Mathematical Proof as Formal Procept in Advanced Mathematical Thinking

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In this paper the notion of "procept" (in the sense of Gray & Tall, 1994) is extended to advanced mathematics by considering mathematical proof as "formal procept". The statement of a theorem or question as a ***symbol*** *may evoke the proof deduction as a* ***process*** *that may contain sequential procedures and require the synthesis of distinct cognitive units or the general notion of the theorem or question as an* ***object*** *like a manipulable entity to be used as inputs to other theorems or questions. Therefore, a theorem, for example, could act as a pivot between a* ***process*** *(method of proof) and the* ***concept*** *(general notion of the theorem). We hypothesise that mature theorem-based understanding (in the sense of Chin & Tall, 2000) should possess the ability to consider a theorem as a "formal procept"***,** *and it takes time to develop this ability. Some empirical evidence reveals that only a minority of the first year mathematics students at Warwick could recognise a relevant theorem as a "concept" and these did not have the theorem with the notion of its proof as a "formal procept". A year later some more successful students showed a concept of the theorem as a "formal procept" and their capability of manipulating the theorem flexibly.*

Introduction

Mathematical proof is one of the most important aspects of formal mathematics. From most mathematics textbooks we can simply see the process of a mathematical proof as the development of a sequence of statements using only definitions and preceding results, such as deductions, axioms, or theorems. The *process* of a mathematical proof occurs when the proof is built up and looked at subsequently as a process of deducing the statement of the theorem from definitions and the specified assumptions. A proof becomes a *concept* when it can be used as an established result in future theorems without the need to unpack it down to its individual steps. We choose to focus on this sequence of proof as a process of deduction becoming encapsulated as a concept of proof in a manner that would seem natural to most mathematicians. We note that there are alternative theories, for example, Dubinsky and his colleagues (Dubinsky, Elterman & Gong, 1988) focus on the use of quantified statements as processes becoming turned into mental objects by applying the quantifiers. Pinto and Tall (2002), in contrast, show how some students are capable of building formal proofs by reconstruction of prototypical imagery used in thought experiments.

Gray and Tall (1994) suggest the notion of "*procept*", which was taken to be characteristic of symbolism in arithmetic, algebra and calculus, defined in the following terms:

> An *elementary procept* is the amalgam of three components: a *process* which produces a mathematical *object*, and a *symbol* which is used to represent either process or object.
>
> A *procept* consists of a collection of elementary procepts which have the same object. (Gray & Tall, 1994, pp.3-4)

The original definition was made in the context where the authors were aware of a wide range of examples and the definition was framed to situate the examples within the definition. In this primary consideration it is a "descriptive definition", in the sense of a definition in a dictionary, rather than a "prescriptive definition", in the sense of an

axiomatic theory. However, if we consider the definition of "procept" in a prescriptive view, it seems applicable to extend the notion of "procept" to the notion of formal proof, which can be called "*formal procept*", by adding the following analysis.
It should be noticed that there are three components of an elementary procept: *process*, *object*, and *symbol*. Now we can put the frame of Gray & Tall's "procept", particularly in the form of an "elementary procept", on the notion of mathematical proof. The *symbol* is the statement of what is going to be proved [which can be a theorem or a question]. The *process* is the deduction of the whole proof. And the *object* is the concept of the notion of proof, i.e. the real meaning of the theorem [or question]. The statement of a theorem [or question] as a *symbol* may evoke the proof deduction as a *process* that may contain sequential procedures and require the synthesis of distinct cognitive units or the general notion of the theorem or question as an *object* like a manipulable entity to be used as inputs to other theorems or questions. Therefore, a theorem, for example, could act as a pivot between a process (method of proof) and the concept (general notion of the theorem). With the above interpretation we could see the role of a symbol as being pivotal not only in elementary mathematical thinking but also in advanced mathematical thinking to allow us to change the channel between using a symbol as a concept to reflect on and link to other concepts and as a process to offer the detailed steps to deduce a proof. However, an immediate argument arises. It seems that the above corollary does not always follow because even mathematicians sometimes use certain theorems without fully understanding their proofs. However, we find this viewpoint an *advantage* to our analysis, for it simply shows that such individuals are *not* using theorems as formal precepts, they only have *part* of the structure, usually the statement of the theorem which they then use as an ingredient in another proof without fully understanding the totality of the structure. Our evidence shows that few students understand the notion of proof as a formal procept, but our empirical research shows that, over time, more students grasp the subtlety of the idea.

