THEORETICAL-COMPUTATIONAL CONFLICTS AND THE CONCEPT IMAGE OF DERIVATIVE

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INTRODUCTION

The aim of this research is to discuss how apparent contradictions between computational representations and associated theoretical formulation can be positively converted to enrich students' concept image of derivative and limit. We consider a *theoretical-computational conflict* as any situation where a computational representation for an object is (at least potentially) contradictory with the associated mathematical theory. In particular, numerical calculation with machine accuracy cannot be performed in a way that corresponds exactly to the mathematical theory of limits. Literature provides some examples in which a *narrowing effect* takes place: the intrinsic characteristics of the computational representation lead to limitations in the concept images developed by learners (see Hunter, Monaghan & Roper (1993)). On the other hand, we hypothesize that, if theoretical-computational conflicts are emphasized instead of avoided, they may contribute not to narrowing, but to enrichment of concept images. In this paper, we present some results of an experiment, in which we observed a sample of six undergraduates students, from a Brazilian university, coping with theoretical-computational conflict situations.

CONCEPT IMAGES AND COGNITIVE UNITS

Tall and Vinner (1981) define *concept image* to be the total cognitive structure associated with a mathematical concept in an individual's mind. It includes all the mental pictures, properties, associations and processes related to a given concept, and is continually constructed as the individual matures, changing with new stimuli and experiences of all kinds. The concept image may (or not) be associated to a statement used to specify that concept, named *concept definition* by Tall and Vinner. A concept definition, in its turn, may (or not) be consistent with the formal mathematical definition (that is, the concept definition usually accepted by the mathematical community). Thus, the individual's concept image may or not include the formally correct definition (see Barnard & Tall (1997), Vinner (1983), Vinner (1991), Tall (2000a)). On the other hand, as many authors claim (see Cornu (1991) and Tall & Vinner (1981)), the main ideas used by human beings to build further theoretical developments often do not come out from formal definitions, but from related intuitive ideas. Therefore, the capacity to recall the formal definition itself is not necessarily associated to a rich concept image.

Tall and Barnard (1997) introduced the term *cognitive unit* for a chunk of the concept image on which an individual focuses attention at a given time. Cognitive units may be symbols, theorems, representations, properties or any other aspects related to the concept. Thurston (1990) observed that the understanding of mathematics involves a process of mental compression of ideas that can then be quickly recalled and used. In this way, a rich concept image should include, not only the formal definition but many linkages within and between cognitive units.

NEGATIVE EFFECTS OF THE USE OF COMPUTERS ON MATHEMATICS TEACHING: NARROWING CONCEPT IMAGES

In this investigation, we focus on the positive use of technology to mathematics learning. However, it is important to remark that research shows that misused computational environments can have negative (or at least innocuous) effects. The theory quoted above suggests, in particular, that teaching the concept of derivative must include different approaches and representations, to enable learners to build up multiple and flexible connections between cognitive units. Each representation gives emphasis to certain aspects of the concept, but also blots out others in the same way. Tall (2000a) affirms that the focus on certain aspects and the negligence of others may result in the atrophy of the neglected ones. For instance, Hunter, Monaghan & Roper (1993) observed that students using software Derive did not need to substitute values to get a table and sketch functions' graphs. As a result, students did not develop the skill of evaluating functions by substitution. Even students who could perform the evaluation before the course seemed to have lost the skill afterwards.

In Brazil, Abrahão (1998) observed the reactions of secondary teachers dealing with function graphs produced by computers and graphic calculators. During the experiment, the teachers hesitated to consider that computers can provide "mistaken" or "incomplete" results, due to software limitations or visualization windows inadequacy. Those results were often accepted by participants as correct without query, even when clearly clashing to their prior knowledge of the topic. Laudares & Lachini (2000) observed the introduction of a computer laboratory for the teaching of Calculus in a large Brazilian university, which had been following a traditional approach before. The interviews with the Calculus teachers showed that most of them believed that laboratory activities would be a waste of time, which should be spent with classroom instruction, and the use of computer should be restricted to very heavy calculations. The authors report that the laboratory activities were restricted to mechanical tasks, unlinked to the theory studied in classroom. As a consequence, students seem to have no understanding of those activities. The authors conclude that the use of technology can constitute a important alternative, however it is necessary to encourage the development of a critical perspective by students.

