting to prove that 'a^2 is divisible by 3' implies 'a is divisible by 3'. *None* of the students could do this unaided. Only one student (Eric, in the university first year) considered the algebraic argument squaring the three cases $a=3n$, $3n+1$ or $3n+2$ (a synthetic connection requiring the coordination of three different possibilities).

A further suggestion focusing on factorization into primes was sufficient to help all the university students and some sixth formers to produce suitable arguments although often expressed in an idiosyncratic manner. Student Teresa, for instance, said:

> ' ... the [square] root of a^2, I mean a, that doesn't involve the factor 3. Therefore you've still got a factor 3 which you can divide into a.'

She seems to be saying that if 3 does not divide one of the a-factors of $a \times a$, then it must divide the other a. This seemed to be a special case of the fact that if 3 (a prime) divides ab, then it must divide a or b.

Student John, in the youngest group of school students, also imagined a^2 as a product of two a factors saying:

> ' that has got repeated factors of that, so you can't get [ten seconds pause] … just imagining how many factors of things. … They're going to have the same factors. So yes, 3 would have to divide a.'

2. 8 Synthesizing the findings

Figure 4 is a representation of some of the typical linkages that occur widely in students' proofs that √2 is irrational, omitting highly idiosyncratic links. It is a collage of links made, and difficulties encountered, where links denoted by often prove more difficult than those denoted by , and those in grey scale are intermediate links which may or may not be evoked in detail. The dashed lines represent recall from a previous time. The letters **A-D** denote major difficulties arising in the detailed analysis