Chin and Tall (Chin, 2002; Chin & Tall, 2000) postulated a hierarchy running through the development of systematic proof, in stages consisting of *concept image-based*, *definition-based*, *theorem-based*, and *compressed concept-based*. These stages show successive compressions of knowledge in the sense suggested by Thurston (1990). The first stage, which is concept image-based sees the student having a concept image of a particular concept built from experience, but very much at an intuitive stage of development. The transition to definition-based involves the first compression. From amongst the many properties of the concept-image, a number of generative ideas are selected and refined down to give the concept-definition. During the definition-based stage, the definitions are used to make deductions, all of which are intended to be based explicitly on the definitions. Many students, however, remain in the concept-image based stage, basing their arguments not on definitions and deductions, but on thought experiments using concept images (Tall & Vinner, 1981, Vinner, 1991). Bills and Tall (1998) introduced the term 'formally operable' definition (or theorem), proposing that:

> A (mathematical) definition or theorem is said to be *formally operable* for a given individual if that individual is able to use it in creating or (meaningfully) reproducing a formal argument.

Tracing the development of five individuals over two terms in an analysis course, focusing on the definition of "least upper bound", they found that many students never have operable definitions, relying only on earlier experiences and inoperable concept images. Furthermore, it was also possible for a student to use a concept without an operable definition in a proof using imagery that happens to give the necessary information required. Thus, we already know that the development from the concept-image based stage to the compressed notion of operable definition is a difficult one for many students. Even so, they are then expected to move on to the next, theorem-based stage, when theorems that have been proved by the process of proof are now regarded as being compressed into *concepts* of proof, to be used as entities in the process of proving new theorems. For this to be fully successful, we hypothesise that students who have developed mature theorem-based understanding should possess the ability to consider a theorem as a “formal procept”. We further hypothesise that individuals with this capacity to use theorems flexibly as processes or concepts are developing a compressed concept level of mathematical thinking that enables them to think with great flexibility and conceptual power.

Empirical Study

The subjects are fifteen second year mathematics students following a course in one of the top five ranked mathematics departments in the UK. Their marks for the first year study are widely distributed — three are over 80, four between 70 to 79, four between 60 to 69, one between 50 to 59, and three between 40 to 49. The same questionnaire on the topic of “equivalence relations & partitions”, which they answered for the first time about a year ago when just having learned the topic for several weeks, was answered again. The fifteen subjects were interviewed during the first term in their second year. The study designed is to investigate whether and how the students’ understanding improved.

“Equivalence Relation” at the Theorem-based Level

The following question is designed to examine if the students improve their understanding from the definition-based level to theorem-based level:

> A relation on a set of sets is obtained by saying that a set X is related to a set Y if there is a bijection $f: X \to Y$. Is this relation an ***equivalence relation***?

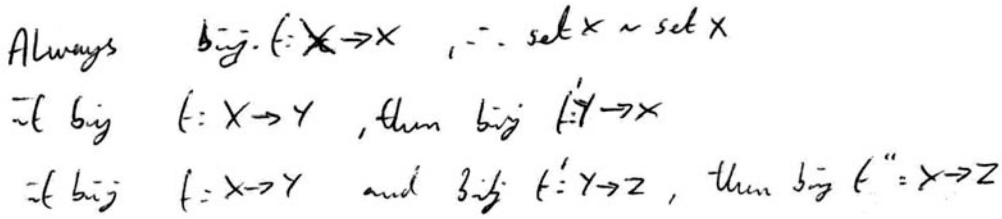
The difference between the students’ former and recent responses is represented in Table 1.

		First Year (N=15)	Second Year (N=15)
Formal perhaps with some informal language	Definition	7	1
	Theorem	3	12
	Partition/Equiv. class	1	0
Informal	Informal definition	3	1
	Misunderstanding	0	0
	Don’t know	0	0
	Others	1	1
	No response	0	0

Table 1: Responses to the formal “bijection” question

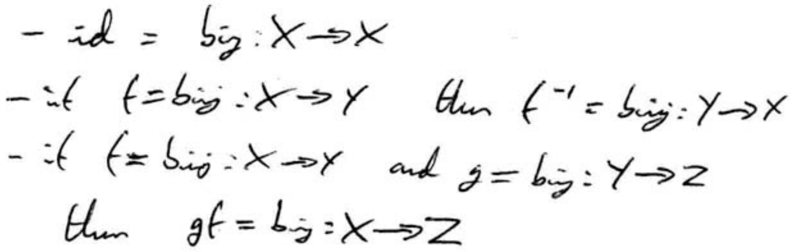
In the first year, ten students (seven “Formal definition” and three “Informal definition”) were able to give a definition-based response. Only three could apply the relevant theorem to make their deductions. One was classified as “Other” because he only answered “Yes” without giving any explanation. PAUOST, again, was the one who responded to the problem in terms of partition.