USING THEORETICAL-COMPUTATIONAL CONFLICTS TO ENRICH CONCEPT IMAGES

Many authors agree that the effects of computers on mathematics learning do not depend only on any inherent feature of the devices themselves. Rather, such effects are consequent from the way they are (mis)used (see, for example, Tall (2000b), Belfort & Guimarães (1998)). The experiment reported by Hunter, Monaghan and

Roper, in particular, has uncovered a phenomenon of narrowing of concept images: *the intrinsic characteristics of the computational representation led to limitations on the concept images developed by learners*. Generally speaking, many limitations of computational representations for mathematical concepts arise from the algorithms' finite structure. Consider the graphs of $f(x)=\dfrac{1}{x-1}$ and $g(x)=\dfrac{1}{(x-1)^2}$ as drawn by Maple (figure 1). Both functions have a vertical asymptote at $x = 1$, but this line only appears on the picture of the graph of *f*. Actually, the software do not identify the existence of the asymptote for either function. The vertical line shown is drawn due to the joining of one point on the left of the discontinuity with one on its right, that is, the software considers the line as *part of the graph*. The same does not occur in the case of *g* because on either side of $x = 1$ the function is positive.

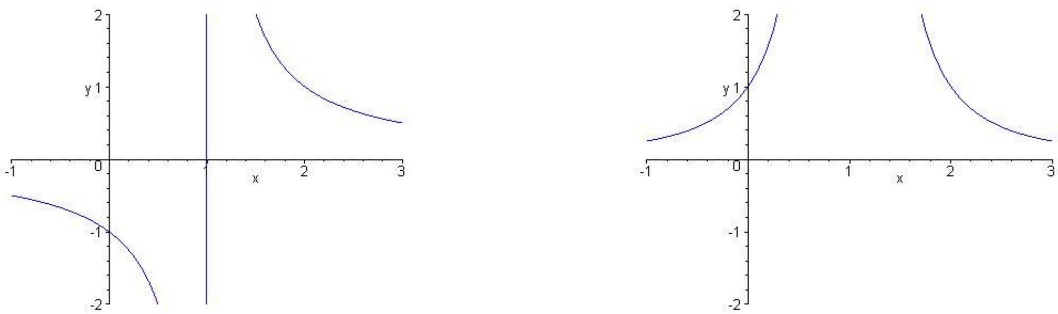


Figure 1. The graphs of $f(x)=\dfrac{1}{x-1}$ **(with a 'fake' asymptote) and** $g(x)=\dfrac{1}{(x-1)^2}$**.**

To focus on such situations, Giraldo (2001) names a **theoretical-computational conflict** to be *any situation in which a computational representation is apparently contradictory to the associated theoretical formulation* (Giraldo & Carvalho, 2002).

Another example of a theoretical-computational conflict is shown on figure 2 displaying the local magnification of $y=2x^2$, around the point $x_0=1$, performed by Maple. Since the curve is differentiable, it should acquire the aspect of a straight line when highly magnified. Rather, due to floating point errors, for very small values of graphic windows ranges (on orders lower than 10^{-6}) it looks like a polygon.