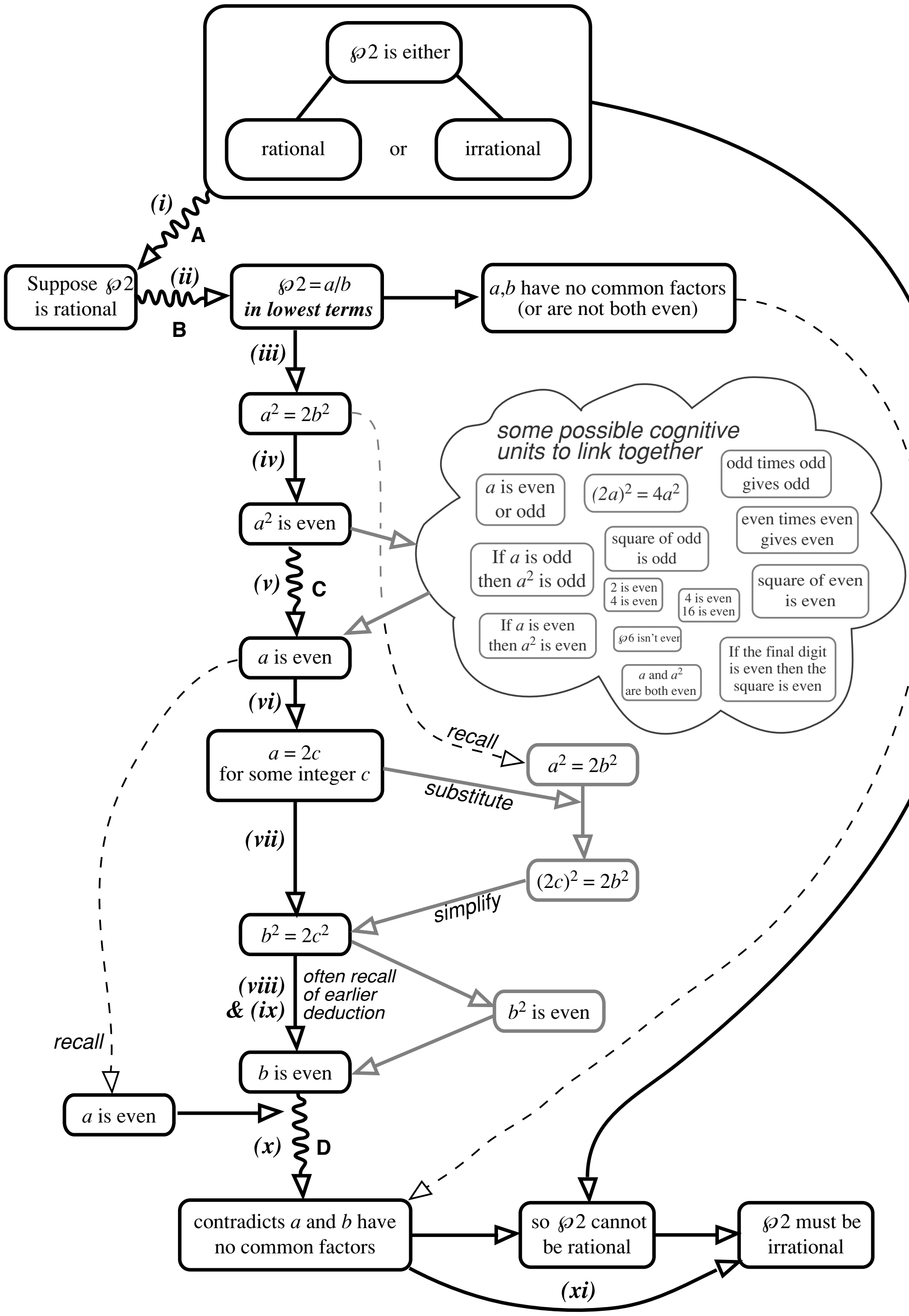

Figure 4: Observed cognitive units and connections in a typical initial proof of √2 irrational

above, summarized as follows:

(*i*) **A** The overall notion of proof by contradiction often causes confusion for the uninitiated and, even when remembered in outline, it is not always used in appropriate detail and may not be properly recalled at the end.

(*ii*) **B** The assumption 'a/b is in lowest terms' is not obvious to students new to the proof, but it could become part of the long-term global strategy.

(*iii*), (*iv*) Translation between familiar terms 'odd and even' and algebraic representations are acceptable, but not always initially evoked by those new to the proof. The algebraic manipulations should be straightforward, but are sometimes remembered episodically with faulty recollection.

(*v*) **C** The step 'a^2 even implies a even' is initially not easy to synthesise and remains so for those in the sample who attempt to remember the proof by rote. For some students the cognitive units 'a^2 even' and 'a even' coexist to produce an informal unit of the form 'a^2 and a are both even at the same time' and the direction of implication is not relevant. For others the unit 'a^2 even' is more strongly linked to 'a even' (which fails to produce the proof) rather than to the operative alternative 'a odd'.

(*vi*)–(*ix*) algebraic simplification and recently used arguments are far easier to recall. Most students readily evoked the recent argument for 'a is even' to assert directly that 'b is even'.

(*x*) **D** The link back to the earlier statement 'a, b have no common factors' (or are not both even) is not always made by those new to the proof. It may not be made by some who have seen it before but remember little of the proof.

(xi) The conclusion of the proof usually follows once the contradiction has been identified.

These difficulties correspond to the items identified at the outset:

- (the strategy of) proof by contradiction (**A**),
- (the statement that) $\sqrt{2}$ is irrational *in its subtle form* written in lowest terms (or cancelled so that numerator and denominator are not both even) (**B**, **D**),
- (the deduction that) a^2 even implies a even (**C**).

By focusing on relevant elements in figure 4, we can now see the essential subtlety of the structure of the proof (figure 5). The overall contradiction proof consists in showing that a and b are both even, hence contradicting the assumption. Algebraic manipulations are required to obtain $a^2=2b^2$, during which time the focus of attention is on the algebra in greater or lesser detail. Then this formula is used to deduce that a^2 is even. The embedded contradiction proof for 'a^2 even implies a even' has a side branch available if challenged.