As was found in the cross-sectional study, the students’ concept images were not solid at that time. Although most of the students knew the relevant theorem, they did not really have a clear idea how to apply it to this practical problem. JULSON was an example offering a definition-based response (as follows) but he vividly expressed in the interview — "I remember I learned it [the theorem] in the lecture a couple weeks ago, but I'm sorry I haven't put it in my head yet."



(JULSON(68%), 1st year)

After being given a period of time to digest what they had been taught, whilst three were theorem-based in the first year test, twelve were able to upgrade their understanding to the theorem-based level in the second year. In addition, compared with their former responses, the quality of their deductions seemed to indicate that the concept of the theorem had become more workable in their concept images. JULSON's recent response (classified as "theorem-based") could offer us some evidence.



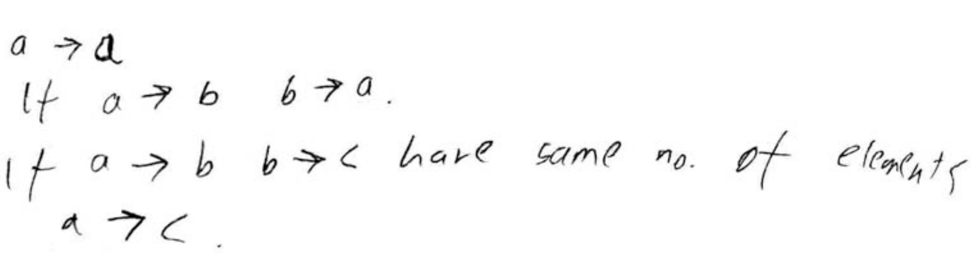
(JULSON(68%), 2nd year)

In the second year, JULSON not only stated the theorem but also explained how the theorem can be proved. He clearly showed that the notion of proof of this theorem had become a "procept" in his concept image as he knew both the method of proof (as process) and the statement of theorem (as concept).

As to the other three, one was classified as "Formal definition" because he answered the question by checking the three axioms without referring to the theorem, another only simply replied "all 3 axioms are satisfied", the other gave a wrong answer by saying "not symmetric or transitive as a bijection does not allow elements to be bijective to one another in Y". These three students' understanding on this question did not seem to be improved.

The following quoted conversations recorded in the interview and e-mail discussion with DIAHUM might offer us some more delicate insight into how the successive moves — from informal to definition-based, then on to theorem-based conceptions — happened with the individual.

DIAHUM gave the following response (classified as "Informal definition") in the first year:



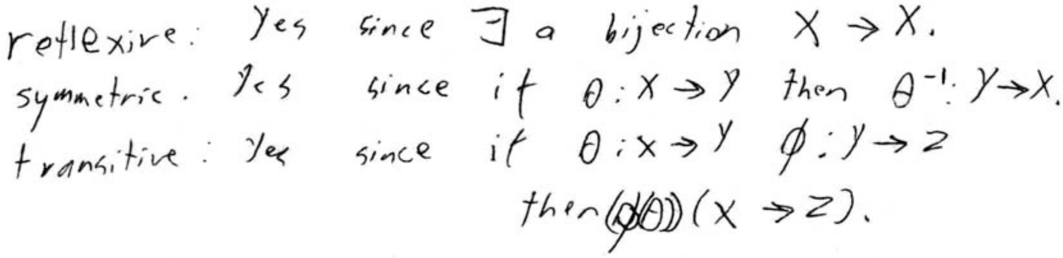
(DIAHUM(48%), 1st year)

He cleared up what he meant in his response through an e-mail.

> I was trying to apply the definition of equivalence relation to make the answer more formal. But I don't think my answer was formal enough because I didn't really know how to apply the definition even though I can remember it. And another problem is I can't recall the definition of bijection. What I can remember is a bijection is one-to-one and onto. That means the two sets have the same number of elements (he explained later that this idea was from what he learned at A-level).

He also expressed that he knew the theorem which is directly relevant to this question. (Please refer to the chapter "Design of Questionnaire".) But the theorem seemed to be something only in his understanding in a theoretical manner rather than in his intuition which can be freely referred to at any time.