We believe the narrowing effect observed in Hunter, Monaghan & Roper's experiment was due not to the occurrence of theoretical-computational conflicts, but, to their absence. Overuse of computational environments—especially when not confronted by other forms of representation— may contribute to the conception that

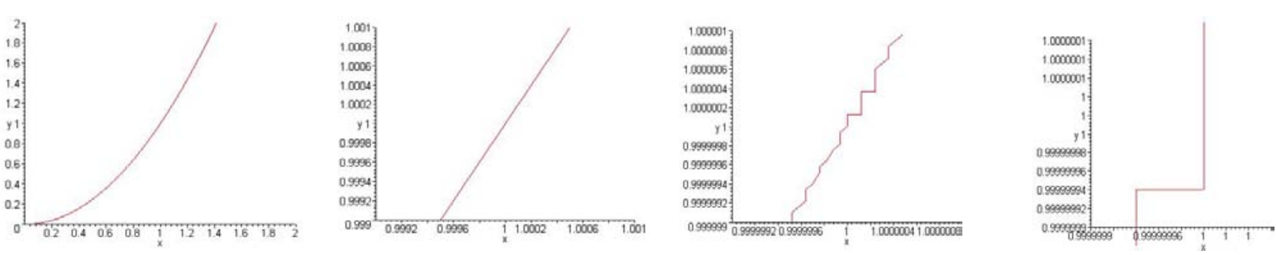


Figure 2. A theoretical-computational conflict observed through local magnification.

the limitations of the representation are characteristics of the mathematical concept itself, leading to the development of narrowed concept images. Sierpinska (1992) remarks that awareness of the limitations of each form of representation, and that they represent the same concept, are fundamental for the understanding of functions.

Our hypothesis is that, if theoretical-computational conflicts are emphasized, rather than avoided, the cognitive role of inherent characteristics of each form of representation may have a positive **conversion**—t*hey may contribute not to the narrowing, but to the enrichment of concept images*.

To investigate this hypothesis, we presented a sample of six first year undergraduate students in Brazil with theoretical-computational conflicts in individual interviews. One question considered the function $h(x)=\sqrt{x^2+1}$ and the graph sketched by Maple for $(x,y)\in[-100,100]^2$ (figure 3). The conflict here is between the appearance of the graph at the origin (which seemed to have a 'corner') and the formula which was differentiable. Students were free to manipulate the software as they wanted. Each was asked the following question:

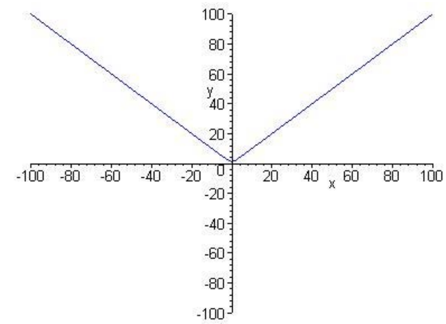
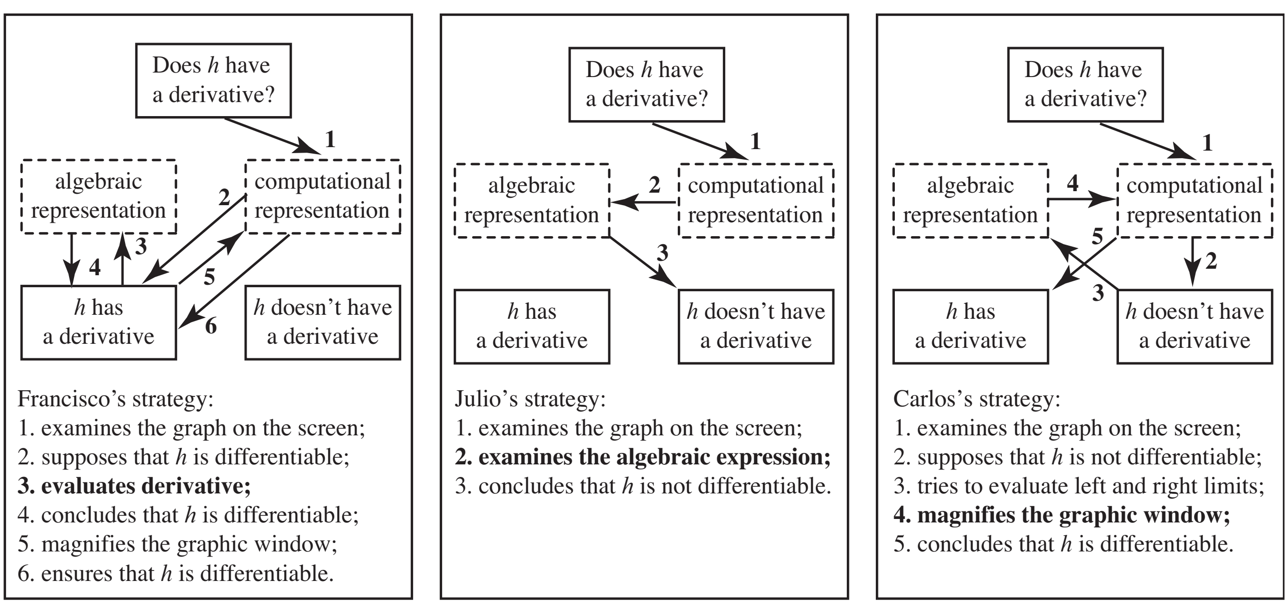