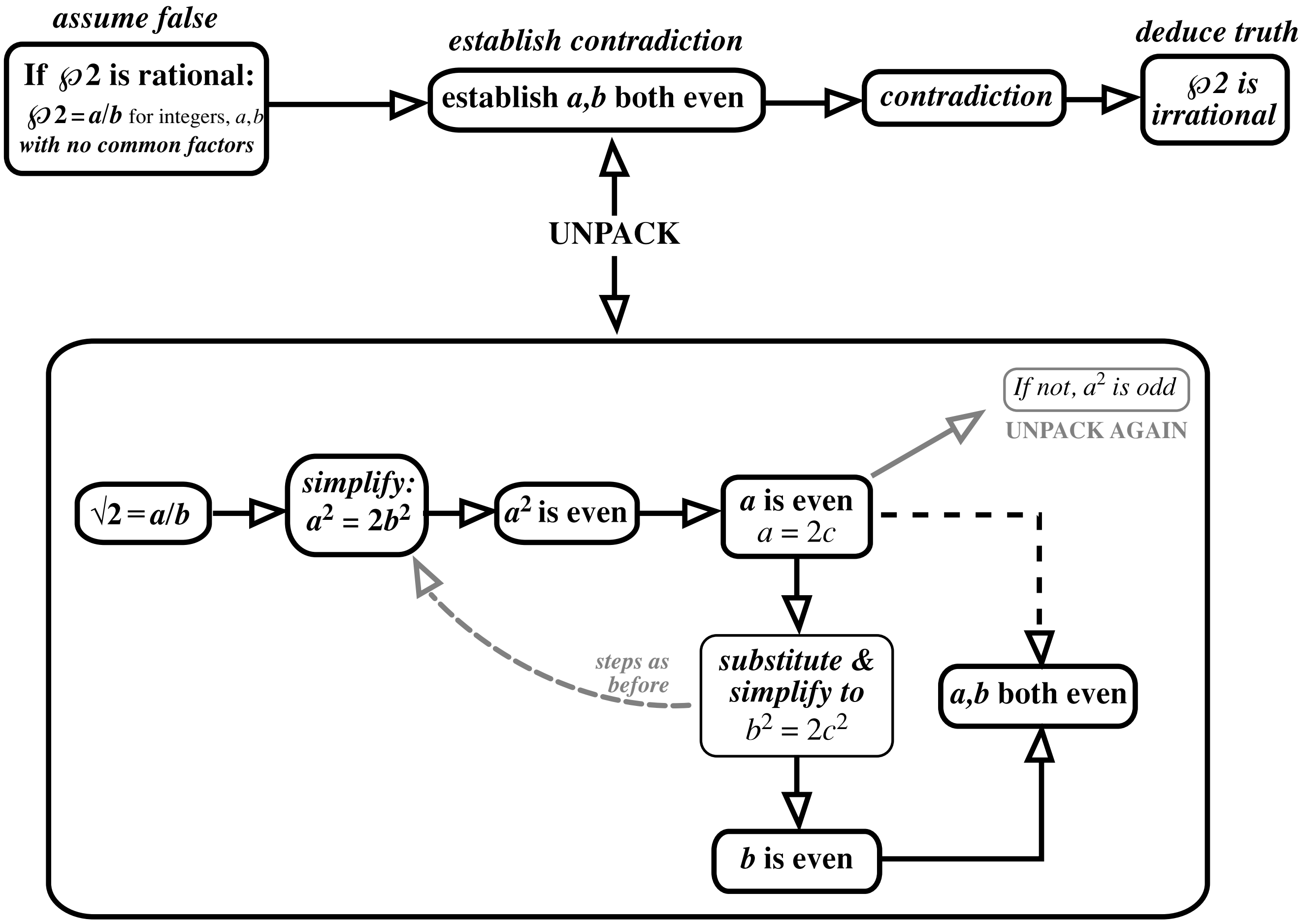

Figure 5: Carrying out the steps of the proof for √2 irrational

The next part of the proof, that b is even, is usually straightforward as the recency effect of performing the first part reveals that it is essentially the same sequence of argument repeated for b instead of a. This contradicts the assumption that a, b have no common factors (so cannot both be even), so √2 is irrational.