In the second year, he responded in terms of the relevant theorems as follows:



reflexive: Yes since $\exists$ a bijection $X \to X$.
symmetric. Yes since if $\theta: X \to Y$ then $\theta^{-1}: Y \to X$.
transitive: Yes since if $\theta: X \to Y$ $\phi: Y \to Z$
then $(\phi\theta)(X \to Z)$.

(DIAHUM(48%), 2nd year)

Although he did not use the term "identity" to mention the bijection mapping from the set *X* to itself, he could precisely write down the composition of two bijections whilst some others mentioned it in the wrong order. When being asked why he answered in this way this time, he gave the following explanation:

> Well, I think it's fairly natural for me to make the deduction like this. When I faced the question, the theorems burst upon my head and I just wrote down the proof.

DIAHUM's case seems to suggest that he cannot freely apply a formal conception until it is assimilated in his concept image as an embodiment. When DIAHUM could only recite the formal definition of equivalence relation but was still struggling with the implication of it, it is natural for him to consult the relevant ideas he learned at school to make his first deduction because they were more embodied in his concept image. Having a year to digest all these notions, the theorem he only knew about before, had been assimilated into his concept image as embodiments that he could recall intuitively in the second test.

Summary of evidence of moving from definition-based to theorem-based

In the students' (written or oral) responses, we can see that most students seemed to apply the relevant theorem directly to this practical question in the second year whilst most of them only gave a definition-based response in the previous year. This kind of result is consistent with the successive move from definition-based conceptions to theorem-based conceptions over time during which the ideas are being used formally. From the improved quality of the students' deductions, I consider, at least for some students, the notion of proof of the theorem seemed to have become a "procept" in their concept images. Since they only seemed to know the concept (statement of the theorem) but not the process (method of proof) of the notion of proof of the theorem before. But, a year later, some students appeared to be able to unpack the notion of the theorem to the proof process and to apply the theorem to the question more flexibly.

Linkage between “Equivalence Relations” and “Partitions” (at the compressed-concept level)

Theoretically the notion of “equivalence relations” is linked to the notion of “partitions” as there is a theorem connecting these two notions together. The following question is asked in order to examine whether the students appreciate the idea practically.

> Write down two different ***partitions*** of the set with four elements, $X=\{a,b,c,d\}$. For the first of these, please write down the ***equivalence relation*** that it determines.

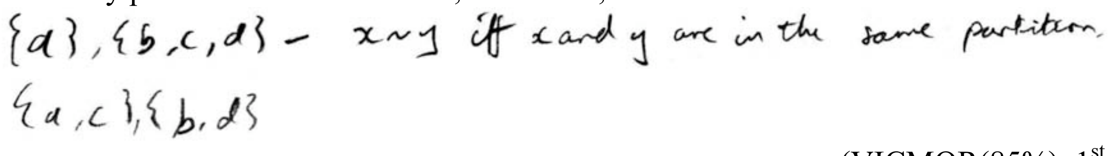
Table 2 shows that most of these fifteen students could handle this practical problem better than a year ago.

		First Year (N=15)	Second Year (N=15)
Correct Partitions	with correct ER	3	8
	with wrong ER	3	1
	without giving ER	7	3
Incorrect Partitions	with correct ER	2	2
	with wrong ER	0	0
	without giving ER	0	1
Don’t know		0	0
No response		0	0

Table 2: Responses to the “link between Equivalence Relation & Partition” question

When being asked to give examples of partitions of a set, thirteen out of fifteen gave two satisfactory partitions in the first year but the number decreased by one in the second year. The two who failed to give correct examples of partitions (in both years) misunderstood partition as a subset of the set because they seemed to follow the linguistic meaning of a partition. This kind of result seems to be consistent with the students’ feeling that they understood partition better than equivalence relation, yet actually gave unsatisfactory formal responses. In addition, please note that all fifteen students said that they remembered they had seen in the lecture the theorem which links the two notions together.