Figure 3. The graph of $\boldsymbol{h(x)=\sqrt{x^2+1}}$, for $\boldsymbol{-100\le x\le 100,\ -100\le x\le 100}$.

> You see on computer's screen the graph of the function $h(x)=\sqrt{x^2+1}$, sketched for $-100\le x\le 100$ and $-100\le x\le 100$. Do you think this function has a derivative?

Figure 4 summarizes the strategies of three of the students. The continuous boxes represent the question – *does h have a derivative* – and its possible answers – *h has a derivative* or *h doesn't have a derivative*. The dashed boxes represent the two given representations for *h* – computational (graph) and algebraic. The arrows indicate the interviewee's actions and are enumerated in chronological order. The boldface arrow indicates interviewee's decisive action, that is the one that led to the conclusion.



We will focus on Francisco's strategy, translated from Portuguese. He said:

> [...] For example, if you made $\sqrt{x^2}$, it'd be $|x|$. It'd have a corner. But you've put +1 there. This +1 complicate things, you can't take it off the square root completely, right? [...] Visually it isn't a corner, then, it'd have a derivative. I'm speaking in visual terms. Now, let's speak algebraically. Indeed, algebraically, if you differentiate, you'll manage to derive, then, we derive it, it's differentiable. [...] Can we zoom in here? [zooms in.] Yes, it looks like a parabola. Zooming in there, you see clearly how it's differentiable.

After concluding about the differentiability of *h*, Francisco spontaneously went on studying the function. He commented:

> That would be a good question. It looks like a [straight] line, or is it a line? [...] I know it has a derivative! I'll try to derive it to see if it is a line or not. [calculates the derivative] Look! This function will have a different slope for each point. It's not like the modulus function, which doesn't have a derivative at 0, but has the same derivative at the positive side of *x* and the same one at the negative side for all the points. This function is different, it will be close to the modulus function at $+\infty$ and $-\infty$. It will be close, but for each point it will have a different derivative. So, it looks like a line, but is not a line.

DISCUSSION

As we may see from the excerpts above, Francisco undertakes flexible connections between computational and algebraic representations in the course of the interview. His conclusion about the differentiability of *h* is grounded on the algebraic representation—he argues his case by applying the formulae. Furthermore, he makes use of the computational representation, by zooming in the graph, to build up a broader understanding of the local function behavior. However, the point we underline is that Francisco spontaneously goes further. After giving the answer for the proposed question, he formulates another question himself: *Is it really a straight line or does it only look like a straight line?* In this new investigation, another cognitive unit is triggered: *If the derivative is not constant, then the primitive function is not a straight line*. The formulation of the question, which activated a new cognitive unit, was motivated by a theoretical-computational conflict—the graph, as seen on the screen, did not match with the given algebraic expression.

A longer version of this article, which is part of a wider investigation using conflict situations, can be downloaded from www.davidtall.com/papers.

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