Summarising the broad development of the proof of the irrationality of √2, we see that there are several initial difficulties that make it a formidable initial challenge. Surprisingly few of those who had seen the proof before were able to immediately recall or reconstruct the proof. They may remember something of the overall strategy (assume it is rational and derive a contradiction) without being able to cope with the detail, particularly the proof that if a^2 is even then a is even.

After the passage of time, proof by contradiction as a strategy may become less problematic as it becomes more familiar, but there is sufficient difficulty to cause a bifurcation in understanding. Some students, such as Lucinda, make meaningful links that allow them to compress the information into richly connected cognitive units that are explanative in nature. For others, the proof is fragmented into smaller detail where the student loses the grasp of all the elements required. Some use episodic memory to remember some of the ideas they were told—even the overall strategy of the proof. They often relying on the authority of their teacher rather than on building meaningful links to reconstruct subtle detail. This may enable them to rote-learn theorems and pass examinations, but they are often aware of the nature of their own coping strategies:

> Maths education at university level, as it stands, is based like many subjects on the system of lectures. The huge quantities of work covered by each course, in such a short space of time, make it extremely difficult to take it in and understand. The pressure of time seems to take away the essence of mathematics and does not create any true understanding of the subject. From personal experience I know that most courses do not have any lasting impression and are usually forgotten directly after the examination. This is surely not an ideal situation, where a maths student can learn and pass and do well, but not have an understanding of his or her subject.
>
> (*Third Year U.K. Mathematics Student*, quoted in Yusof & Tall, 1999)

The two successful students (exemplified by Lucinda) have rich cognitive units that enable them to see the proof in large chunks. The strategy of proof by contradiction controls their proof structure, the fundamental assumption that √2 be assumed rational *in lowest terms* is firmly fixed in their memory, and the embedded proof by contradiction is handled by asserting its result and unpacking further if challenged. However, the step in the proof relating divisibility by 2 to evenness and using the terms 'odd and even' is so firmly rooted that even Lucinda needs further assistance and encouragement to reconstruct her knowledge to deal with the irrationality of √3.

There are, of course, many other ways of showing that √2 is irrational. Tall (1979) offered students a *generic proof* by showing that, when a number is factorized into prime factors, squaring it doubles the number of each prime factor. Thus a rational is a square if its prime factors all occur an even number of times. Unlike the difficulty of generalizing the standard contradiction proof from √2 to √3, this generic proof easily generalises from √2 to √(5/8) or to any other square root where the rational under the root sign does not have all primes occurring an even number of times.

The existence of more generalizable proofs is important, but it does not alter the fact that many proofs in mathematics require a contradiction argument. The need to understand a proof by contradiction early in formal mathematics is therefore essential. The contradiction proof that '√2 is irrational' is often put forward as an introduction to such a strategy, but the additional complication of one contradiction proof embedded within another makes it over-complicated for its chosen purpose.

3. Concluding remarks

The challenge in mathematics is how we can help students to construct appropriately linked cognitive units that are flexible and precise to help them build mathematics as a coherent and meaningful structure. These cognitive units arise naturally in human thinking and take on a wide range of roles—general strategies, specific information, routinized sequences of steps—linked together to produce mathematical thinking. Without cognitive units of appropriate manipulable size, thinking becomes diffuse and imprecise and is far less likely to be successful. An analysis of student thinking in terms of cognitive units has the potential of giving insight into why some students think in a manner that is simultaneously both simple and sophisticated, whilst others are overwhelmed by the huge complexity of disparate pieces of information.

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