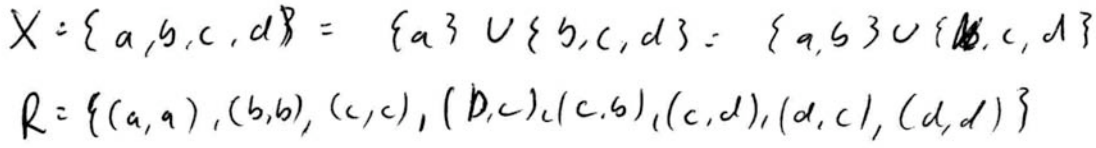
In the first year, seven students gave satisfactory partitions but did not manage to write down the corresponding equivalence relations that their given partition determined. Two gave a wrong equivalence relation because they seemed to get confused when dealing with too many pairs whilst the other one, VICMOR, tried to define an exact relation as follows:



(VICMOR(85%), 1st year)

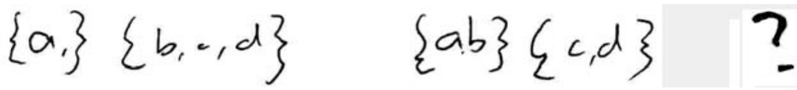
As I have mentioned in the chapter of cross-sectional study, VICMOR thought the question was not to ask them to write down something like *a~b*, *b~c*,…,etc. because it was trivial. He had no problem offering the equivalence relation determined by his given partition in the interview. However, being a very successful student, he immediately recognised his own mistake saying “in the same *partition*” when reviewing his answered questionnaire. He said that he did not notice he answered the question in that way. It seems that this kind of idea came out from his thinking intuitively. VICMOR has said that he always focuses on the formal definition to grasp the implication of it first then he tries to accommodate his (informal) embodiments to fit into the formal concept in his concept image. This seems to suggest that we should be more careful about employing embodiments especially when we are constructing a concept image for a new formal concept because the embodied objects may turn up in our thinking without noticing.

In the second year, eight students could successfully give two correct partitions with a correct corresponding equivalence relation. The only one who gave correct partitions but with a wrong corresponding equivalence relation was VICMOR. He rashly missed the two pairs (*b*,*d*), (*d*,*b*) in his equivalence relation. However, he used different symbols precisely in his response.



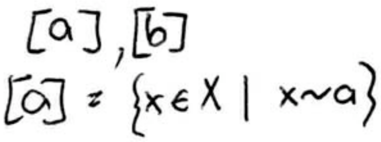
(VICMOR(85%), 2nd year)

Three students within the seven a year ago kept getting confused in determining the equivalence relation defined by a given partition. The same two students still had the misconception: that the term "partition" referred to each individual subset, not to the collection of all subsets. The only student, MAUHAM, who gave wrong partitions without the corresponding equivalence relation was able to give two correct partitions followed by a big question mark in the first test.



(MAUHAM (71%), 1st year)

In the second year, he gave the following response:



(MAUHAM (71%), 2nd year)

In the interview he confessed he just copied the notation from the following question. He had no idea of the formal notion of a partition so he could not answer the question.

Summary of evidence of developing a compressed concept relating equivalence relation and partition

Six students seemed to have improved their understanding and became able to link the notion of partition to the notion of equivalence relation in the second test. However, because of having no idea of the formal definition of "partitions", another six appeared to be drowned in their improper conceptions so that they still could not make the theoretical linkage operable.

The result of this question appears to parallel QB3 in some instances. All the students sensed the relevant theorem linking the two notions together but only a few could practically apply the theorem to the question in the first year. A year later, some students' understanding had progressed to reach a more mature theorem-based level. The theorem was no longer a "concept" only but also a "process" which suggests the method of proof to make the whole notion of proof of the theorem as a "procept" in their concept images.

Conclusions

The concept image-based thinking is based on the embodiments embedded in the individual's concept image. As this kind of informal thinking is more strongly linked to the real world, it is very natural to be concealed in any mode of understanding with or without being realised.

At the level of definition-based thinking, the role of the definition and whether it is *operable* is considered (in the sense of Bills & Tall, 1998). The students need time to digest the implications of the definition so that they can develop their understanding from *making* to *having* the definition. That means the definition is operable in their concept images so that they are able to use it to make relevant formal deductions.

The next, theorem-based level, moves on to using the results of theorems as elements in succeeding proofs. Whether the theorems involved are "*procepts*" is considered. I notice that only a relatively low level of proof as procept appeared even in the high quality novice students studying at Warwick. However, over the period of a year there is a movement from informal thought to definition-based deductions and on to theorem-based deductions.

The concepts introduced through definitions may be handled as "cognitive units" which are thinkable elements that can be manipulated as entities. These include operable definitions, theorems as procepts, and concepts such as equivalence relation and partition which are mathematically equivalent and may therefore be compressed to a single cognitive unit. Only a few students display understanding at this level.

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